All about “cosine” and truncated cosine in CT

Discrete Time Systems

Luca Martino — luca.martino@urjc.es — http://www.lucamartino.altervista.org

Fourier series of the cosine “with Euler”

Fourier series of the cosine "with Euler"

$$x(t) = \cos(\omega_0 t)$$

- For Euler…

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

- Sum of complex exponentials… this is already the Fourier Series !!!

Fourier series of the cosine “with Euler”

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

- Sum of complex exponentials… this is already the Fourier Series !!!

$$a_1 = \frac{1}{2}, \qquad a_{-1} = \frac{1}{2},$$

$$a_k = 0 \text{ for all } k \neq -1, 1$$

Fourier series of the cosine “with definition” of a_k

Fourier series of the cosine "with definition" of a_k

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

Note that \omega_0 inside the cosine is exactly \omega_0, that is, the fundamental frequency of the signal…

$$= \frac{1}{T_0}\int_{T_0} \cos(\omega_0 t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{2T_0}\int_{T_0} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-jk\omega_0 t}dt$$

$$= \frac{1}{2T_0}\int_{0}^{T_0} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-jk\omega_0 t}dt$$

Fourier series of the cosine "with definition" of a_k

$$a_k = \frac{1}{2T_0}\int_0^{T_0} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2T_0}\int_0^{T_0} \left(e^{j(1-k)\omega_0 t} + e^{-j(1+k)\omega_0 t}\right) dt$$

$$= \frac{1}{2T_0}\int_0^{T_0} e^{j(1-k)\omega_0 t} dt + \frac{1}{2T_0}\int_0^{T_0} e^{-j(1+k)\omega_0 t} dt$$

Now we focus on this integrals….

Fourier series of the cosine “with definition” of a_k

we can do more calculus (and also divide in cosine and sine, if you need/desire), the pure complex exponential…

but if we recognize we are integrating *virtually* a cosine/sine signal in its period then we can easily write:

$$(\text{Recall that } \omega_0 = \frac{2\pi}{T_0})$$

$$\int_0^{T_0} e^{j(1-k)\omega_0 t} dt = 0 \quad \text{for } k \neq 1$$

$$\int_0^{T_0} e^{-j(1+k)\omega_0 t} dt = 0 \quad \text{for } k \neq -1$$

Fourier series of the cosine "with definition" of a_k

For k=1, and k=-1:

$$\int_0^{T_0} e^{j(1-1)\omega_0 t} dt = \int_0^{T_0} 1 \cdot dt = T_0$$

$$\int_0^{T_0} e^{-j(1-1)\omega_0 t} dt = \int_0^{T_0} 1 \cdot dt = T_0$$

Fourier series of the cosine "with definition" of a_k

Finally, we obtain:

$$a_k = \frac{1}{2T_0}\int_0^{T_0} e^{j(1-k)\omega_0 t}dt + \frac{1}{2T_0}\int_0^{T_0} e^{-j(1+k)\omega_0 t}dt$$

$$a_1 = \frac{1}{2T_0}T_0 + 0 = \frac{1}{2} + 0 = \frac{1}{2}$$

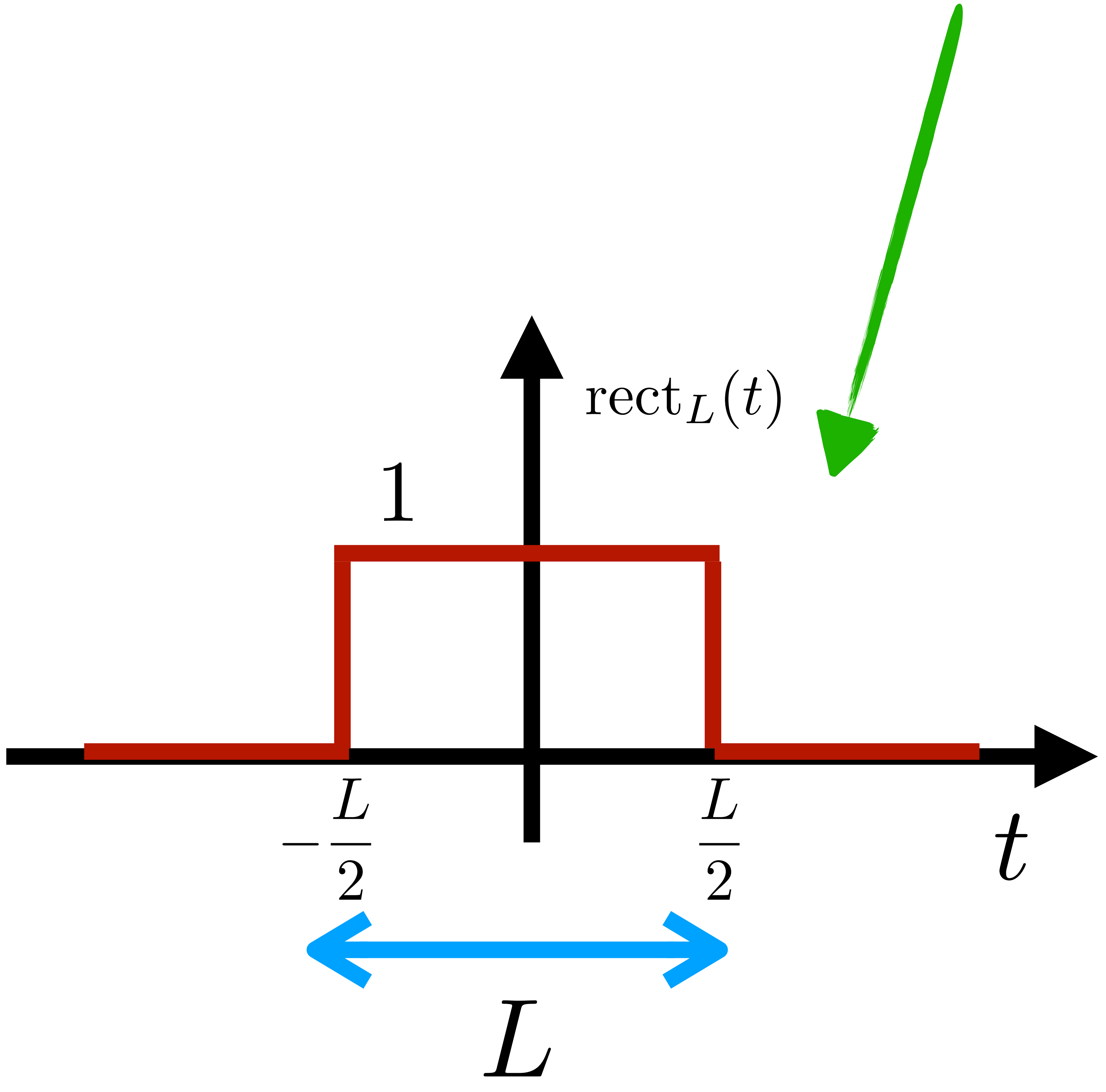
$$a_{-1} = 0 + \frac{1}{2T_0}T_0 = 0 + \frac{1}{2} = \frac{1}{2}$$

$$a_k = 0, \qquad \text{for all } k \neq 1, -1$$

Truncated cosine signal
(method 1: from the "definition" of Standard FT)

Truncated cosine signal

$$x(t) = \cos(\omega_0 t)\text{rect}_L(t)$$



Truncated cosine signal: method 1

$$X(\omega) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos(\omega_0 t) e^{-j\omega t} dt$$

$$= \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right) e^{-j\omega t} dt$$

$$= \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{j(\omega_0 - \omega)t} dt + \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-j(\omega_0 + \omega)t} dt$$

Truncated cosine signal: method 1

$$X(\omega) = \frac{1}{2}\left[\frac{-1}{j(\omega-\omega_0)}e^{j(\omega_0-\omega)t}\right]_{-\frac{L}{2}}^{\frac{L}{2}} + \frac{1}{2}\left[\frac{-1}{j(\omega+\omega_0)}e^{-j(\omega_0+\omega)t}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$X(\omega) = \frac{1}{2}\left[\frac{-e^{j(\omega_0-\omega)\frac{L}{2}}}{j(\omega-\omega_0)} + \frac{e^{-j(\omega_0-\omega)\frac{L}{2}}}{j(\omega-\omega_0)}\right] + \frac{1}{2}\left[\frac{-e^{-j(\omega_0+\omega)\frac{L}{2}}}{j(\omega+\omega_0)} + \frac{e^{j(\omega_0+\omega)\frac{L}{2}}}{j(\omega+\omega_0)}\right]$$

$$X(\omega) = \frac{1}{2}\left[\frac{-e^{-j(\omega-\omega_0)\frac{L}{2}} + e^{j(\omega-\omega_0)\frac{L}{2}}}{j(\omega-\omega_0)}\right] + \frac{1}{2}\left[\frac{-e^{-j(\omega_0+\omega)\frac{L}{2}} + e^{j(\omega_0+\omega)\frac{L}{2}}}{j(\omega+\omega_0)}\right]$$

Truncated cosine signal: method 1

$$X(\omega) = \frac{1}{2}\left[\frac{-e^{-j(\omega-\omega_0)\frac{L}{2}} + e^{j(\omega-\omega_0)\frac{L}{2}}}{j(\omega-\omega_0)}\right] + \frac{1}{2}\left[\frac{-e^{-j(\omega_0+\omega)\frac{L}{2}} + e^{j(\omega_0+\omega)\frac{L}{2}}}{j(\omega+\omega_0)}\right]$$

(for any doubts, recall also that $\sin(a) = \sin(-a)$)

$$X(\omega) = \frac{1}{2}\left[\frac{2j\sin((\omega-\omega_0)\frac{L}{2})}{j(\omega-\omega_0)}\right] + \frac{1}{2}\left[\frac{2j\sin((\omega+\omega_0)\frac{L}{2})}{j(\omega+\omega_0)}\right]$$

$$X(\omega) = \frac{\sin((\omega-\omega_0)\frac{L}{2})}{\omega-\omega_0} + \frac{\sin((\omega+\omega_0)\frac{L}{2})}{\omega+\omega_0}$$

Truncated cosine signal: method 1

This is already the solution:

$$X(\omega) = \frac{\sin((\omega - \omega_0)\frac{L}{2})}{\omega - \omega_0} + \frac{\sin((\omega + \omega_0)\frac{L}{2})}{\omega + \omega_0}$$

That can be written as sum of two “octupus”/pulpos: (i.e., two sinc functions)

$$X(\omega) = \frac{1}{2}\mathrm{sinc}_{\frac{L}{2}}(\omega - \omega_0) + \frac{1}{2}\mathrm{sinc}_{\frac{L}{2}}(\omega + \omega_0)$$

Truncated cosine signal: method 1

$$X(\omega) = \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega - \omega_0) + \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega + \omega_0)$$

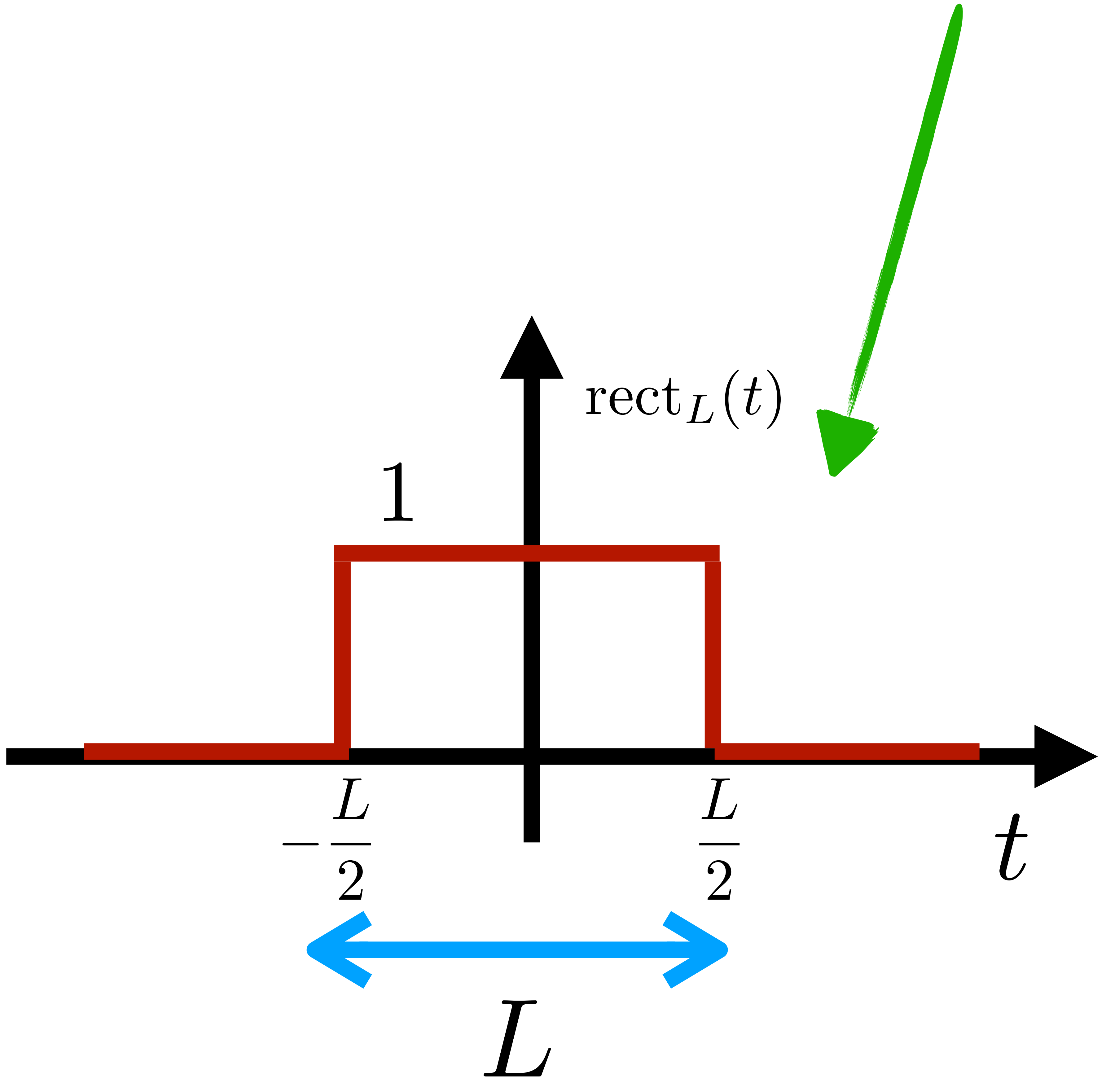
where:

$$\text{sinc}_{\frac{L}{2}}(\omega) = \frac{2\sin(\omega\frac{L}{2})}{\omega}$$

Truncated cosine signal
(method 2: using properties…)

Truncated cosine signal

$$x(t) = \cos(\omega_0 t)\text{rect}_L(t)$$



Truncated cosine signal: method 2

$z(t)$ $f(t)$

$$x(t) = \cos(\omega_0 t)\text{rect}_L(t)$$

product in time ==> convolution in frequency

$$X(\omega) = \frac{1}{2\pi} Z_G(\omega) * F(\omega)$$

Truncated cosine signal: method 2

$$X(\omega) = \frac{1}{2\pi} Z_G(\omega) * F(\omega)$$

$$Z_G(\omega) = \pi \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

$$F(\omega) = \mathrm{sinc}_{L/2}(\omega) = \frac{2\sin(\omega \frac{L}{2})}{\omega}$$

Truncated cosine signal: method 2

Using the property of the convolution with deltas:

$$X(\omega) = \frac{1}{2\pi}\pi \,\mathrm{sinc}_{\frac{L}{2}}\left(\omega - \omega_0\right) + \frac{1}{2\pi}\pi \,\mathrm{sinc}\, c_{\frac{L}{2}}\left(\omega + \omega_0\right)$$

and we obtain again:

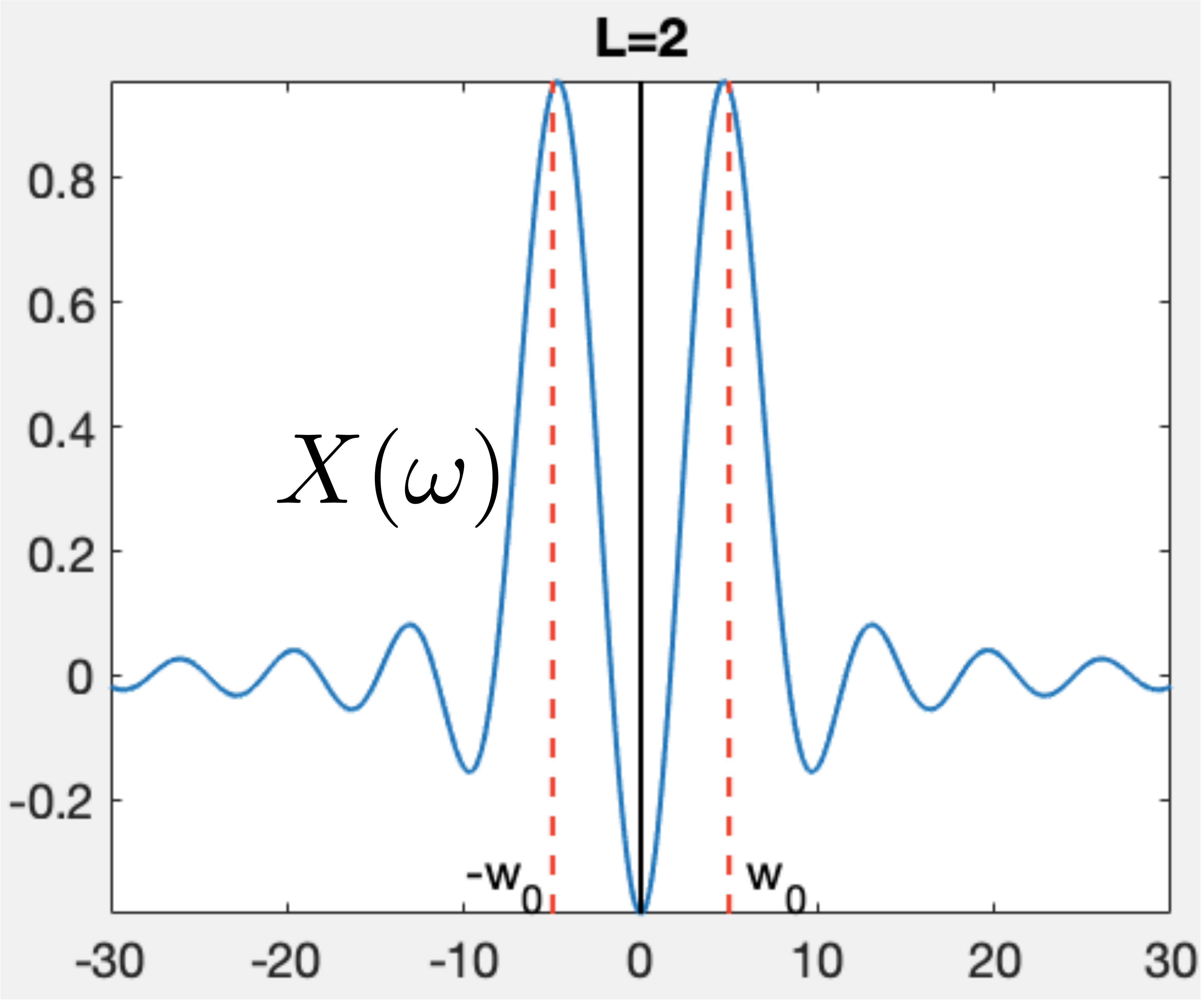
$$X(\omega) = \frac{1}{2}\mathrm{sinc}_{\frac{L}{2}}(\omega - \omega_0) + \frac{1}{2}\mathrm{sinc}_{\frac{L}{2}}(\omega + \omega_0)$$

$$X(\omega) = \frac{\sin((\omega - \omega_0)\frac{L}{2})}{\omega - \omega_0} + \frac{\sin((\omega + \omega_0)\frac{L}{2})}{\omega + \omega_0}$$

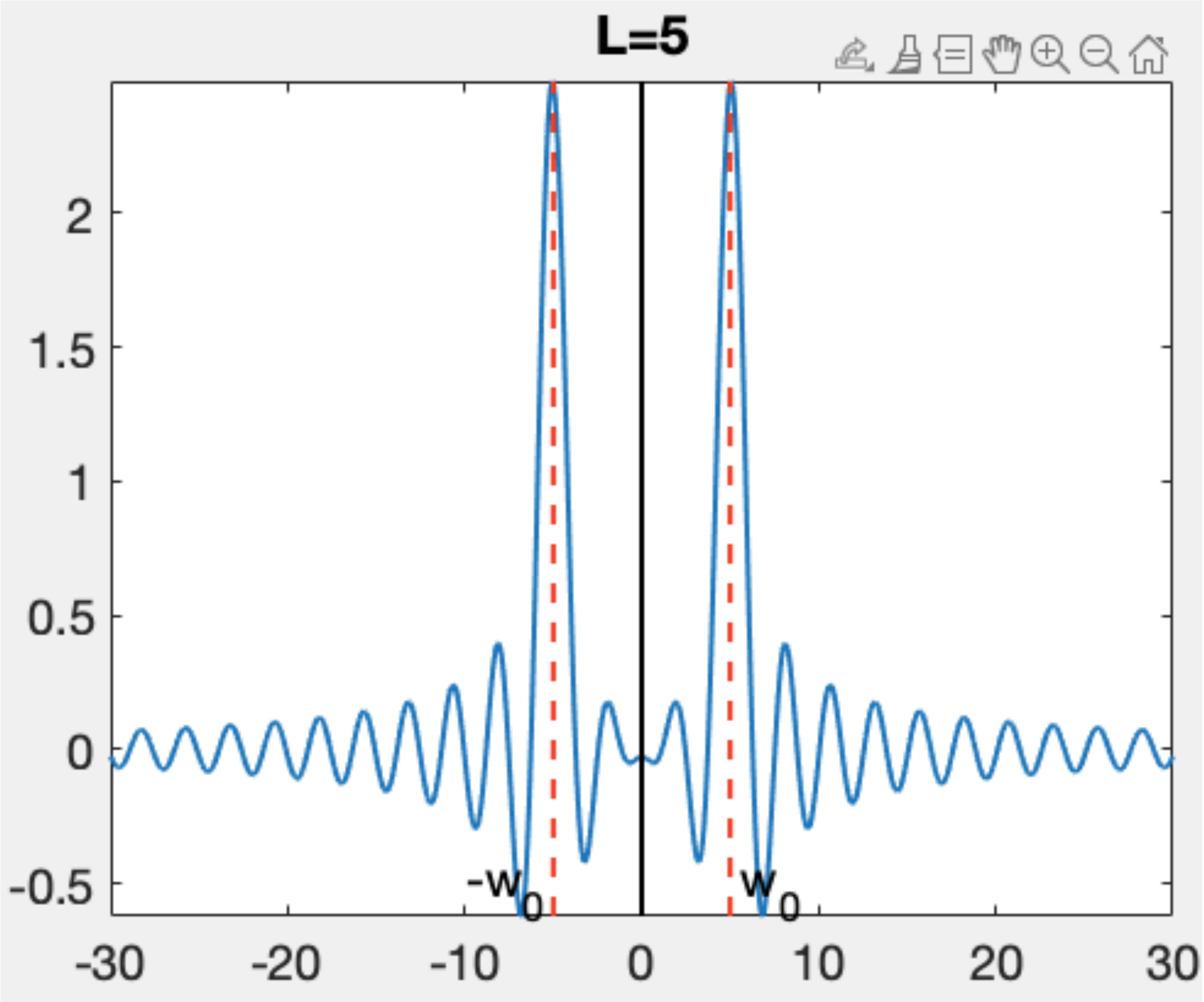
Plotting the Stand. FT of the truncated cosine signal

Plotting X(w)

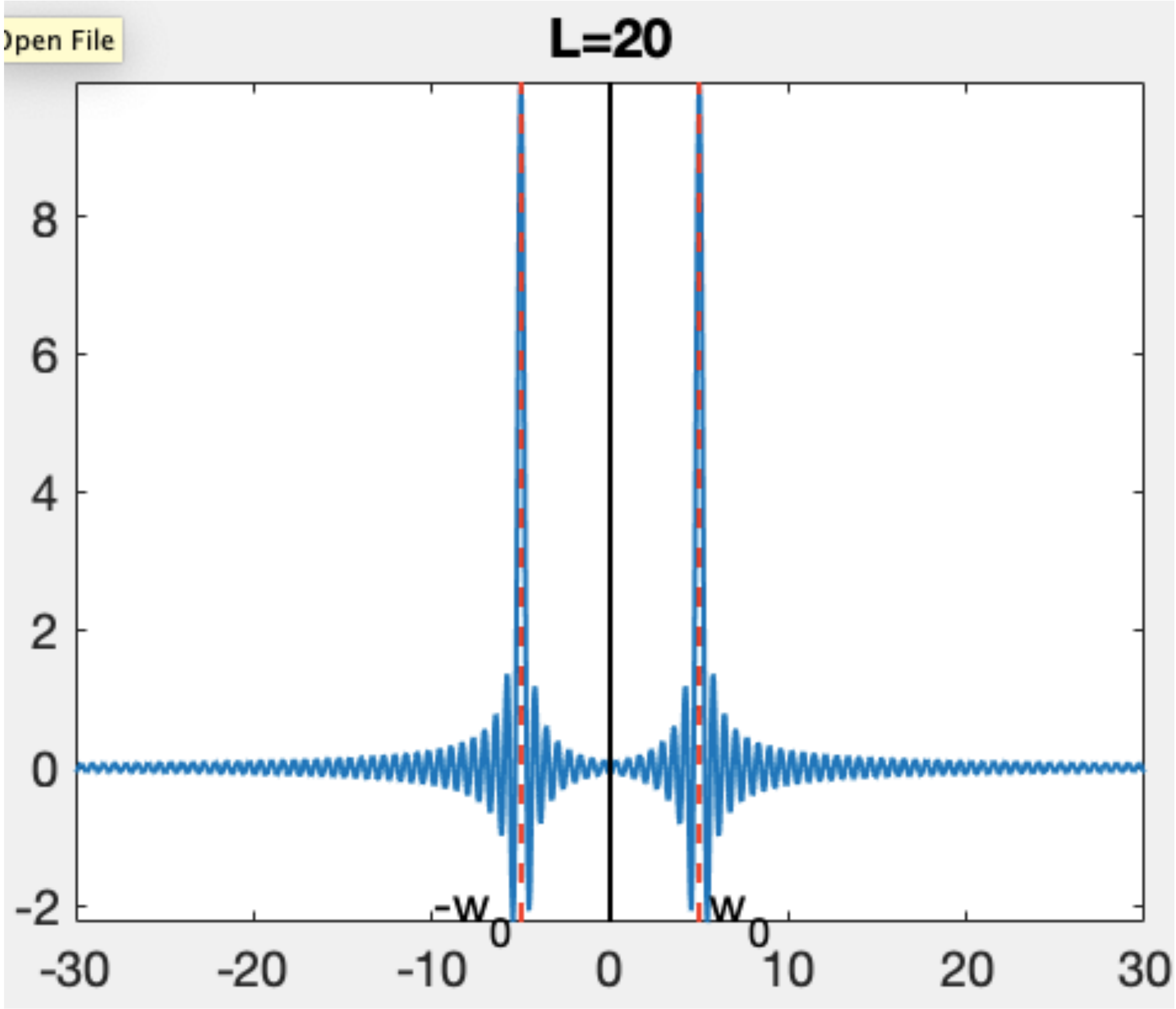
Since X(w) is real in this case, we can plot directly X(w) (without taking the module)



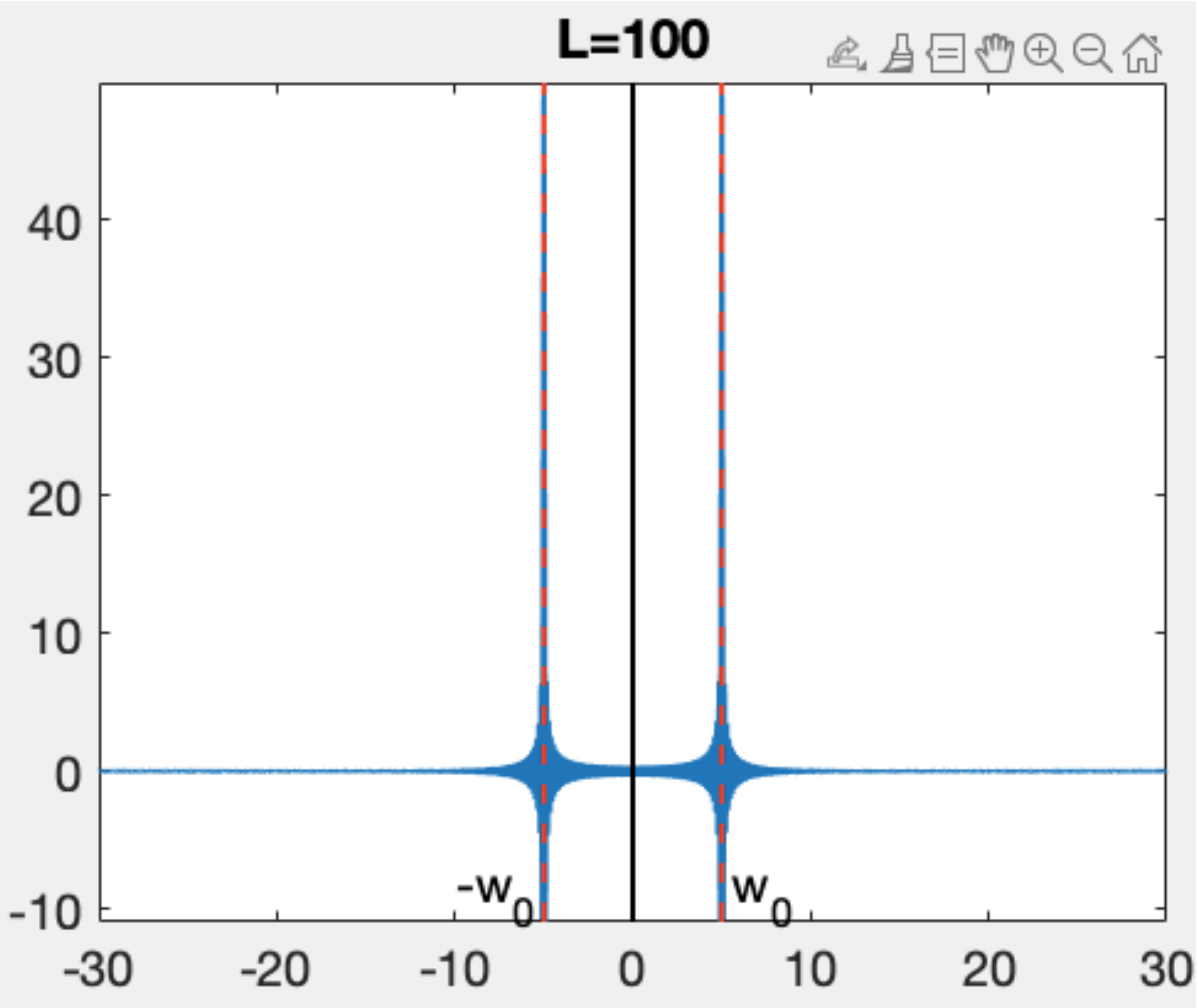
Plotting X(w)



Plotting X(w)



Plotting X(w)



Plotting X(w)

Note that, as L grows to infinity, we have:

$$X(\omega) \longrightarrow \underbrace{\pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)}_{Z_G(\omega)}$$

Since the rectangle f(t) tends to be f(t)=1 (a constant signal).

Questions?