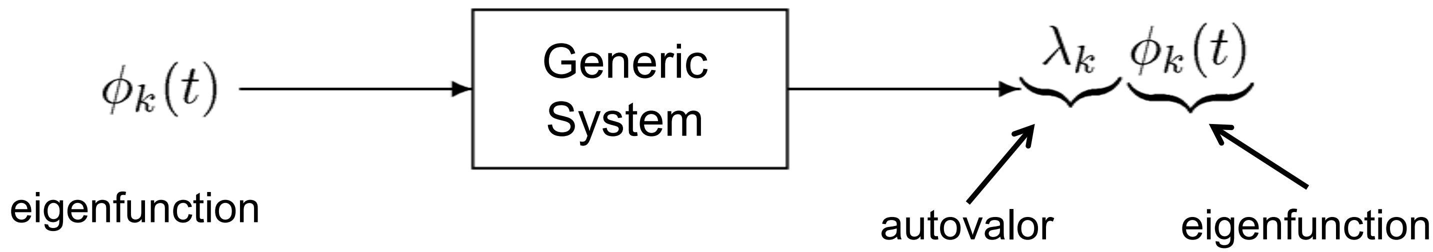
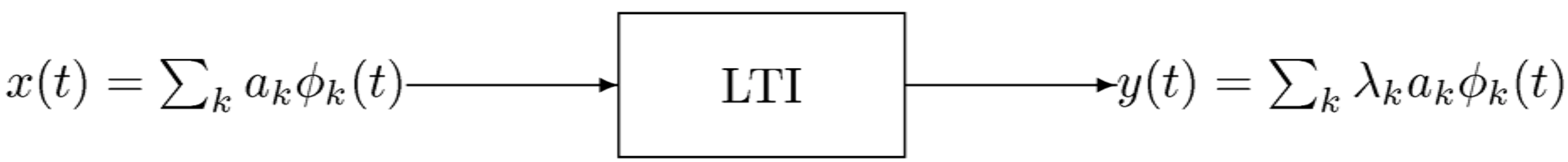
Eigenfunctions of LTI systems: exponentials

➢ What is an eigenfunction?



- Output with input an eigenfunction → the same eigenfunction multiplied for a scalar number/factor (called eigenvalue)
- For the linearity of LTI systems:



Exponentials as eigenfunctions of LTI systems

Very important slide….

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

s: complex number

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = [\int_{-\infty}^{\infty} h(\tau) e^{s\tau} d\tau] e^{st} = H(s) e^{st}$$

eigenvalue

eigenfunction

H(s) is also complex number, in general

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

Laplace Transform

- If x(t) is a combination of exponential complex signals we get :

$$x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

Zeros, poles…. We can understand this study

Properties of the LTI systems in DT

- The output y[n] of a LTI system with input a complex exponential, is other complex exponential with the **same frequency** and, generally, different phase and amplitude:

$$x[n] = Ae^{j(\Omega_0 n+\phi)} \rightarrow \boxed{\text{LTI system}} \rightarrow y[n]$$

$$y[n] = A'e^{j(\Omega_0 n+\phi')} = \frac{A'}{A} Ae^{(j\Omega_0 n+\phi)} e^{j(\phi'-\phi)} = \left(\frac{A'}{A} e^{j(\phi'-\phi)}\right) x[n]$$

- The complex exponentials are **eigenfunctions** for the LTI systems.
- The multiplying factor is the correspoding **eigenvalue** (corresponding to frecuencia Ω_0)

$$(A_0 e^{j\phi_0}) = \left(\frac{A'}{A} e^{j(\phi'-\phi)}\right)$$

Properties of the LTI systems in DT

- The output of an LTI systems when the input is a sum of complex exponentials with different frequencies, is the sum of the **same** complex exponentials with different phase and amplitude

$$x[n] = \sum_{k=0}^{K} e^{j\Omega_k n} \rightarrow \boxed{\text{LTI systems}} \rightarrow y[n] = \sum_{k=0}^{K} A_k e^{j\phi_k} \cdot e^{j(\Omega_k n)}$$

- Then, the output **y[n] NEVER will be contain a frequency that is not contained in the input x[n].**

Frequency response of LTI systems

The response of LTI systems to complex exponentials

- Consider a continuous time LTI system, characterized by $h(t)$.
- Suppose that the LTI system input is a complex exponential $x(t) = e^{s_0 t}$, being $s_0 = \sigma + j\omega$.
- The LTI system output is calculated by means of the convolution method:

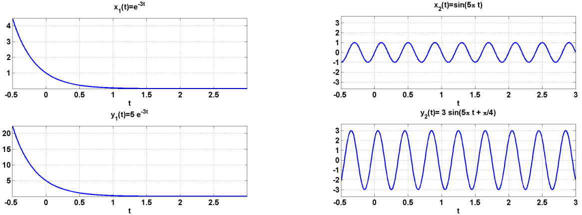
$$y(t) = x(t) * h(t) = e^{s_0 t} * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau =$$

$$= \int_{-\infty}^{\infty} e^{s_0(t-\tau)}h(\tau)d\tau = e^{s_0 t}\boxed{\int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau} = x(t)H(s_0)$$

- $H(s_0)$ is a (complex) constant, that depends on the impulse response and on the exponent of the system input (the exponential function).
- Complex exponential signals are known as *eigenfunctions* of the LTI systems, as the system output to these inputs equals the input multiplied by a constant factor. Both amplitude and phase may change, but the frequency does not change.

Frequency response of LTI systems

Response to real exponential functions and to sinusoids



System function

- If we represent the factor scales for any s_0, we obtain the system function:

LAPLACE transform

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

- Note that this function includes the system response to any complex exponential function.
- Also note that this function depends on the impulse response, that includes all the information related to the LTI system.

1. Introducción: autovalores y autofunciones

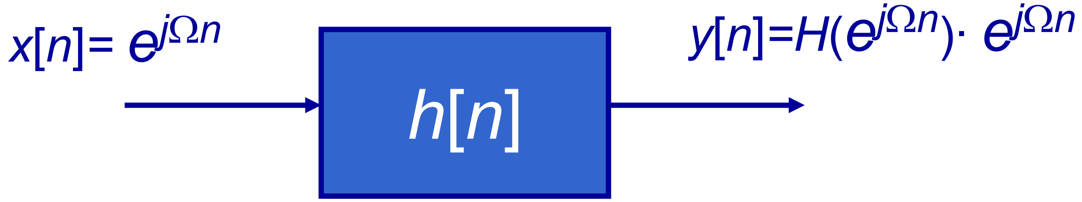
- Para un sistema LTI con respuesta al impulso $h[n]$ la respuesta a una exp. compleja es otra exp. compleja:

$$x[n] = z_0^{\,n}$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z_0^{\,n-k} = z_0^{\,n} \sum_{k=-\infty}^{\infty} h[k]z_0^{\,-k}$$

$$y[n] = z_0^{\,n} H(z_0)$$

- Siendo:
  - $z_0^n = |z_0|^n e^{j\Omega_o n}$ Ξ AUTOFUNCIÓN
  - $H(z_0)$ Ξ AUTOVALOR ∈ ¢
  - Por ser z_0^n *autofunción,* también lo es $e^{j\Omega n}$ (z^n *con* $|z|=1$)
  - Por ser $H(z_0)$ *autovalor,* también lo es $H(e^{j\Omega_o n})$



2. Respuesta de sistemas LTI a señales exponenciales complejas

- Supongamos que la entrada es una combinación lineal de exponenciales:

$$x[n] = \sum_{k=1}^{N} a_k \, e^{j\Omega_k n} \Rightarrow$$

$$y[n] = h[n] * x[n] = \sum_{k=1}^{N} a_k h[n] * e^{j\Omega_k n}$$

$$y[n] = \sum_{k=1}^{N} a_k H\left(e^{j\Omega_k}\right) e^{j\Omega_k n} = \sum_{k=1}^{N} b_k \, e^{j\Omega_k n}$$

$H(e^{j\Omega_k})$ Ξ $H(\Omega_k) \in$ ¢

$a_k, b_k \in$ ¢

- La respuesta es otra combinación lineal de las mismas exponenciales.
- Esto es considerablemente más sencillo que realizar la convolución.
- Por ello vamos a estudiar qué tipo de señales se pueden representar mediante combinación lineal de exponenciales complejas.