Before Topic 3 – symmetries, formulas and more

….Just FT in continuous time for simplicity…

In these slides, We will consider just the Fourier Transform in continuous time, for the sake of simplicity (avoiding to repeat the same concept or formula several times). However, all the formulas can be extended to the other kind of Fourier transformations and/or series.

EVEN and ODD FUNCTIONS

Even and Odd Functions

A function, f, is *even* (or *symmetric*) when

$$f(x) = f(-x).$$

A function, f, is *odd* (or *antisymmetric*) when

$$f(x) = -f(-x).$$

EVEN and ODD FUNCTIONS

Theorem Any function can be written as a sum of even and odd functions.

$$f(t) = \frac{1}{2}\left[f(t) + \underbrace{f(-t) - f(-t)}_{0} + f(t)\right]$$

$$= \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{f_e} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{f_o}$$

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

$$f(t) = f_e(t) + f_o(t)$$

EVEN and ODD FUNCTIONS

f_e is even because $f_e(t) = f_e(-t)$:

$$\begin{aligned} f_e(t) &= f(t) + f(-t) \\ &= f(-t) + f(t) \\ &= f_e(-t). \end{aligned}$$

We omits the factor 1/2

f_o is odd because $f_o(t) = -f_o(-t)$:

$$\begin{aligned} f_o(t) &= f(t) - f(-t) \\ &= -[f(-t) - f(t)] \\ &= -f_o(-t). \end{aligned}$$

We omits the factor 1/2

VERY IMPORTANT THEOREM

Theorem The integral of the product of odd and even functions is zero.

$$\int_{-\infty}^{\infty} f_e(t) f_o(t) dt = 0$$

Proof

We change "t" for "x"....it is just a "label"!

$$\int_{-\infty}^{\infty} f_e(x) f_o(x) dx =$$

Just splitting in two parts:

$$\int_{-\infty}^{0} f_e(x) f_o(x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx.$$

Substituting $-x$ for x and $-dx$ for dx in the first term yields:

$$\int_{\infty}^{0} -f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} \left[f_e(-x) f_o(-x) + f_e(x) f_o(x) \right] dx.$$

Proof

We change "t" for "x"....it is just a "label"!

Substituting $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ yields:

$$\int_0^{\infty} \underbrace{[f_e(-x)f_o(-x) - f_e(-x)f_o(-x)]}_{0}\, dx.$$

Proved !!!

RECALL:
in general, everything is complex

$$f(t) = a(t) + jb(t) \in \mathbb{C}$$

$$f_e(t) = a_e(t) + jb_e(t) \in \mathbb{C}$$

$$f_o(t) = a_o(t) + jb_o(t) \in \mathbb{C}$$

$$F(\omega) = R_f(\omega) + jI_f(\omega) \in \mathbb{C}$$

All real functions

All real functions

Alternative formula for FT and considerations

$$f(t) = a(t) + jb(t) \in \mathbb{C}$$

$$f(t) = f_e(t) + f_o(t) \in \mathbb{C}$$

$$f_e(t) = a_e(t) + jb_e(t) \in \mathbb{C}$$

$$f_o(t) = a_o(t) + jb_o(t) \in \mathbb{C}$$

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) \cos(j\omega t) dt - j \int_{-\infty}^{\infty} f\ (t) \sin(j\omega t) dt \end{aligned}$$

Alternative formula for FT and considerations

$$F(\omega) = \int_{-\infty}^{\infty} f_e(t)\cos(j\omega t)dt + \int_{-\infty}^{\infty} f_o(t)\cos(j\omega t)dt$$

$$- j\int_{-\infty}^{\infty} f_e(t)\sin(j\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(j\omega t)dt$$

0

0

$$F(\omega) = \int_{-\infty}^{\infty} f_e(t)\cos(j\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(j\omega t)dt$$

Alternative formula for FT and considerations

$$F(\omega) = \int_{-\infty}^{\infty} f_e(t)\cos(j\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(j\omega t)dt$$

$$F_e(\omega) \in \mathbb{C} \qquad F_o(\omega) \in \mathbb{C}$$

$$f_e(t) = a_e(t) + jb_e(t) \in \mathbb{C}$$

$$f_o(t) = a_o(t) + jb_o(t) \in \mathbb{C}$$

$$F(\omega) = F_e(\omega) + F_o(\omega)$$

IMPORTANT THEOREM

Theorem The Fourier transform of a real even function is real. $F(\omega) = F(\omega)^*$

If f(t) is even and real → then F(w) is real (and also even) !!!!

If $f(t) = f(-t) = f_e(t) \in \mathbb{R}$ **i.e., also** $f(t) = f(t)^*$

Then,
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$
$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)dt$$

0

$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt \in \mathbb{R}$$

Repeating: When the Fourier Transform F(w) is a real function?

$$F(\omega) = F(\omega)^*$$

When the signal f(t) is even and real !!!

==> $f(t) = f(-t) = f_e(t)$

==> $f(t) = f(t)^*$

Moreover:

Theorem The Fourier transform of a real odd function is imaginary. **(and also "odd")**

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t) dt - j \int_{-\infty}^{+\infty} f(t) \sin(\omega t) dt$$

0

$$= -j \int_{-\infty}^{+\infty} f(t) \sin(\omega t) dt \in \mathbb{C}$$

just imaginary part…. the real part=0

$$F(\omega) = -F(\omega)^*$$

We already know that If the signal f(t) is “just” is real, we have

$$f(t) = f(t)^* \longrightarrow F(\omega) = F(-\omega)^*$$

<u>Hermitian Symmetry</u>

If the signal f(t) is "just" is even:

Theorem The Fourier transform of an even function is even.

$$f(t) = f(-t) \longrightarrow F(\omega) = F(-\omega)$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

since $f(t)$ is even

$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t) dt - j \int_{-\infty}^{+\infty} f(t)\sin(\omega t) dt$$

0

$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t) dt = \int_{-\infty}^{+\infty} f(t)\cos(-\omega t) dt = F(-\omega)$$

$$\cos(\omega t) = \cos(-\omega t)$$

If the signal f(t) is "just" is odd:

Theorem The Fourier transform of an odd function is odd.

$$f(t) = -f(-t) \longrightarrow F(\omega) = -F(-\omega)$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

since $f(t)$ is odd

$$= \int_{-\infty}^{+\infty} f(t) \cos(\omega t) dt - j \int_{-\infty}^{+\infty} f(t) \sin(\omega t) dt$$

0

$$= -j \int_{-\infty}^{+\infty} f(t) \sin(\omega t) dt = +j \int_{-\infty}^{+\infty} f(t) \sin(-\omega t) dt = -F(-\omega)$$

$$\sin(\omega t) = -\sin(-\omega t)$$

SUMMARY

Even		**Even**
$f(t) = f(-t)$	→	$F(\omega) = F(-\omega)$
Odd		**Odd**
$f(t) = -f(-t)$	→	$F(\omega) = -F(-\omega)$
Real		**Hermitian**
$f(t) = f(t)^*$	→	$F(\omega) = F(-\omega)^*$
Real and even		**Real and even**
$f(t) = f(t)^*$ $f(t) = f(-t)$	→	$F(\omega) = F(\omega)^*$ $F(\omega) = F(-\omega)$

SUMMARY

Real and odd

$$f(t) = f(t)^*$$
$$f(t) = -f(-t)$$

→

Imaginary and odd

$$F(\omega) = -F(\omega)^*$$
$$F(\omega) = -F(-\omega)$$