Problems - Examples: Fourier Transform

Linear systems and circuit applications and Señales y Sistemas

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

Example 9

Consider the following linear differential equation with constant coefficients (and null initial conditions):

$$(1-j)\frac{d^3y(t)}{dt^3} + 3\frac{dy(t)}{dt} = 3\frac{d^2x(t)}{dt^2} + j\frac{dx(t)}{dt}$$

Find the Laplace Transform, H(s), and the FT, H(w), of the impulse response h(t) of the LTI system.

Example 9

We will use the properties:

$$\frac{d^k x(t)}{dt^k} \Longleftrightarrow s^k X(s)$$

For the Laplace Transform (bilateral)

$$s = \sigma + j\omega \qquad \sigma = 0$$

$$\frac{d^k x(t)}{dt^k} \Longleftrightarrow (j\omega)^k X(\omega)$$

For the Fourier Transform

Example 9

Let us do with Laplace, first,

$$(1-j)s^3Y(s) + 3sY(s) = 3s^2X(s) + jsX(s)$$

$$((1-j)s^3 + 3s)Y(s) = (3s^2 + js)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s^2 + js}{(1-j)s^3 + 3s}$$

This is the solution

Example 9

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s^2 + js}{(1-j)s^3 + 3s}$$

With Fourier, we set sigma=0,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3(j\omega)^2 + j(j\omega)}{(1-j)(j\omega)^3 + 3(j\omega)}$$

This is already the solution

Example 9

With Fourier, we set sigma=0,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3(j\omega)^2 + j(j\omega)}{(1-j)(j\omega)^3 + 3(j\omega)}$$

$$H(\omega) = \frac{3j^2\omega^2 + j^2\omega}{j^3\omega^3 - j^4\omega^3 + 3j\omega}$$

$$H(\omega) = \frac{-3\omega^2 - \omega}{-j\omega^3 - \omega^3 + 3j\omega}$$

the same solution
in different forms

$$H(\omega) = \frac{-\omega(3\omega + 1)}{-\omega^3(j+1) + 3j\omega}$$

$$H(\omega) = \frac{\omega(3\omega + 1)}{\omega^3(j+1) - 3j\omega}$$

Example 9

$$H(\omega) = \frac{\omega(3\omega + 1)}{\omega^3(j+1) - 3j\omega}$$

the same solution in different forms

$$H(\omega) = \frac{3\omega + 1}{\omega^2(j+1) - 3j} \qquad \text{for } \omega \neq 0$$

Example 10

Consider the following generic linear differential equation with constant coefficients (and null initial conditions):

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

Hay que tener en cuenta las condiciones iniciales

Find the Laplace Transform, H(s), and the FT, H(w), of the impulse response h(t) of the LTI system.

Example 10

We will use again the properties:

$$\frac{d^k x(t)}{dt^k} \Longleftrightarrow s^k X(s)$$

For the Laplace Transform (bilateral)

$$s = \sigma + j\omega \qquad \sigma = 0$$

$$\frac{d^k x(t)}{dt^k} \Longleftrightarrow (j\omega)^k X(\omega)$$

For the Fourier Transform

Example 10

T. LAPLACE (BILATERA): (Si las condiciones iniciales son nulas, es lo mismo bilatera o unilatera)

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{m=0}^{M} b_m s^m X(s)$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{m=0}^{M} b_m s^m\right) X(s)$$

Example 10

T. LAPLACE (BILATERA):

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{m=0}^{M} b_m s^m X(s)$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{m=0}^{M} b_m s^m\right) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{k=0}^{N} a_k s^k}$$

Example 10

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{k=0}^{N} a_k s^k}$$

$$s = \sigma + j\omega \qquad \downarrow \qquad \sigma = 0$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Example 11

Consider the following Laplace Transform, H(s), of the impulse response h(t) of an LTI system:

$$H(s) = \frac{s-5}{s^2 - s + 1}$$

Find the corresponding differential equation representing the LTI system.

Example 11

$$H(s) = \frac{s-5}{s^2 - s + 1}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-5}{s^2 - s + 1}$$

$$(s^2 - s + 1)Y(s) = (s-5)X(s)$$

Example 11

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-5}{s^2-s+1}$$

$$(s^2 - s + 1)Y(s) = (s-5)X(s)$$

$$s^2Y(s) - sY(s) + Y(s) = sX(s) - 5X(s)$$

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 5x(t)$$

This is the solution

Example 12

Consider the following FT, H(w), of the impulse response h(t) of an LTI system:

$$H(\omega) = \frac{\omega - 5}{\omega^2 - \omega + 1}$$

Find the corresponding differential equation representing the LTI system.

Example 12

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\omega - 5}{\omega^2 - \omega + 1}$$

$$\omega^2 Y(\omega) - \omega Y(\omega) + Y(\omega) = \omega X(\omega) - 5X(\omega)$$

$$\frac{j^2}{j^2}\omega^2 Y(\omega) - \frac{j}{j}\omega Y(\omega) + Y(\omega) = \frac{j}{j}\omega X(\omega) - 5X(\omega)$$

$$\frac{1}{j^2}(j\omega)^2 Y(\omega) - \frac{1}{j}(j\omega) Y(\omega) + Y(\omega) = \frac{1}{j}(j\omega) X(\omega) - 5X(\omega)$$

Example 12

$$\frac{1}{j^2}(j\omega)^2 Y(\omega) - \frac{1}{j}(j\omega)Y(\omega) + Y(\omega) = \frac{1}{j}(j\omega)X(\omega) - 5X(\omega)$$

$$-(j\omega)^2 Y(\omega) + j(j\omega)Y(\omega) + Y(\omega) = -j(j\omega)X(\omega) - 5X(\omega)$$

$$-\frac{d^2 y(t)}{dt^2} + j\frac{dy(t)}{dt} + y(t) = -j\frac{dx(t)}{dt} - 5x(t)$$

This is the solution

Example 13

Conpute the stand. FT of the signal:

$$x(t) = e^{-3|t|}\sin(2t)$$

Example 13

By the definition of Stand. FT:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-3|t|} \sin(2t) e^{-j\omega t} dt$$

$$= \underbrace{\int_{0}^{\infty} e^{-3t} \sin(2t) e^{-j\omega t} dt}_{X_1(\omega)} + \underbrace{\int_{-\infty}^{0} e^{3t} \sin(2t) e^{-j\omega t} dt}_{X_2(\omega)}$$

let focus of this first integral

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

Example 13

$$X_1(\omega) = \int_0^\infty e^{-3t} \sin(2t) e^{-j\omega t} dt$$

$$X_1(\omega) = \int_0^\infty e^{-3t} \left[\frac{1}{2j} (e^{2jt} - e^{-2jt}) \right] e^{-j\omega t} dt$$

$$X_1(\omega) = \frac{1}{2j} \int_0^\infty e^{(-3+j(2-\omega))t} dt - \frac{1}{2j} \int_0^\infty e^{(-3-j(2+\omega))t} dt$$

Example 13

$$X_1(\omega) = \frac{1}{2j}\int_0^\infty e^{(-3+j(2-\omega))t}dt - \frac{1}{2j}\int_0^\infty e^{(-3-j(2+\omega))t}dt$$

$$X_1(\omega) = -\frac{1}{2j}\frac{1}{-3+j\left(2-\omega\right)} + \frac{1}{2j}\frac{1}{-3-j\left(\omega+2\right)}$$

$$X_1(\omega) = \frac{1}{2j}\left[\frac{1}{3-2j+j\omega} - \frac{1}{3+j\omega+2j}\right]$$

Example 13

with similar steps, it is possible to show:

$$X_2(\omega) = \frac{-1}{2j}\left[\frac{1}{3-2j-j\omega} - \frac{1}{3-j\omega+2j}\right]$$

...and finally:

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

Example 14

Considere la respuesta de un sistema LTI con respuesta al impulso

$$h(t) = e^{-at}u(t), \quad a > 0,$$

a la señal de entrada

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Compute the FT of the output of the system, Y(w).

Example 14

En lugar de calcular $y(t) = x(t) * h(t)$ directamente, transformemos el problema en el dominio de la frecuencia. Del ejemplo 4.1, las transformadas de Fourier de $x(t)$ y $h(t)$ son

$$X(\omega) = \frac{1}{b + j\omega}$$

y

$$H(\omega) = \frac{1}{a + j\omega}.$$

Por tanto,

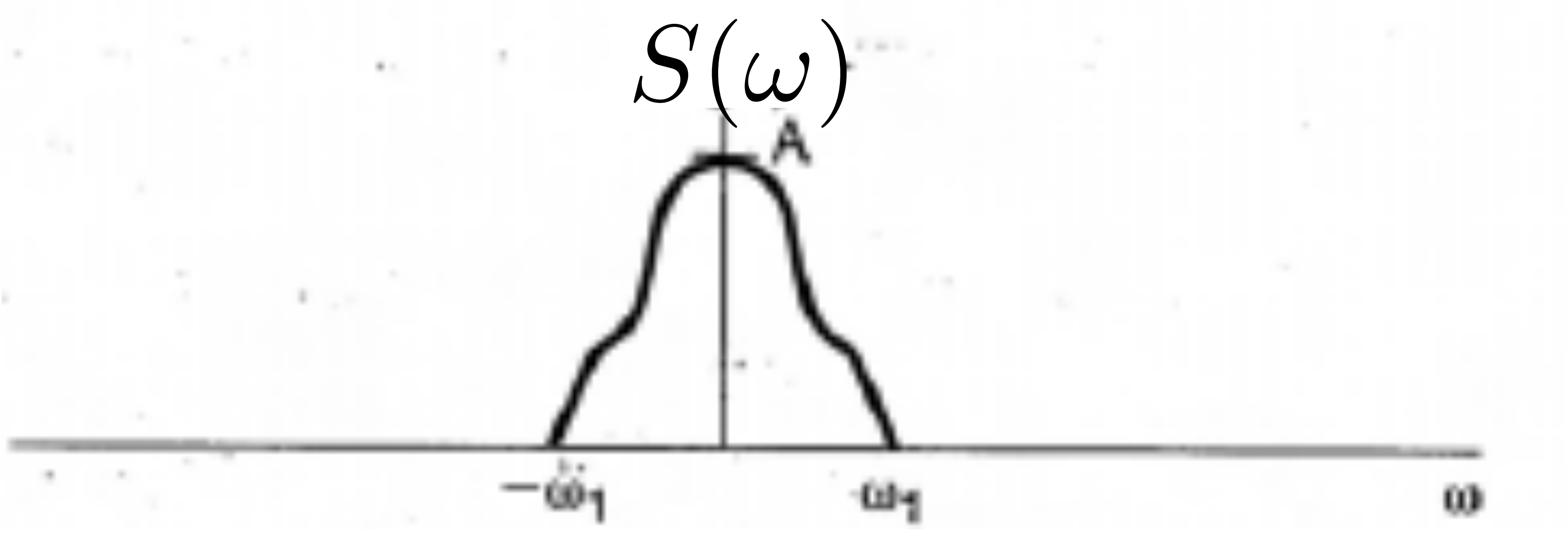
$$Y(\omega) = \frac{1}{(a + j\omega)(b + j\omega)}.$$

$$Y(\omega) = H(\omega)X(\omega)$$

This is the solution

Example 15

Sea $s(t)$ una señal cuyo espectro $S(\omega)$ se ha trazado en la figura



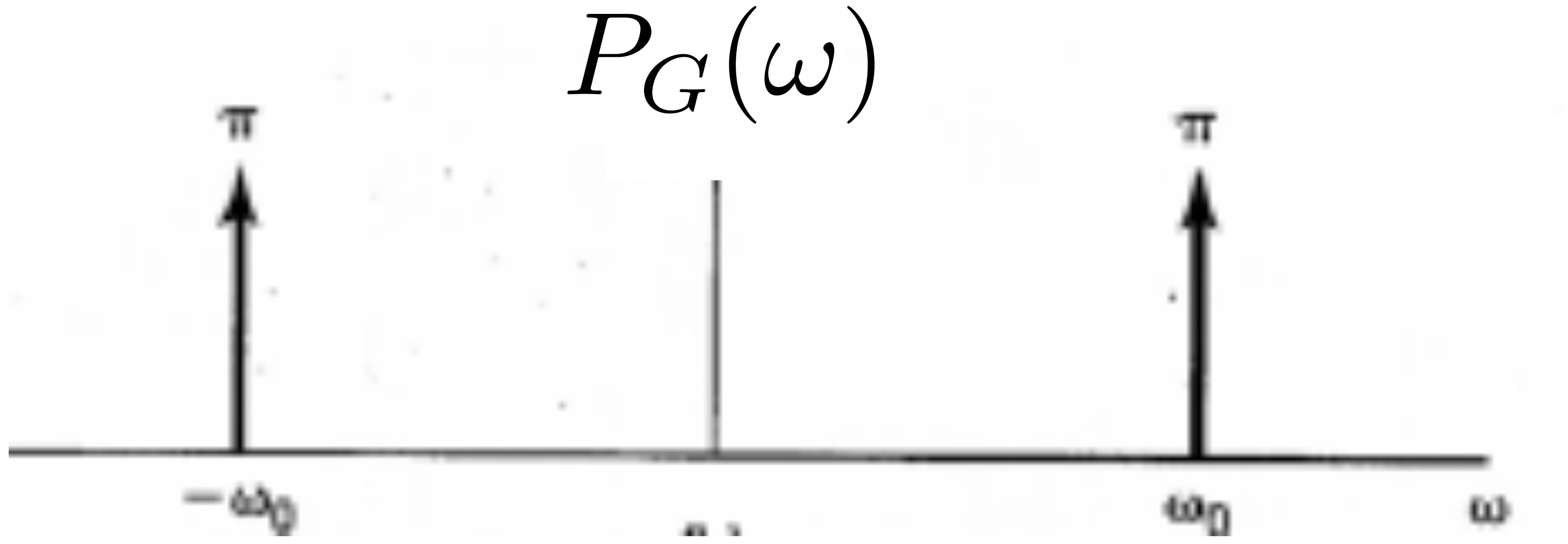
Then, also consider the following signal: $p(t) = \cos(\omega_0 t)$

Find the FT of the signal r(t)=s(t)p(t)

Example 15

The GFT of p(t):

$$P_G(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$



and using the property:

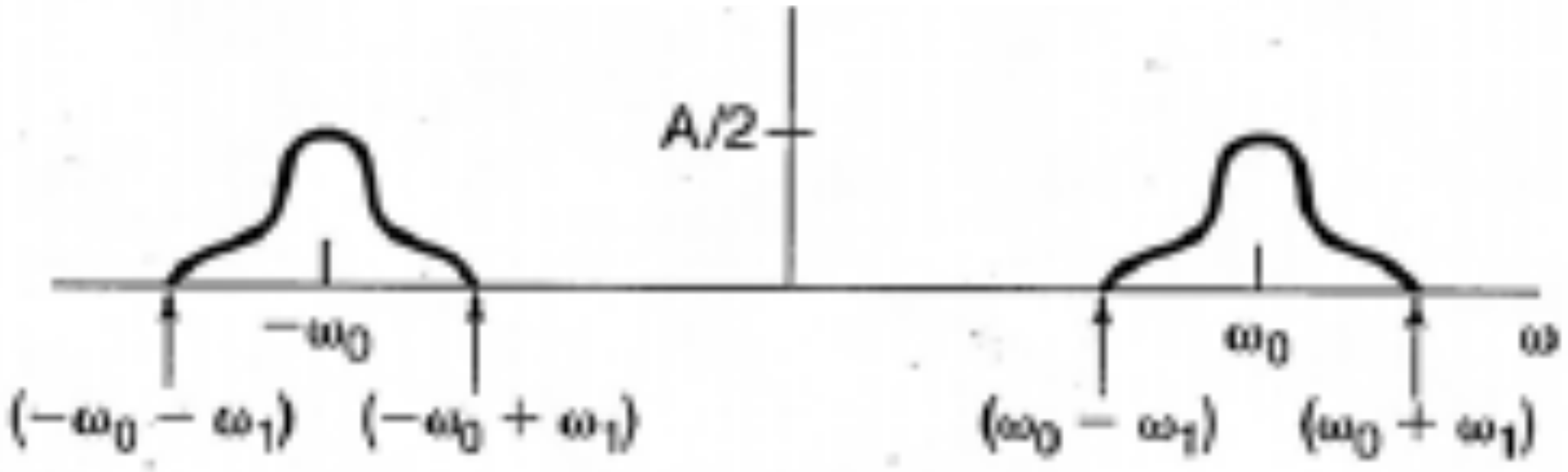
$$r(t) = s(t)p(t) \Longleftrightarrow \frac{1}{2\pi}[S(\omega) * P_G(\omega)]$$

Example 15

and using the property:

$$r(t) = s(t)p(t) \Longleftrightarrow \frac{1}{2\pi}[S(\omega) * P_G(\omega)]$$

$$R(\omega) = \frac{1}{2\pi}[S(\omega) * P_G(\omega)]$$



Questions?