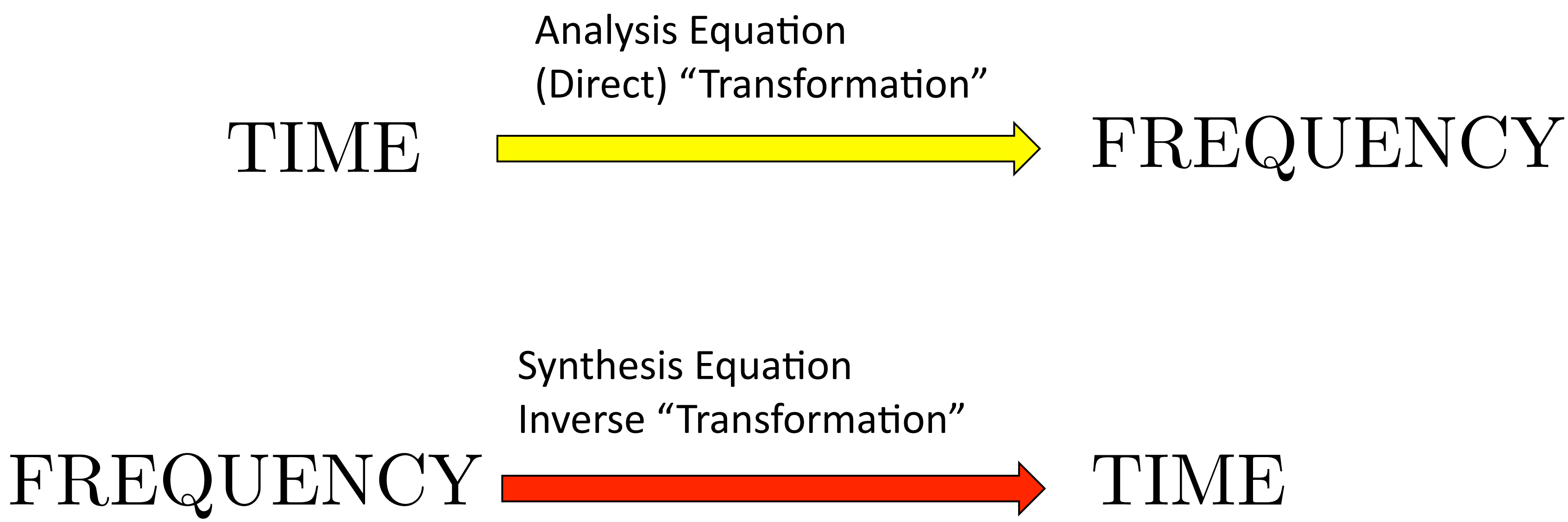
Before Topic 3 –
Summary, Important formulas and expressions

Clarification/recall



Summary: “Fourier so far”

	Periódica en el tiempo	No periódica en el tiempo	
Continua en el tiempo	**CTFS** Analysis $a_k = X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$ Synthesis – CT Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	**CTFT** Synthesis – Inverse Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ Analysis – CT Fourier Transform $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	**No periódica en frecuencia**
Discreta en el tiempo	**DTFS** Synthesis – DT Fourier Series $x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$ Analysis $a_k = X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\Omega_0 n}$	**DTFT** Synthesis – Inverse Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$ Analysis – DT Fourier Transform $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$	**Periódica en frecuencia**
	Discreta en frecuencia	**Continua en frecuencia**	

- did you have studied something more?

Generalized Fourier Transforms (mathematically…is …)

- for periodic signals (including constants)

$$a_k = X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$\omega_0 = \frac{2\pi}{T}$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$a_k = X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\Omega_0 n}$$

$$\Omega_0 = \frac{2\pi}{N}$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\Omega - k\Omega_0)$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\Omega - k\frac{2\pi}{N}\right)$$

Generalized Fourier Transforms (mathematically…is …)

- The Generalized Fourier Transform can be also used for SUM OF PERIODIC SIGNALS
- even if the signal defined as sum of signals is not periodic

$$\underset{\text{non-periodic}}{z(t)} = \underset{\text{periodic}}{x_1(t)} + \underset{\text{periodic}}{x_2(t)}$$

$$Z_G(\omega) = X_{1G}(\omega) + X_{2G}(\omega)$$

Also some other non-periodic signals of infinite energies can have a Generalized TF…

- does z(t) admit Fourier Series? NO!

If x(t), x[n] is real

$$x(t) = x(t)^*$$

$$x[n] = x[n]^*$$

if also periodic, i.e., $x(t) = x(t+T)$

$$a_k = a_{-k}^*$$

if also periodic, i.e., $x[n] = x[n+N]$

$$a_k = a_{-k}^*$$

$$X(\omega) = X(-\omega)^*$$

$$X(\Omega) = X(-\Omega)^*$$

If x(t), x[n] is real

This implies:

even module and real part

$$|a_k| = |a_{-k}|$$
$$|X(\omega)| = |X(-\omega)|$$

$$\text{Re}\{a_k\} = \text{Re}\{a_{-k}\}$$
$$\text{Re}\{X(\omega)\} = \text{Re}\{X(-\omega)\}$$

This implies:

even module and real part

$$|a_k| = |a_{-k}|$$
$$|X(\Omega)| = |X(-\Omega)|$$

$$\text{Re}\{a_k\} = \text{Re}\{a_{-k}\}$$
$$\text{Re}\{X(\Omega)\} = \text{Re}\{X(-\Omega)\}$$

odd phase and imag. part

$$\text{Im}\{a_k\} = -\text{Im}\{a_{-k}\}$$
$$\text{Im}\{X(\omega)\} = -\text{Im}\{X(-\omega)\}$$

odd phase and imag. part

$$\text{Im}\{a_k\} = -\text{Im}\{a_{-k}\}$$
$$\text{Im}\{X(\Omega)\} = -\text{Im}\{X(-\Omega)\}$$

ONLY for discrete time (x[n])

ONLY FOR DISCRETE TIME!!!!

$$a_k = a_{k+N}$$ PERIODIC !!

$$X(\Omega) = X(\Omega + 2\pi)$$ PERIODIC !!

SPOILER: THERE IS an IMPORTANT EXCEPTION !
See the end of these slides….

VERY IMP SLIDE:

Discrete time → periodicity in frequency

$x[n]$ —Transform→ periodicity in frequency

if periodic

$x[n]$ → $a_k = a_{k+N}$ PERIODIC !!

$x[n]$ → $X(\Omega) = X(\Omega + 2\pi)$ PERIODIC !!

Convolution in time....product in...

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)H(\omega)$$

$$y[n] = x[n] * h[n]$$

$$Y(\Omega) = X(\Omega)H(\Omega)$$

Derivatives, differences….

$$\frac{d^i x(t)}{dt^i}$$

$$\Downarrow$$

$$(j\omega)^i X(\omega)$$

$$x[n-d]$$

$$\Downarrow$$

$$e^{-j\Omega d} X(\Omega)$$

$$t x(t)$$

$$\Downarrow$$

$$j \frac{dX(\omega)}{d\omega}$$

$$n x[n] \longleftrightarrow j \frac{dX(\Omega)}{d\Omega}$$

Parseval Relationships

Continuous Time

periodic

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

non-periodic

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Discrete Time

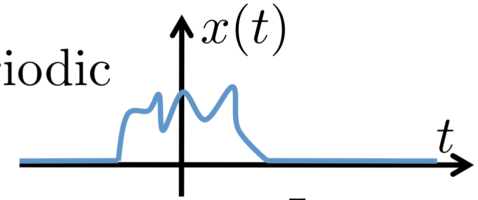
periodic

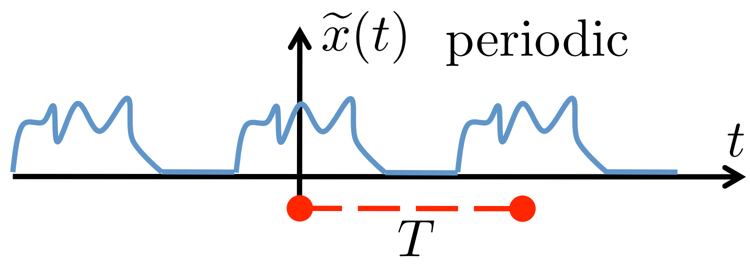
$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

non-periodic

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 d\Omega$$

Finite-length signals, and their periodic "brothers"…





In one period:

$$x(t) = \widetilde{x}(t) \qquad -\frac{T}{2} < t < \frac{T}{2}$$

$X(\omega) \quad X(\Omega)$

a_k

Continuous Time
$$a_k = \frac{1}{T} X(k\omega_0) = \frac{1}{T} X(\omega)\Big|_{\omega = k\omega_0} \qquad \omega_0 = \frac{2\pi}{T}$$

Discrete Time
$$a_k = \frac{1}{N} X(k\Omega_0) = \frac{1}{N} X(\Omega)\Big|_{\Omega = k\Omega_0} \qquad \Omega_0 = \frac{2\pi}{N}$$

Derivation of Impulse Response in frequency of LTI systems

Discrete Time

$$\sum_{i=0}^{L} b_i y[n-i] = \sum_{r=0}^{R} c_r x[n-r]$$

Linear difference equation with constant coefficients

$$\mathcal{F}\left\{\sum_{i=0}^{L} b_i y[n-i]\right\} = \mathcal{F}\left\{\sum_{r=0}^{R} c_r x[n-r]\right\}$$

$$\sum_{i=0}^{L} b_i \mathcal{F}\{y[n-i]\} = \sum_{r=0}^{R} c_r \mathcal{F}\{x[n-r]\}$$

$$\left(\sum_{i=0}^{L} b_i e^{-j\Omega i}\right) Y(\Omega) = \left(\sum_{r=0}^{R} c_r e^{-j\Omega r}\right) X(\Omega)$$

Derivation of Impulse Response in frequency of LTI systems

Discrete Time

$$\left(\sum_{i=0}^{L} b_i e^{-j\Omega i}\right) Y(\Omega) = \left(\sum_{r=1}^{R} c_r e^{-j\Omega r}\right) X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{r=0}^{R} c_r e^{-j\Omega r}}{\sum_{i=0}^{L} b_i e^{-j\Omega i}}$$

Derivation of Impulse Response in frequency of LTI systems

Continuous Time

$$\sum_{i=0}^{L} b_i \frac{d^i y(t)}{dt^i} = \sum_{r=0}^{R} c_r \frac{d^r x(t)}{dt^r}$$

Linear differential equation with constant coefficients

$$\mathcal{F}\left\{\sum_{i=0}^{L} b_i \frac{d^i y(t)}{dt^i}\right\} = \mathcal{F}\left\{\sum_{r=0}^{R} c_r \frac{d^r x(t)}{dt^r}\right\}$$

$$\sum_{i=0}^{L} b_i \mathcal{F}\left\{\frac{d^i y(t)}{dt^i}\right\} = \sum_{r=0}^{R} c_r \mathcal{F}\left\{\frac{d^r x(t)}{dt^r}\right\}$$

$$\left(\sum_{i=0}^{L} b_i \left(j\omega\right)^i\right) Y(\omega) = \left(\sum_{r=0}^{R} c_r \left(j\omega\right)^r\right) X(\omega)$$

Derivation of Impulse Response in frequency of LTI systems

Continuous Time

$$\left(\sum_{i=0}^{L} b_i \left(j\omega\right)^i\right) Y(\omega) = \left(\sum_{r=0}^{R} c_r \left(j\omega\right)^r\right) X(\omega)$$

$$\Downarrow$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{r=0}^{R} c_r \left(j\omega\right)^r}{\sum_{i=0}^{L} b_i \left(j\omega\right)^i}$$

Extension: Laplace and Zeta transforms

LAPLACE TRANSFORM

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

$$X(\omega) = X(s)\Big|_{s=j\omega}$$

$$\sigma = 0$$

ZETA TRANSFORM

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$z = re^{j\Omega}$$

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}}$$

$$r = 1$$

Derivatives, differences (again): Laplace and Zeta

$$\frac{d^i x(t)}{dt^i}$$

$$\Downarrow$$

$$s^i X(s)$$

$$x[n-d]$$

$$\Downarrow$$

$$z^{-d} X(z)$$

Derivation of Impulse Response in frequency of LTI systems (again)

Continuous Time

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{r=0}^{R} c_r s^r}{\sum_{i=0}^{L} b_i s^i}$$

Discrete Time

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{R} c_r z^{-r}}{\sum_{i=0}^{L} b_i z^{-i}}$$

DELTA FUNCTIONS
in time and in frequency

DELTA FUNCTIONS in frequency:

you already know =>

Generalized Fourier Transform

Come back to
Generalized Fourier Transform

- for periodic signals (including constants and others…); - but mathematically…
- Examples:

$$x(t) = 1 \Longrightarrow a_0 = 1 \Longrightarrow X_G(\omega) = 2\pi\delta(\omega)$$

$$x(t) = e^{jm\omega_0 t} \Longrightarrow a_m = 1 \Longrightarrow X_G(\omega) = 2\pi\delta(\omega - m\omega_0)$$

$$a_k = 0 \quad k \neq m$$

Come back to Generalized Fourier Transform

Other examples:

$$x(t) = e^{j\omega_0 t} + e^{-j3\omega_0 t} \Longrightarrow a_1 = 1 \;\; a_{-3} = 1 \text{ (the rest are zero)}$$

$$\Longrightarrow X_G(\omega) = 2\pi\delta(\omega + 3\omega_0) + 2\pi\delta(\omega - \omega_0)$$

(maybe non-periodic; see previous slides)

$$x(t) = -je^{j\omega_1 t} + 2e^{-j3\omega_2 t}$$

where w_1, w_2 are fundamental frequencies

$$\Longrightarrow X_G(\omega) = -2\pi j\delta(\omega - \omega_1) + 4\pi\delta(\omega + 3\omega_2)$$

DELTA FUNCTIONS
in time domain

(considering the duality, and Fourier properties you already know)

Delta functions in time

Assuming that we can do Fourier transform…

$$x(t) = \delta(t)$$

$$\Longrightarrow X(\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$x(t) = \delta(t) \Longrightarrow X(\omega) = 1$$

Delta functions in time

Assuming that we can do Fourier transform…

$$x(t) = \delta(t - t_0)$$

$$\Longrightarrow X(\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

$$x(t) = \delta(t - t_0) \Longrightarrow X(\omega) = e^{-j\omega t_0}$$

Sum of deltas in time

$$x(t) = \delta(t - t_0) + \delta(t - t_1)$$

$$\Downarrow$$

$$X(\omega) = e^{-j\omega t_0} + e^{-j\omega t_1}$$

Sum of deltas in time

Let us consider

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

$$\mathcal{F}\left\{ \sum_{m=-\infty}^{+\infty} \delta(t - mT) \right\} = \sum_{m=-\infty}^{+\infty} \mathcal{F}\left\{\delta(t - mT)\right\}$$

$$X(\omega) = \sum_{m=-\infty}^{+\infty} e^{-jmT\omega}$$

Sum of deltas in time

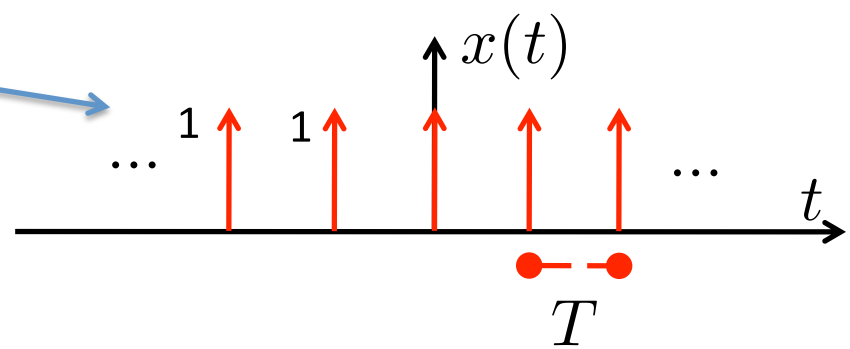
Let us consider AGAIN

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

NOTE THAT IS PERIODIC

with period T

The coefficients are 1...



...we can do the Fourier Series...

Sum of deltas in time

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

NOTE THAT IS PERIODIC with period T

$$\omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\frac{2\pi}{T}t}dt$$

$$a_k = \frac{1}{T}$$ for all k

Sum of deltas in time

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

Then, the Fourier series is:

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

note that this expression is still in time domain

hence, we obtain a first equality... →

First equality with sum of deltas (in time)

$$\sum_{m=-\infty}^{\infty} \delta(t - mT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

We did the Fourier series;

then we can also write

the Generalized Fourier Transform

GFT of a periodic sum of deltas

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

We have obtained another expression of the transform of $x(t)$!!

(Very important slide! Please pay attention....)

Summary: Fourier of a sum of deltas

$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

$$X(\omega) = \sum_{m=-\infty}^{+\infty} e^{-jmT\omega}$$

$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

We would like to be the same….

Second equality (in frequency)

$$X(\omega) = X_G(\omega)$$

$$\sum_{m=-\infty}^{+\infty} e^{-jmT\omega} = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Very important observation

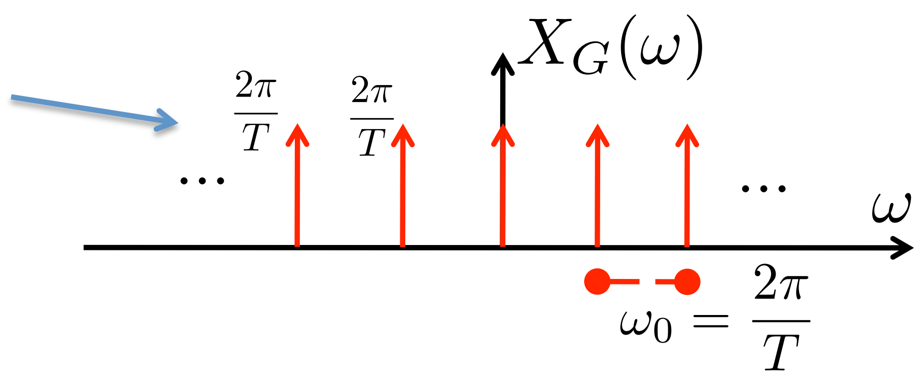
$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Note that also $X_G(\omega)$ is also periodic!!

with PERIOD ω_0 (YES, in this case is "period"...we are in the frequency domain!)

The coefficients...



But before we have said:
ONLY (sure?) discrete time → periodicity in frequency

$x[n]$ —Transform→ periodicity in frequency

if periodic

$x[n]$ → $a_k = a_{k+N}$ PERIODIC !!

$x[n]$ → $X(\Omega) = X(\Omega + 2\pi)$ PERIODIC !!

A sum of deltas in **continuous time** has also a periodic GFT.

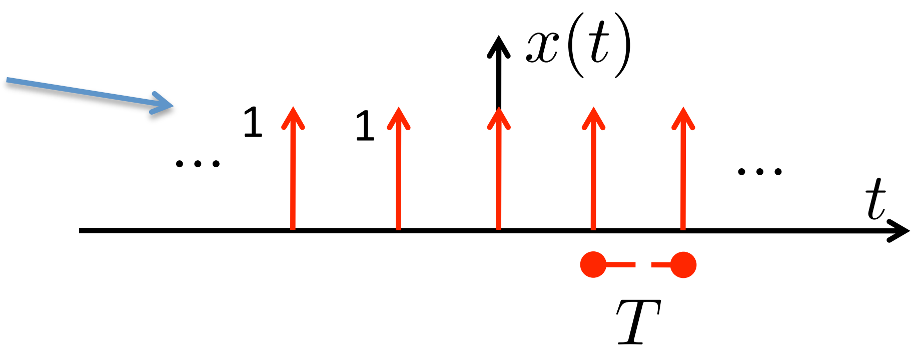
$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT)$$

$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Why?

Have a look:

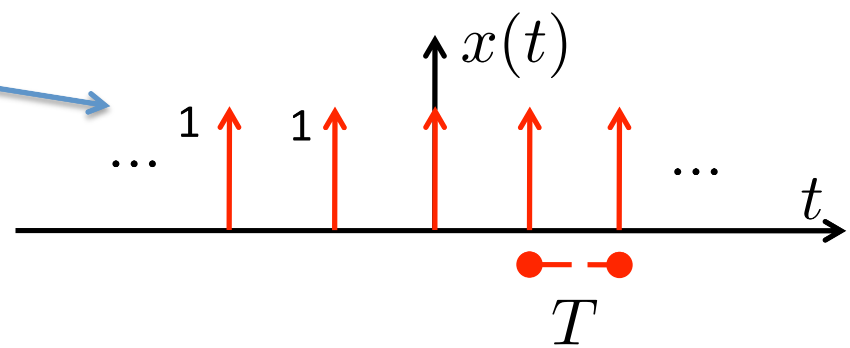
The coefficients are 1…



It is defined in continuous time, But this signal seems defined in a discrete time….

Have a look:

The coefficients are 1…



It is "something" in between continuous and discrete time…

it is defined everywhere but looks defined on a discrete space…