Topic 2 - Part 3:
SYSTEMS IN THE TIME DOMAIN

Linear systems and circuit applications

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

What we saw

- An LTI system in continuous time can be expressed (mathematically):
- Forced L-ODE *with constant coefficients* and *with NULL initial conditions*
- Convolution integral where h(t) is the impulse response

In this slides…

- properties of the convolution

Convolution

- To obtain the LTIS response for any particular signal $x(t)$, we use the convolution:

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau}$$

- So that, to get the output signal we only need to solve an integral.

  - if we know h(t)

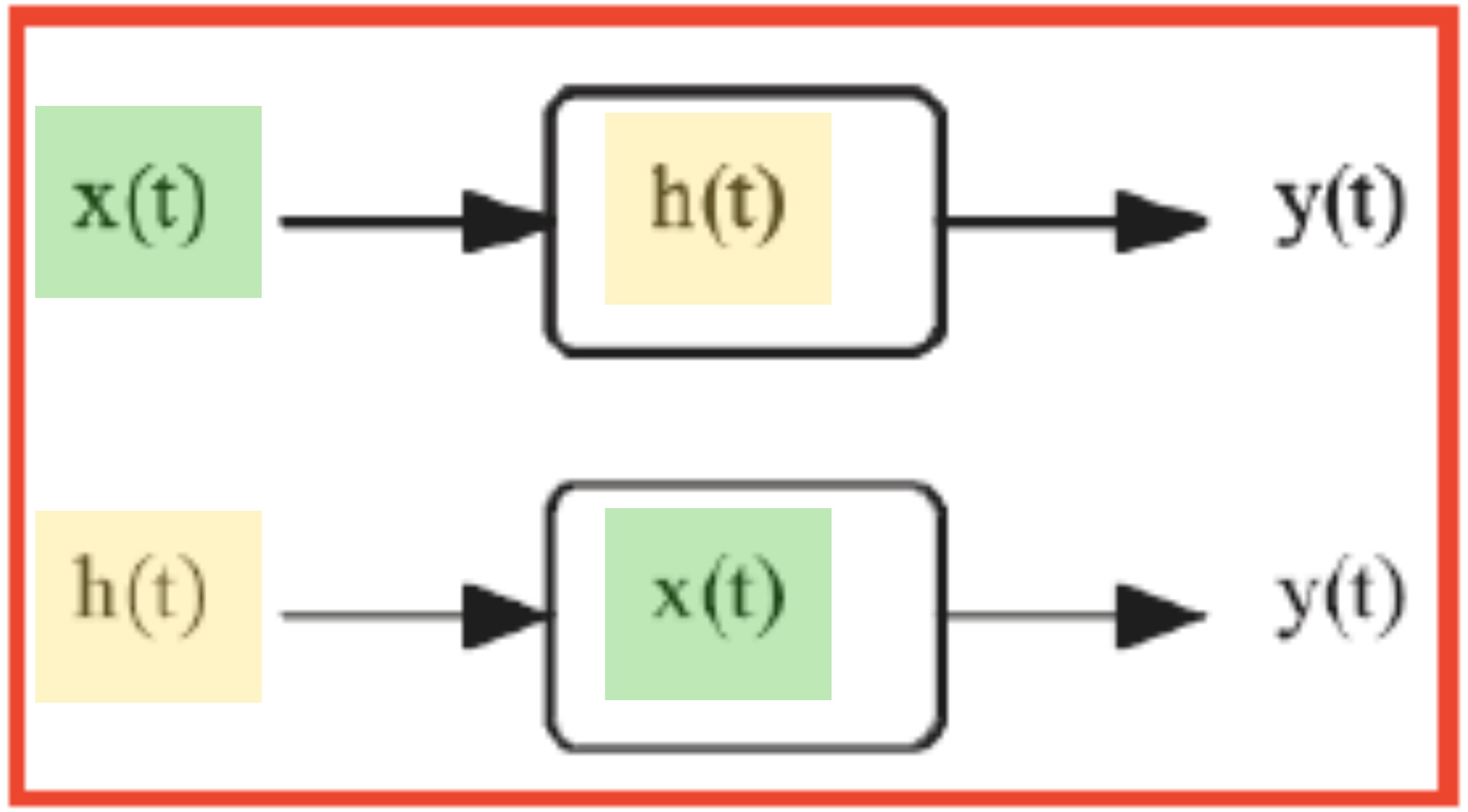
Property: Commutative

Conmutative

- Changin the order of the signals does not change the result of the convolution:

$$x(t) * h(t) = h(t) * x(t)$$

- In terms of systems and signals, the output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI systems with input $h(t)$ and unit impulse response $x(t)$.



Property: Commutative - proof

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- Change of variable:

$$v = t - \tau \implies \tau = t - v \implies d\tau = -dv$$

$$\tau \to -\infty \implies v \to \infty$$

$$\tau \to +\infty \implies v \to -\infty$$

Property: Commutative - proof

- Replacing:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = -\int_{+\infty}^{-\infty} x(t-v)h(v)dv$$

$$= \int_{-\infty}^{+\infty} x(t-v)h(v)dv$$

$$= \int_{-\infty}^{+\infty} h(v)x(t-v)dv$$

$$= h(t) * x(t)$$

Property: Commutative

- Then we finally obtain:

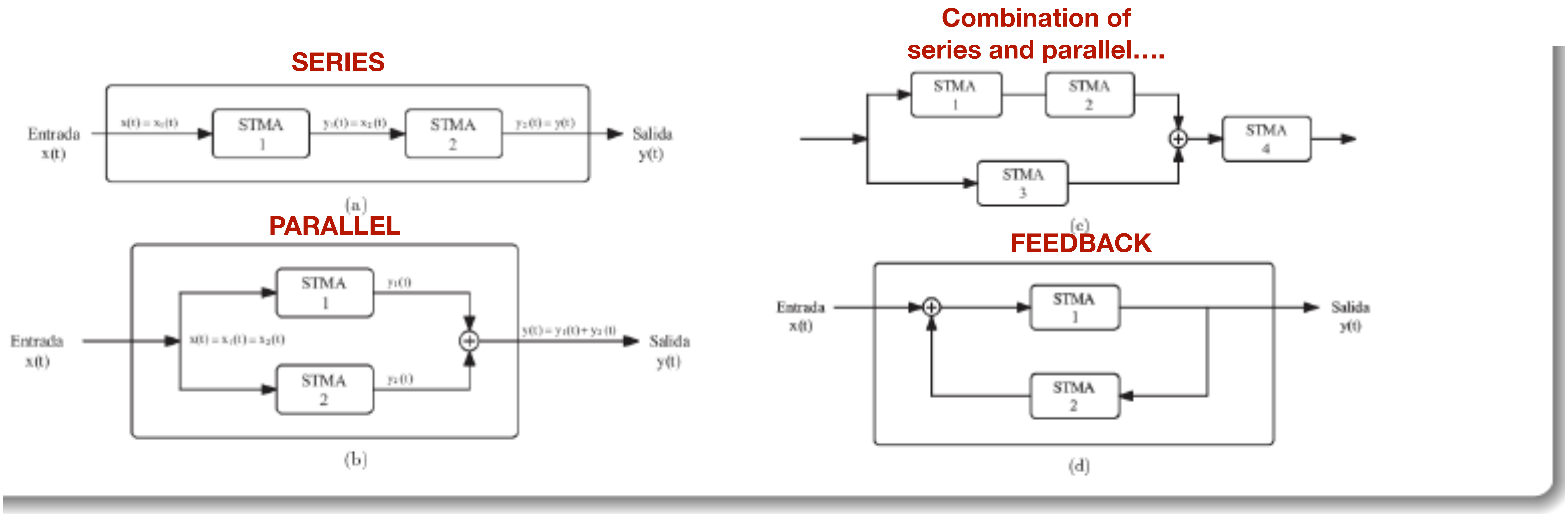
$$x(t) * h(t) = h(t) * x(t)$$

Connections among systems

Innterconnections of Systems (I)

- Many real systems are built as interconnections of different simple subsystems to create a complex system. There are several basic system interconnections:
  1. **Series interconnection.** The output of one systems is the input of the following system.
  2. **Parallel interconneciton** The same input is applied to the interconnected systems, and the output is the sumn of the individual ouputs.
  3. **Combination** We can combine series and parallel interconnections to create more complicated systems.
  4. **Feedback interconnection** The output of the system is feeded back to the input.

Connections among systems



Property: Associative

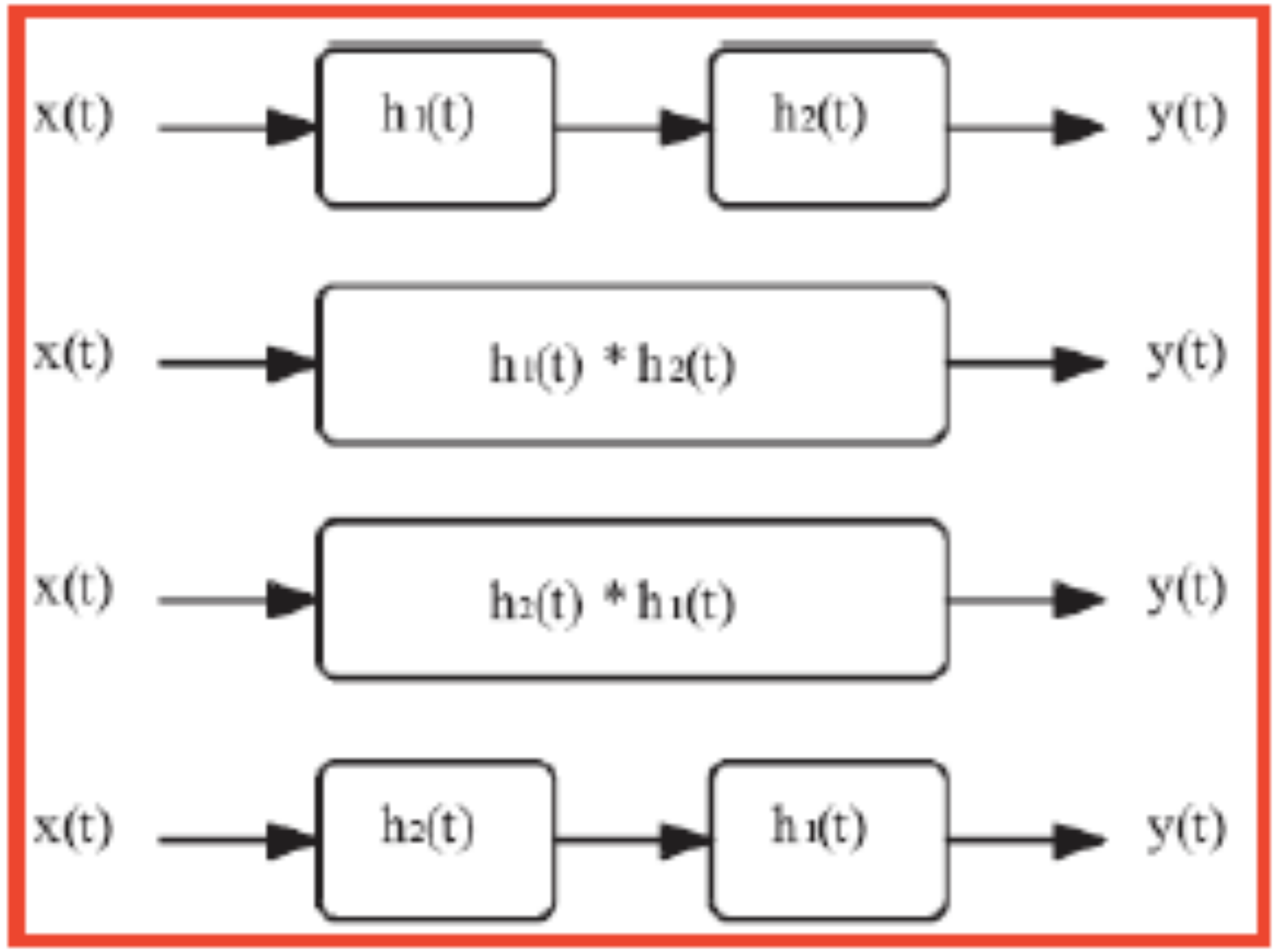
- Think on integral properties…

Associative

- El orden en que se realizan las convoluciones no altera el resultado:

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

$$h_{eq}(t) = h_1(t) * h_2(t)$$



- We can also switch the positions of h_1(t) and h_2(t), and nothing does not change.

Property: Associative

SERIES of Systems

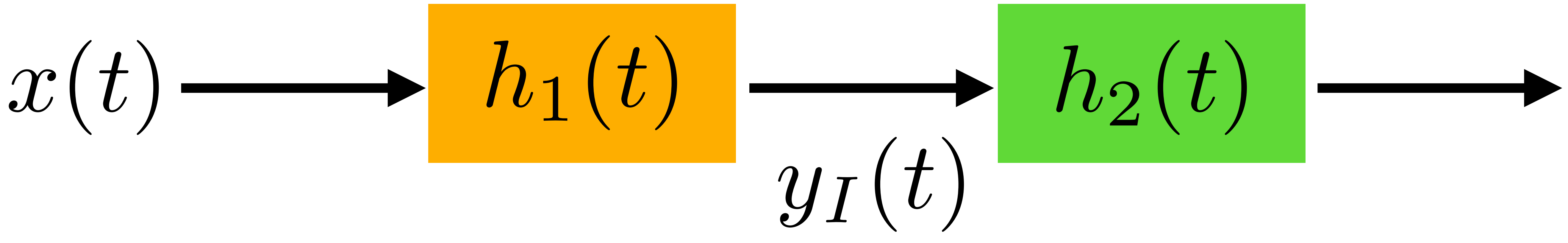
$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow[y_I(t)]{} \boxed{h_2(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_{eq}(t) = h_1(t) * h_2(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h_{eq}(t)$$

Series of systems: equivalent impulse response

SERIES of Systems



$$y(t) = y_I(t) * h_2(t)$$

$$y_I(t) = x(t) * h_1(t)$$

$$y(t) = (x(t) * h_1(t)) * h_2(t)$$

Series of systems: equivalent impulse response

$$y(t) = (x(t) * h_1(t)) * h_2(t)$$

$$y(t) = x(t) * \boxed{h_1(t) * h_2(t)}$$

$$h_{eq}(t) = h_1(t) * h_2(t)$$

$$y(t) = x(t) * \boxed{h_{eq}(t)}$$

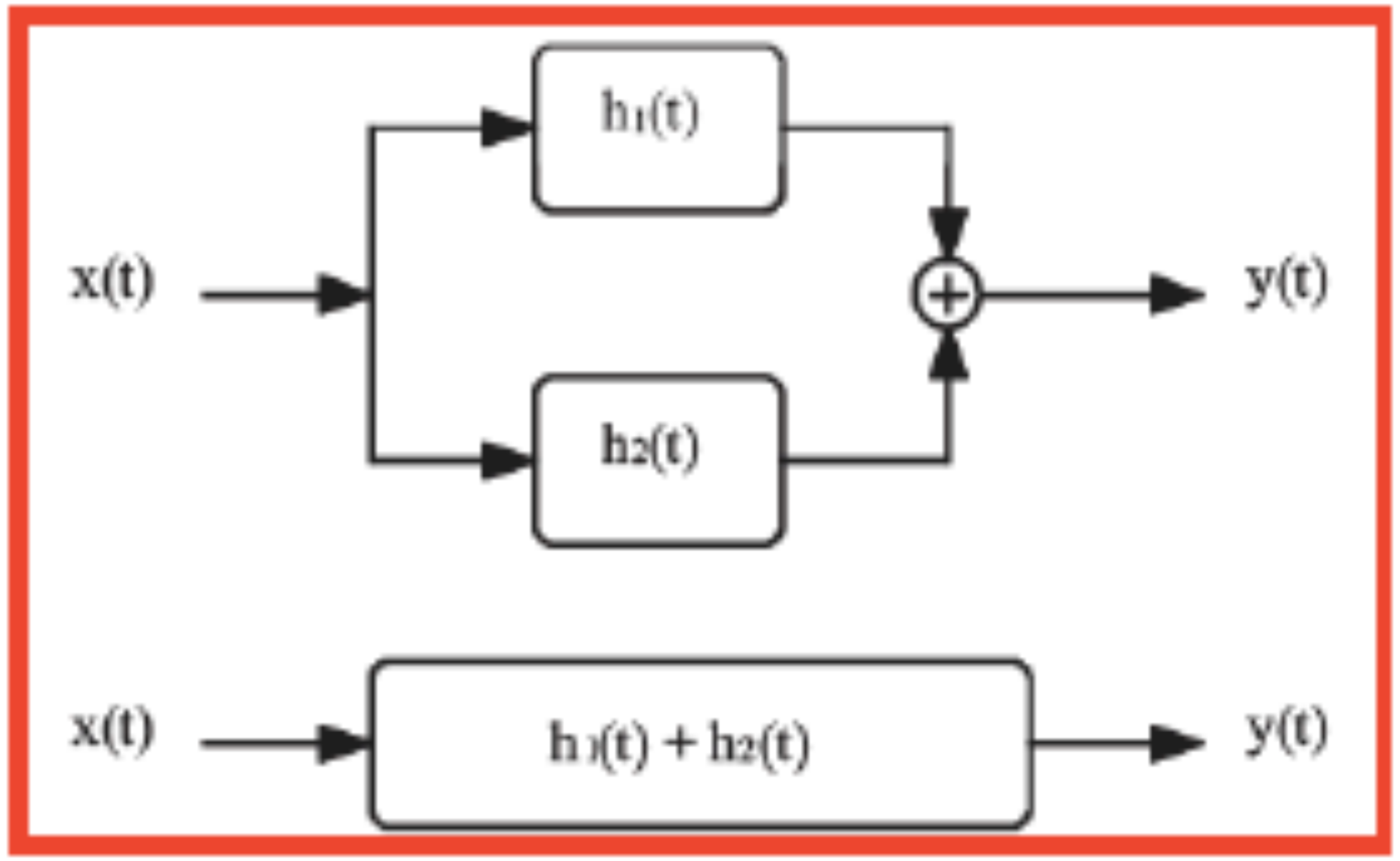
Property: Distributive

- Think on integral properties…

Distributive

- Convolution has the distributive property over addition:

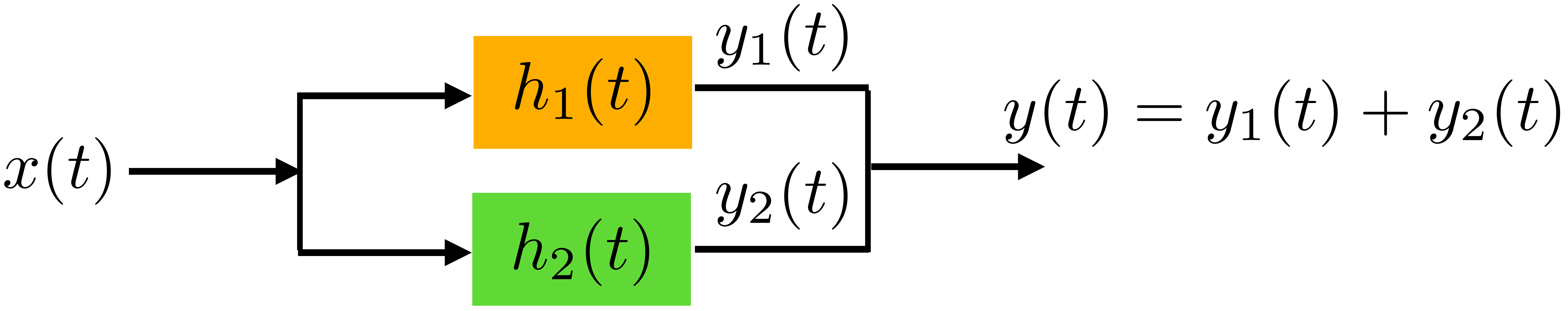
$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

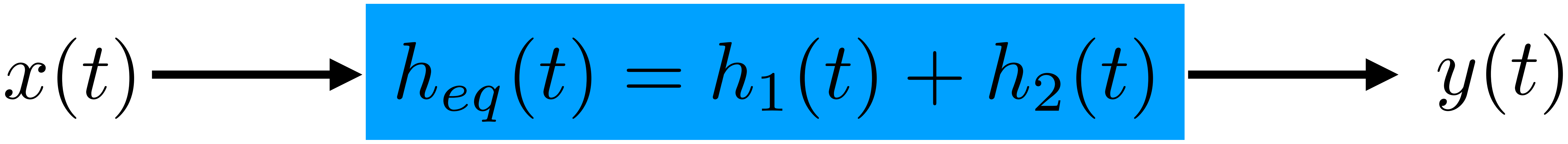


- Distributive property can be interpreted in systems language. LTI interconnected in parallel are equivalent a signle LTI systems with unit impulse response the sum of the original unite impulse responses. $h_{eq}(t) = h_1(t) + h_2(t)$.

Property: Distributive

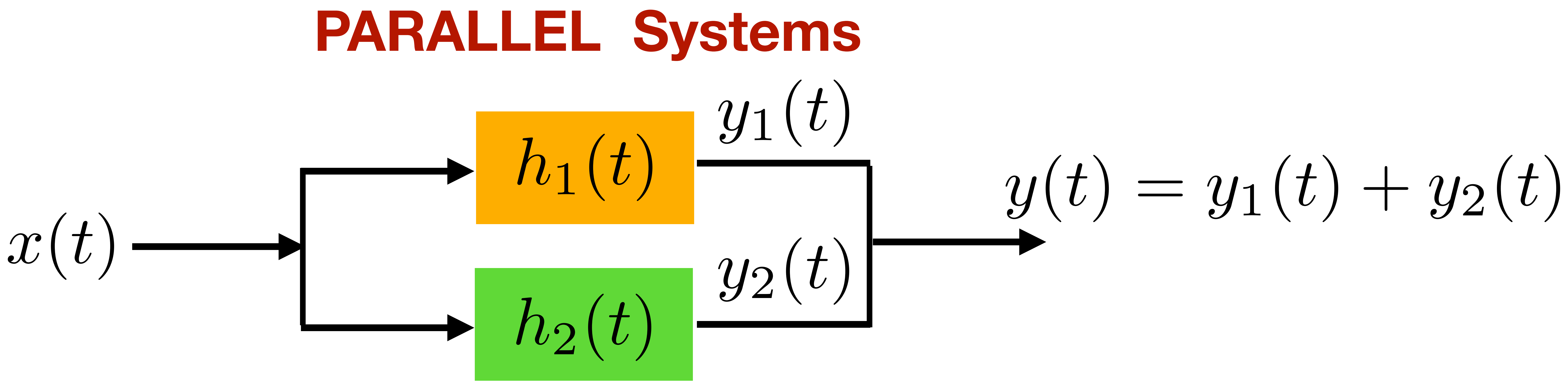
PARALLEL Systems





$$y(t) = x(t) * h_{eq}(t)$$

Parallel systems: equivalent impulse response



$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$

Parallel systems: equivalent impulse response

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$

$$\boxed{h_{eq}(t) = h_1(t) + h_2(t)}$$

DISTRIBUTIVE

$$y(t) = x(t) * \boxed{h_{eq}(t)}$$

Property: convolution with a delta function

Identity element of the convolution

- The unit impulse is the identity elemento of the convolution:

$$x(t) \xrightarrow{} \boxed{\delta(t)} \xrightarrow{} x(t)$$

$$x(t) * \delta(t) = x(t)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} \delta(\tau)x(t-\tau)d\tau = x(t)$$

- Recall the delta properties…

Property: convolution with a delta function

Convolution with a delayed impulse

- The convolution of a signal with a delayed unit impulse is the same original signal delayed the same amount of time:

$$\boxed{x(t) * \delta(t - t_0) = x(t - t_0)}$$

IMP!!!!

- Proof:

$$y(t) = x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - t_0 - \tau)d\tau$$

The impulse is not zero at $t - t_0 - \tau = 0 \Rightarrow \tau = t - t_0$, therefore:

$$y(t) = \int_{-\infty}^{\infty} x(t - t_0)\delta(t - t_0 - \tau)d\tau = x(t - t_0)\int_{-\infty}^{\infty} \delta(t - t_0 - \tau)d\tau = x(t - t_0)$$

- **Recall the delta properties…**

Property: convolution with a delta function

- **VERY IMPORTANT:**

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Convolution with a delta function: example

Compute $y(t) = x(t) * h(t)$ with $h(t) = \delta(t-2) - \delta(t+2)$

$$\begin{aligned} y(t) &= x(t) * [\delta(t-2) - \delta(t+2)] \\ &= x(t) * \delta(t-2) - x(t) * \delta(t+2) \\ &= x(t-2) - x(t+2) \end{aligned}$$

Convolution with a delta function: example

Example: LTI systems and impulse response

- Let be an LTI system with $h(t) = \delta(t) - \delta(t-2)$. An alternativa representation can be obtained usign the input $x(t)$ and computing the output $y(t)$, that is:

$$y(t) = x(t) * h(t) = x(t) - x(t-2)$$

Although the equations are different (convolution and sum of delayed versions of the input) the LTI systems is the same.

Recall that:

Properties of LTI Systems

- The properties of the LTI systems are completely determined by its impulse response.
- If two LTI systems have the same impulse response, then they are the same system. This only holds for LTI systems.

- **Then h(t) summarizes all the properties of LTI systems ==>**

Then:

- We can study the properties of the system looking h(t) ==>

Property of an LTI system: memory

always memory,
except the case of
h(t)=A delta(t)

Memory in LTI Systems

- Let be an LTI system with $h(t)$. The output is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, the systems it memoryless iff , $h(\tau) = 0$ for all $\tau \neq 0$. That is, the only LTI systems memoryless are those with:

$$h(t) = A\delta(t)$$

Property of an LTI system: memory

Examples: LTI systems and memory

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ has memory.
- The LTI system with $h(t) = u(t) - u(t-1)$ has memory.

Property of an LTI system: causality

Commutative property

Causality for LTI Systems

- Let be an LTI systems with $h(t)$. The output is given by:

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

- Therefore, the impulse response of a causal LTI systems requieres to satisfy, $h(\tau) = 0$ for all $\tau < 0$, that is, causal LTI systems has the following type of impulse response:

$$h(t) = \cancel{h(t)}u(t)$$

$h\_1(t)$

Property of an LTI system: causality

Examples: LTI systems and causality

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ is causal.
- The LTI system with $h(t) = u(t) - u(t-1)$ is causal.
- The LTI system with $h(t) = u(t+1) - u(t)$ is anticausal.
- The LTI system with $h(t) = u(t+1) - u(t-1)$ is noncausal.

Property of an LTI system: stability

Stability for LTI systems

- Let be an LTI systems with $h(t)$. The output is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, in order the system to be stable, bounded inputs should lead to bounded outputs.
- This can be expressed as:

$$|x(t)| \leq k_x \Rightarrow |y(t)| = |x(t) * h(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right| \leq$$

$$\leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq$$

$$\leq \int_{-\infty}^{\infty} k_x|h(\tau)|d\tau \leq k_x \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

integrable

Therefore, an LTI system is stable iff the impulse response is *absolutely ~~integrabel~~*:

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau = k_h \overset{<}{\cancel{\leq}} \infty$$

Property of an LTI system: stability

Examples: Stability and LTI systems

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ is stable.
- The LTI system with $h(t) = u(t) - u(t-1)$ is stable.
- The LTI system with $h(t) = u(t)$ is unstable.
- The LTI system with $h(t) = e^{-t}$ is unstable.
- The LTI system with $h(t) = \sin(t) * u(t)$ is unstable.

We will see…

- examples of convolutions
- LTI systems in a transformed domain

Questions?