Problems - Examples: Fourier Transform

Linear systems and circuit applications and Señales y Sistemas

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

SUMMARY

Even		**Even**
$f(t) = f(-t)$	$\longrightarrow$	$F(\omega) = F(-\omega)$
Odd		**Odd**
$f(t) = -f(-t)$	$\longrightarrow$	$F(\omega) = -F(-\omega)$
Real		**Hermitian**
$f(t) = f(t)^*$	$\longrightarrow$	$F(\omega) = F(-\omega)^*$
Real and even		**Real and even**
$f(t) = f(t)^*$ $f(t) = f(-t)$	$\longrightarrow$	$F(\omega) = F(\omega)^*$ $F(\omega) = F(-\omega)$

SUMMARY

Real and odd

$$f(t) = f(t)^*$$
$$f(t) = -f(-t)$$

→

Imaginary and odd

$$F(\omega) = -F(\omega)^*$$
$$F(\omega) = -F(-\omega)$$

Example 1

$$X(t) = e^{-at}u(t), \qquad a > 0.$$

(a) Compute the Standard FT, $X(\omega) =?$

(b) and compute $X(2) =?$

(c) Compute also the Stand. FT of

$$X(t) = e^{-a(t-3)}u(t-3), \qquad a > 0.$$

Example 1

Considere la señal

$$x(t) = e^{-at}u(t) \quad a > 0.$$

Using the direct FT transformation:

$$X(\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}.$$

due to the step function ("escalón")

Esto es,

$$X(\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

Example 1

$$X(\omega) = \frac{1}{a + j\omega}$$

It is NOT required but, to understand more, we can try to plot it:

$$X(\omega) = \frac{(a - j\omega)}{(a - j\omega)(a + j\omega)} = \frac{a - j\omega}{a^2 + \omega^2}$$

$$= \frac{a}{a^2 + \omega^2} - j\frac{\omega}{a^2 + \omega^2}$$

Example 1

It is NOT required but, to understand more, we can try to plot it:

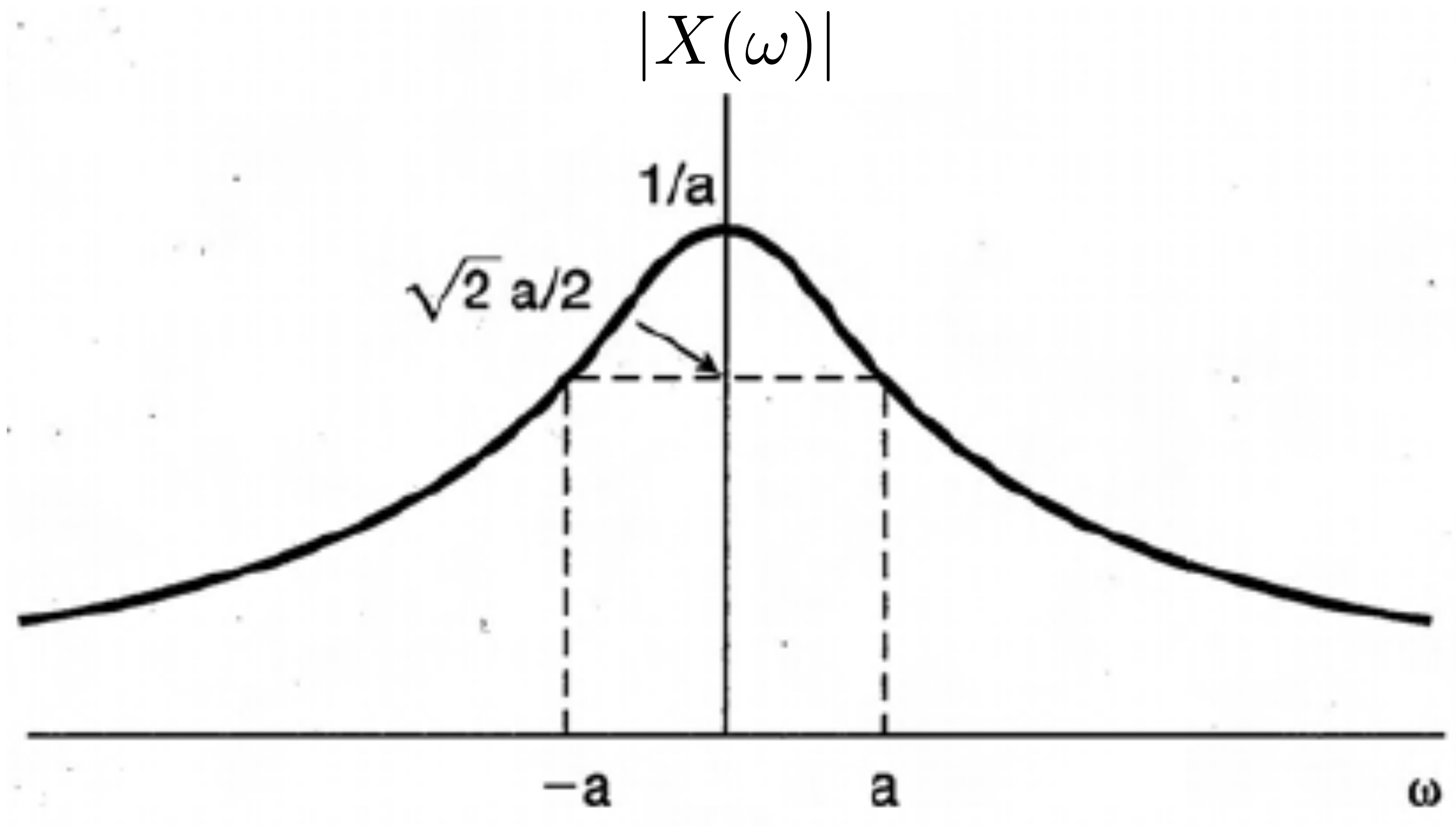
$$X(\omega) = \frac{a}{a^2 + \omega^2} - j\frac{\omega}{a^2 + \omega^2}$$

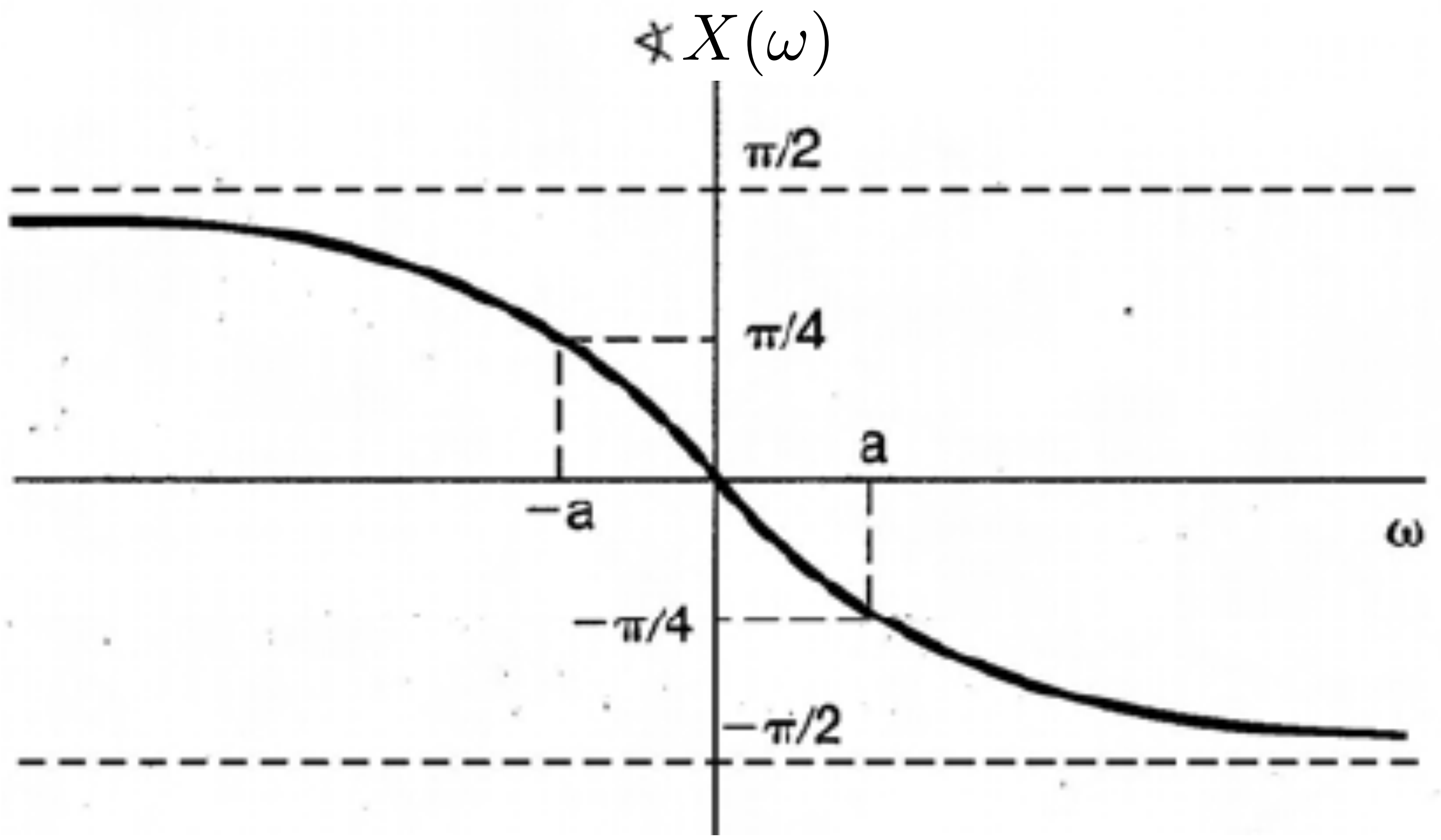
$$|X(\omega)| = \sqrt{\frac{a^2}{(a^2 + \omega^2)^2} + \frac{\omega^2}{(a^2 + \omega^2)^2}} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\text{phase}(X(\omega)) = \arctan\left[\frac{\text{Im}(X(\omega))}{\text{Real}(X(\omega))}\right] = \arctan\frac{\omega}{a}$$

Example 1

It is NOT required but, to understand more, we can try to plot it:





Example 1

$$X(\omega) = \frac{1}{a + j\omega}$$

$$X(2) = \frac{1}{a + 2j}$$

Example 1

$$X(t) = e^{-a(t-3)}u(t-3), \qquad a > 0.$$

You can redo the same of the previous part, or use a property of the Fourier Transform:

$$x(t - t_0) \Longleftrightarrow e^{-j\omega t_0} X(\omega)$$

$$X(\omega) = e^{-j3\omega} \frac{1}{a + j\omega}$$

Example 2

$$x(t) = e^{-a|t|}, a > 0.$$

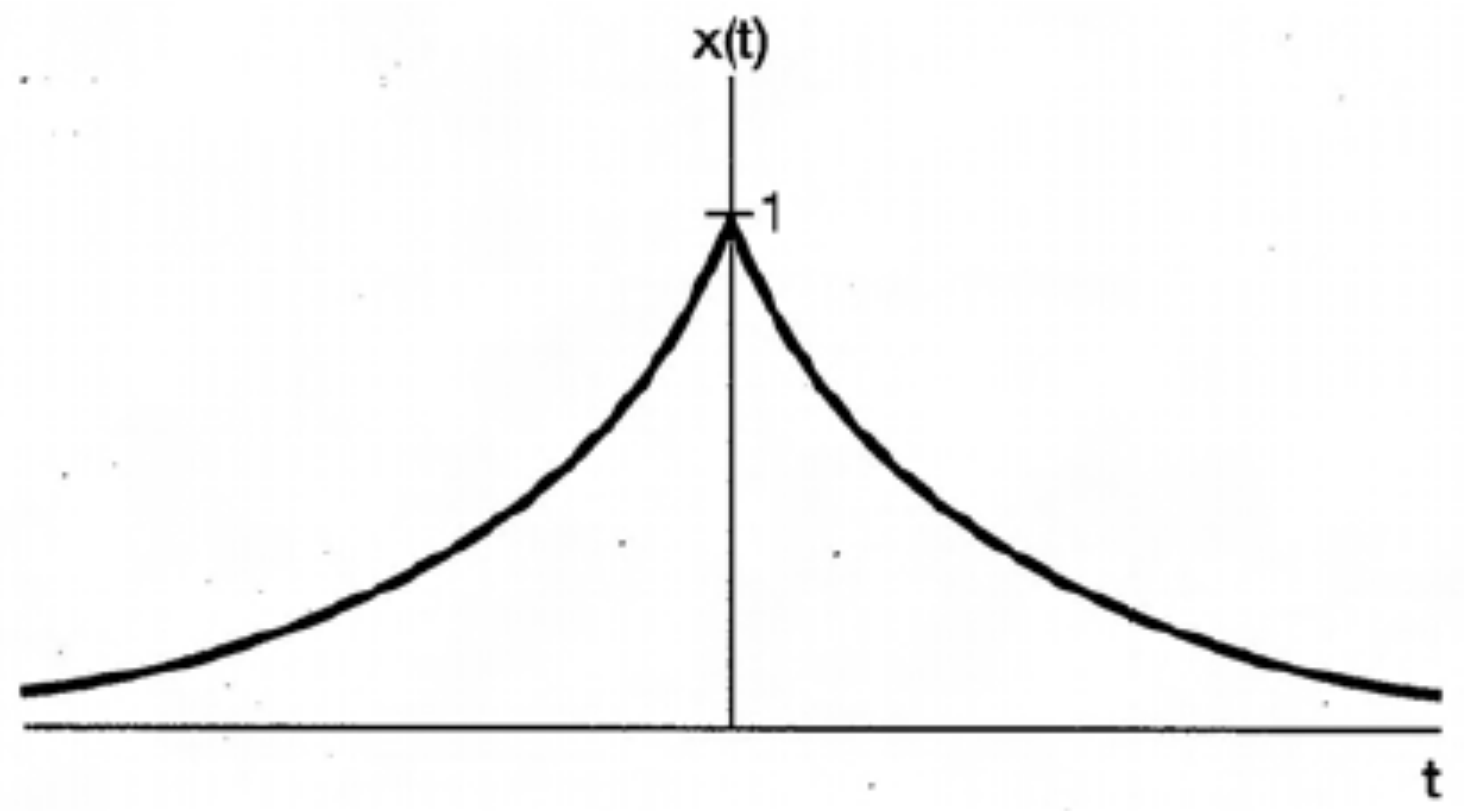
(a) Discuss the properties that you expect of $X(\omega)$, after just looking/studying $x(t)$.

(b) Compute the Standard FT, $X(\omega) = ?$

Example 2

(a) Discuss the properties that you expect of $X(\omega)$, after just looking/studying $x(t)$.

Since x(t) is real and even, then the FT is real and even



Example 2

$$X(\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

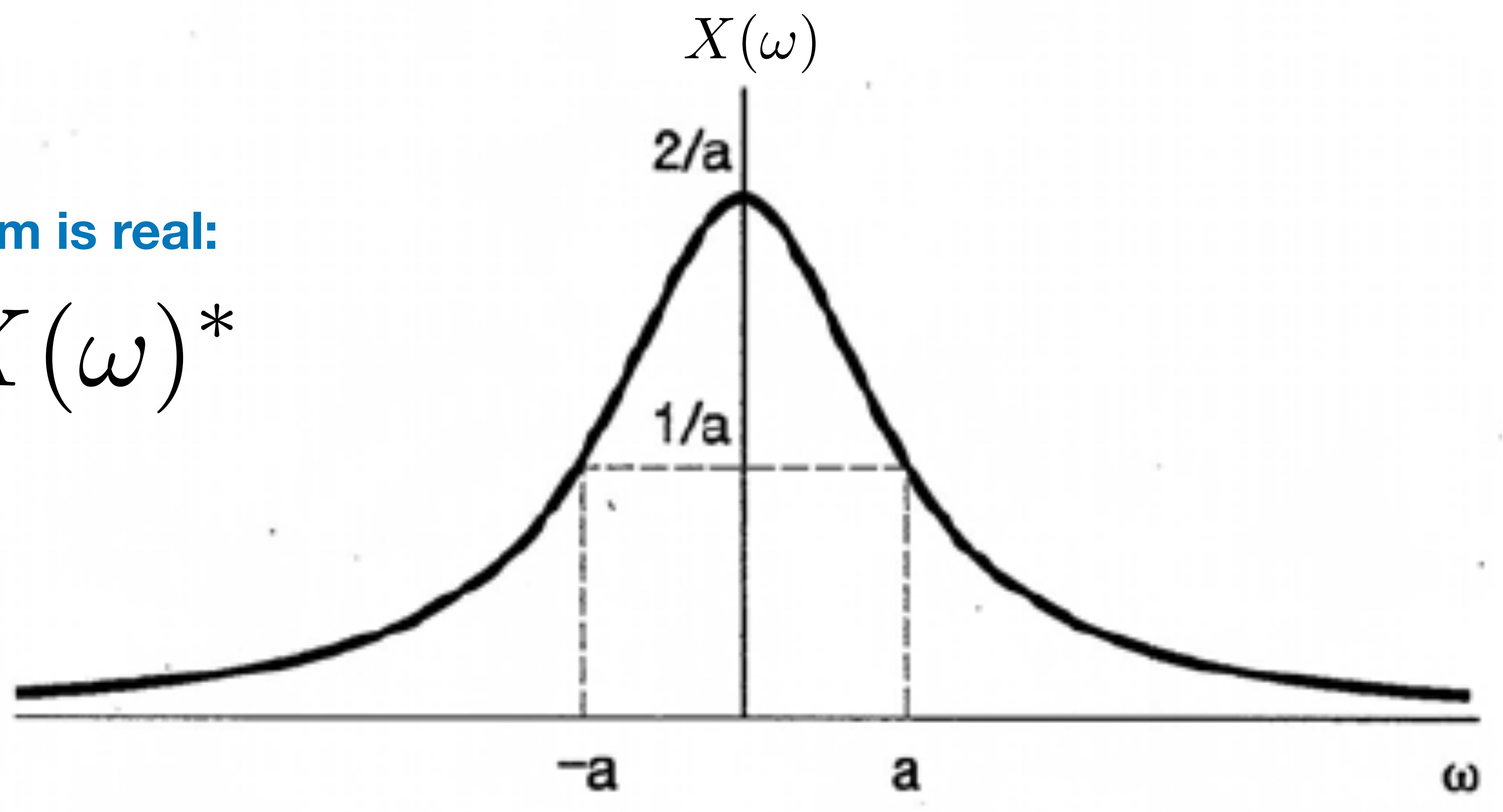
$$= \frac{2a}{a^2 + \omega^2}.$$

En este caso, $X(\omega)$ es real y se ilustra en la figura →

Example 2

The Fourier Transform is real:

$$X(\omega) = X(\omega)^*$$



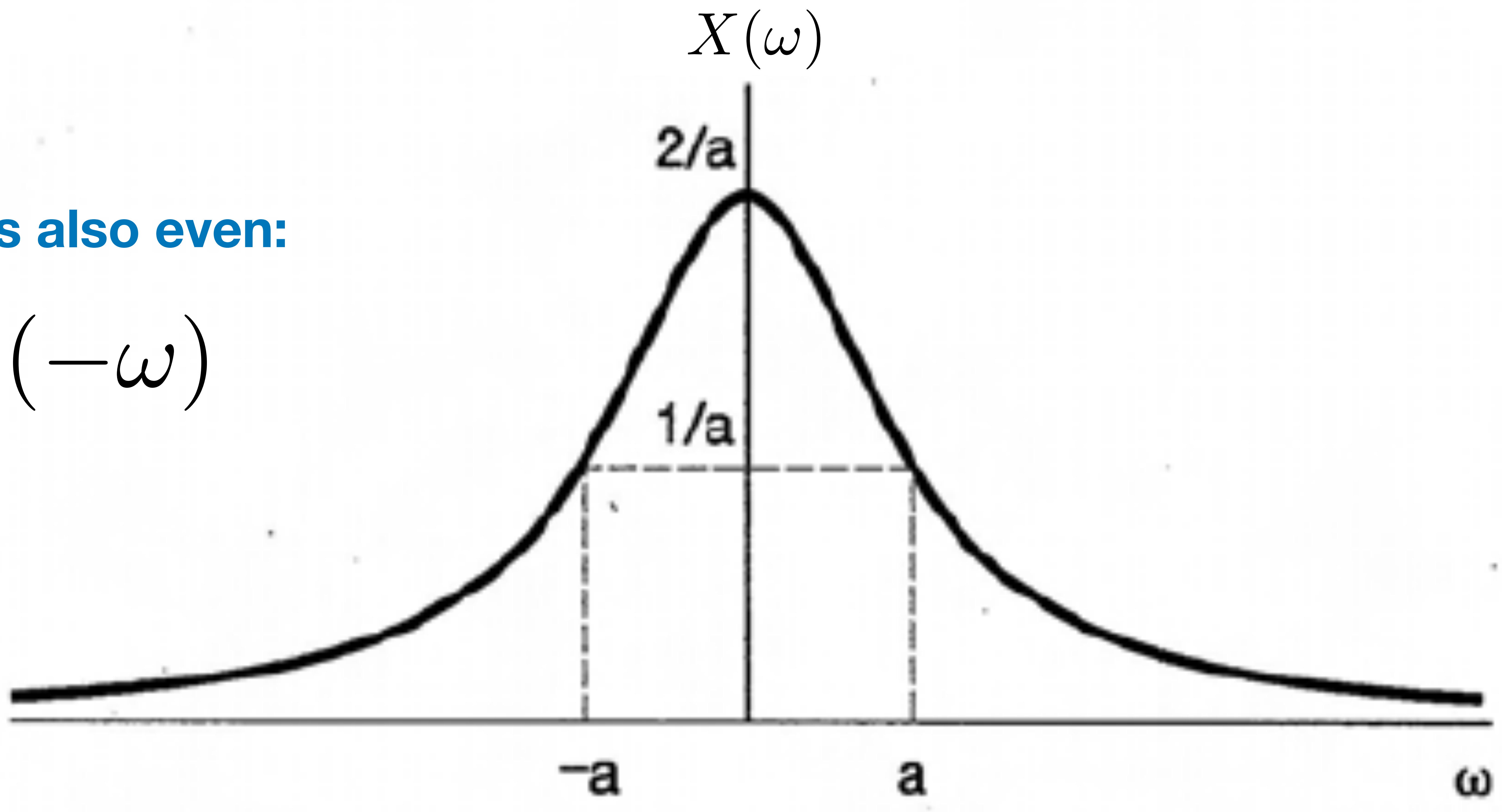
The Fourier Transform is real, other consequences are:

$$X(\omega) = |X(\omega)| = \mathrm{Real}\{X(\omega)\}$$

Example 2

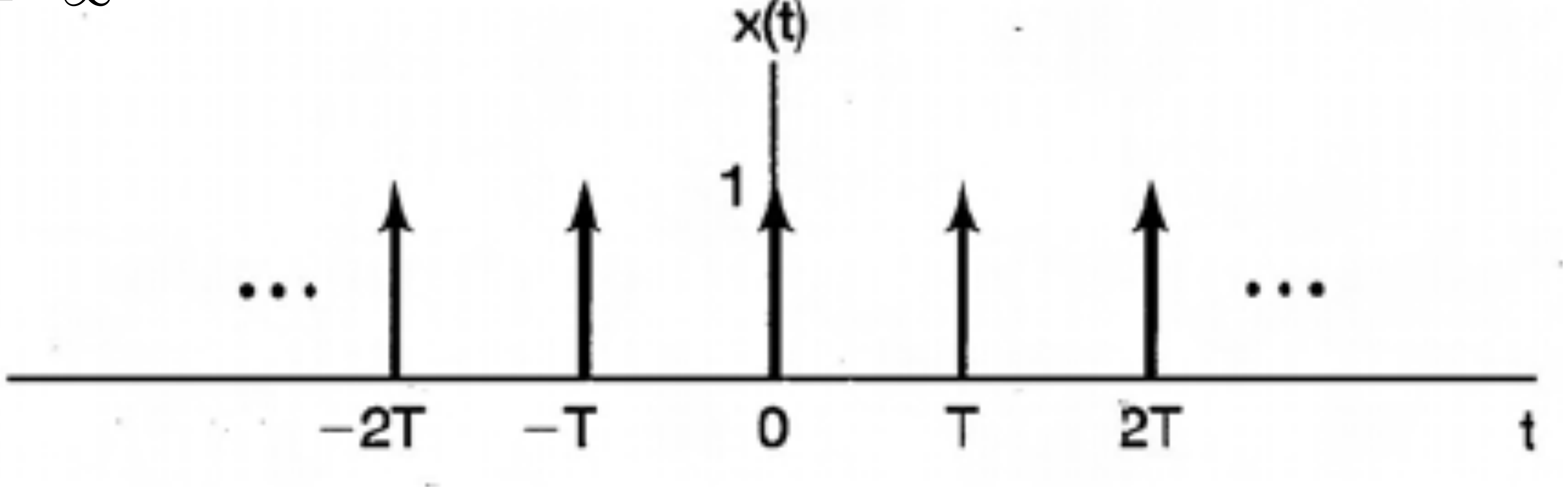
The Fourier Transform is also even:

$$X(\omega) = X(-\omega)$$



Example 3

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta\left(t - kT\right)$$



- Compute the Generalized Fourier Transform (GTF)

Example 3

Una señal que será en extremo útil en nuestro análisis de sistemas de muestreo en el capítulo 7 es el tren de impulsos

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

el cual es periódico con periodo T, como se indica en la figura 4.14(a). Los coeficientes de la serie de Fourier para esta señal se calcularon en el ejemplo 3.8 y están dados por

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}.$$

Example 3

Esto es, cada coeficiente de Fourier del tren de impulsos periódico tiene el mismo valo
$1/T$. Sustituyendo este valor para a_k en la ecuación obtenemos

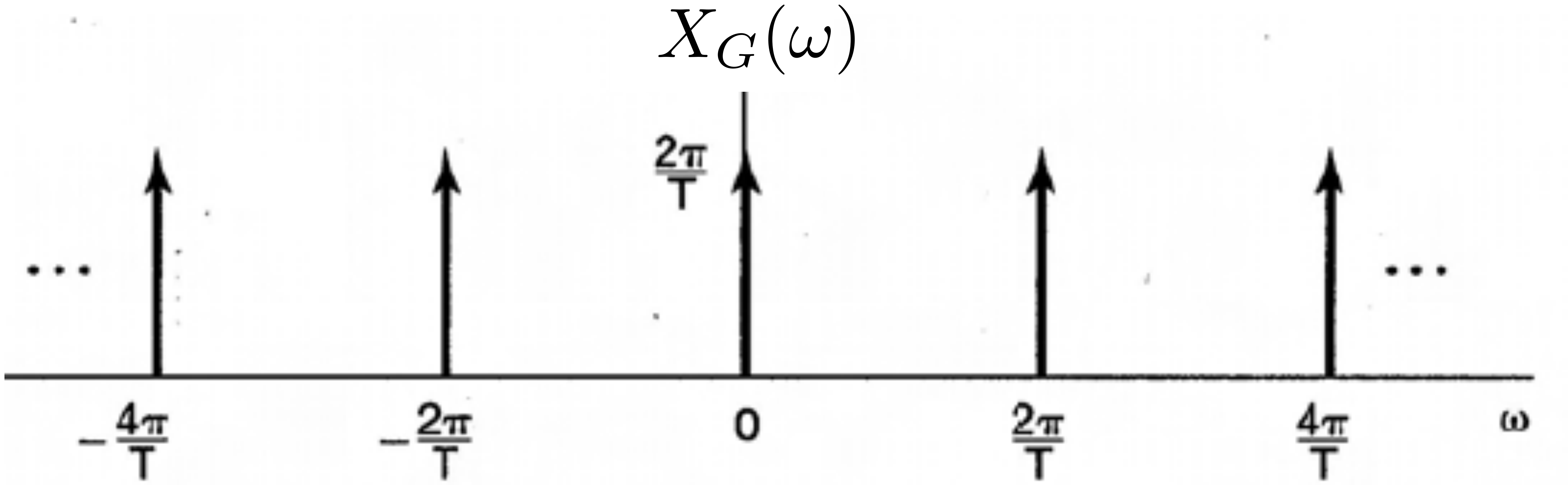
$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

For periodic signals

$$X_G(\omega) = 2\pi \sum_{-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

Example 3

$$X_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$



Example 4

Let us consider

$$x(t) = e^{at}u(-t), \quad a > 0$$

Compute the Standard FT, $X(\omega) = ?$

Example 4

Solution very similar to Example 1

$$x(t) = e^{at}u(-t), \quad a > 0$$

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{at}u[-t]e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}dt \\
&= \left[\frac{1}{a-j\omega}e^{(a-j\omega)t}\right]_{-\infty}^{0} = \frac{1}{a-j\omega}
\end{aligned}
$$

Example 5

$$x(t) = e^{-at} \qquad\qquad a > 0$$

Say if this signal has (can have) a standard FT

Example 5

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$\int_{-\infty}^{\infty} e^{-(a+j\omega)t} dt = \left[\frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right]_{-\infty}^{\infty}$$

$$= 0 - \infty$$

This signal has not standard FT
(also has not GFT)

Example 5

$$x(t) = u(t)$$

Say if this signal has (can have) a standard FT and if you know a GTF for this signal

Example 5

$$\int_0^{\infty} e^{-j\omega t} dt = \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_0^{\infty}$$

$$= \nexists - \frac{1}{j\omega} = \nexists$$

This signal has not standard FT

it has a GFT (following the “books”) - see slides about GTF

Example 7

$$x(t) = e^{-at}\cos(\omega_0 t)u(t) \qquad\qquad a > 0$$

Compute the Standard FT, $X(\omega) = ?$

Example 7

$$X(\omega) = \int_0^\infty e^{-at} \cos(\omega_0 t)\, e^{-j\omega t} dt$$

$$X(\omega) = \frac{1}{2}\int_0^\infty e^{-at} e^{j\omega_0 t} e^{-j\omega t} dt + \frac{1}{2}\int_0^\infty e^{-at} e^{-j\omega_0 t} e^{-j\omega t} dt$$

$$= \frac{1}{2}\left[\frac{1}{-a - j(\omega - \omega_0)} e^{-a-j(\omega-\omega_0)t}\right]_0^\infty + \frac{1}{2}\left[\frac{1}{-a - j(\omega + \omega_0)} e^{-(a+j\omega_0+j\omega)t}\right]_0^\infty$$

$$X(\omega) = -\frac{1}{2}\frac{1}{-a - j(\omega - \omega_0)} - \frac{1}{2}\frac{1}{-a - j(\omega + \omega_0)}$$

This is already the solution

Example 7

$$X(\omega) = -\frac{1}{2}\frac{1}{-a - j(\omega - \omega_0)} - \frac{1}{2}\frac{1}{-a - j(\omega + \omega_0)}$$

$$X(\omega) = \frac{1}{2}\frac{1}{a + j(\omega - \omega_0)} + \frac{1}{2}\frac{1}{a + j(\omega + \omega_0)}$$

$$X(\omega) = \frac{1}{2}\frac{1}{a - j\omega_0 + j\omega} + \frac{1}{2}\frac{1}{a + j\omega_0 + j\omega}$$

this is the same solution written in other way

Example 8

Let consider the following stand. FT,

$$X(\omega) = \omega^2 - \omega$$

(a) Let "think" that this FT corresponds to a signal x(t) with finite length. Consider a periodic "brother" of x(t) of period To, find its coefficients of the corresponding Fourier series. **(b)** Write also the corresponding GTF of the periodic "brother" of x(t).

Example 8

We use the formula

$$a_k = \frac{1}{T_0}[X(\omega)]_{\omega=k\omega_0} \qquad a_k = \frac{1}{T_0}X(k\omega_0)$$

Example 8

$$a_k = \frac{1}{T_0} X(k\omega_0)$$

$$X(\omega) = \omega^2 - \omega$$

$$a_k = \frac{1}{T_0} X(k\omega_0) = \frac{1}{T_0}\left((k\omega_0)^2 - k\omega_0\right)$$

This is already the solution

Example 8

Since $\omega_0 = \frac{2\pi}{T_0}$ → $T_0 = \frac{2\pi}{\omega_0}$

↓

$$a_k = \frac{1}{T_0} X(k\omega_0) = \frac{1}{T_0}\left((k\omega_0)^2 - k\omega_0\right)$$

↓

$$a_k = \frac{\omega_0}{2\pi}\left((k\omega_0)^2 - k\omega_0\right)$$

Example 8

let denote the periodic "brother" signal as $\widetilde{x}(t)$

Its GTF is

$$\widetilde{X}_G(\omega) = 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$\widetilde{X}_G(\omega) = 2\pi \sum_{-\infty}^{\infty} \left(\frac{\omega_0}{2\pi} \left((k\omega_0)^2 - k\omega_0 \right) \right) \delta(\omega - k\omega_0)$$

$$\widetilde{X}_G(\omega) = \omega_0 \sum_{-\infty}^{\infty} \left((k\omega_0)^2 - k\omega_0 \right) \delta(\omega - k\omega_0)$$

This is already the solution

Questions?