Topic 2 - Part 4: SYSTEMS IN THE TIME DOMAIN

Linear systems and circuit applications

Luca Martino — luca.martino@urjc.es — http://www.lucamartino.altervista.org

Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

In this slides…

- **Examples - problems - exercises**

Example 1

$$x(t) = e^{-\alpha t},$$

$$h(t) = \delta(t - t_0), \quad t_0 \text{ is a constant}$$

$$y(t) = x(t) * h(t) = ?$$

Example 1

$$
\begin{aligned}
y(t) &= e^{-\alpha t} * \delta(t - t_0), \\
&= e^{-\alpha(t - t_0)}
\end{aligned}
$$

Example 2

$$x(t) = e^{-\alpha t}u(t), \quad \alpha \neq 0$$

$$h(t) = e^{-\beta t}u(t), \quad \beta \neq 0$$

$$y(t) = x(t) * h(t) = ?$$

Example 2

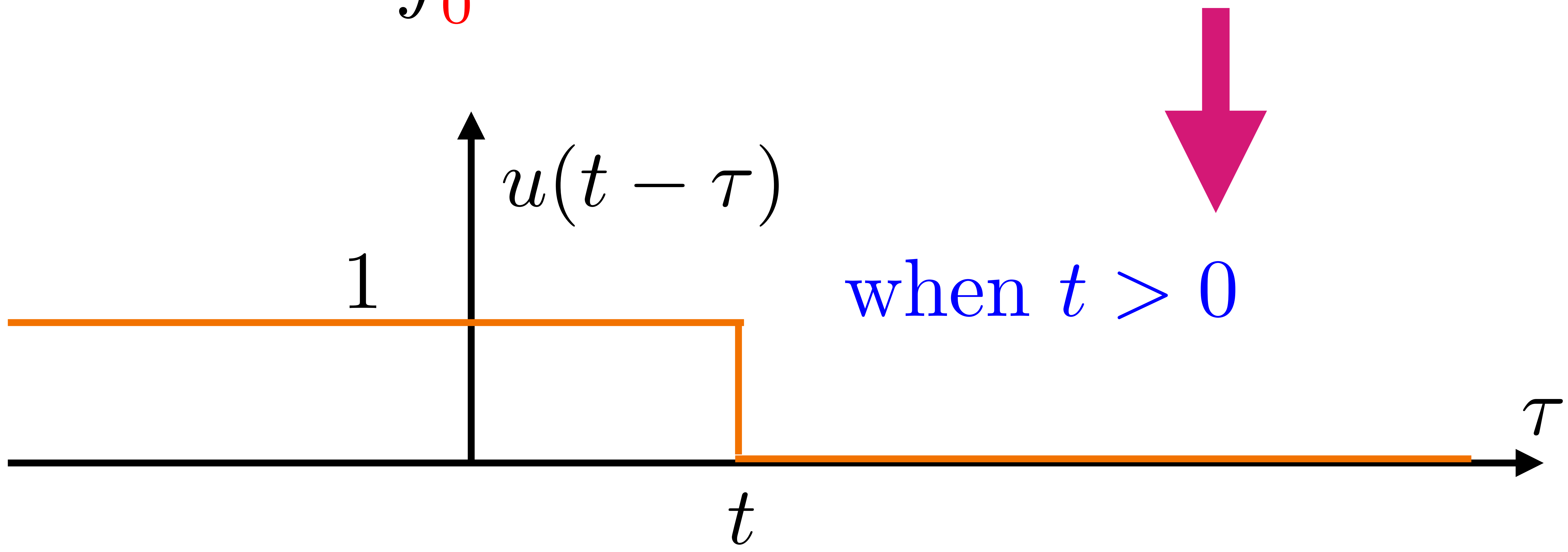
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

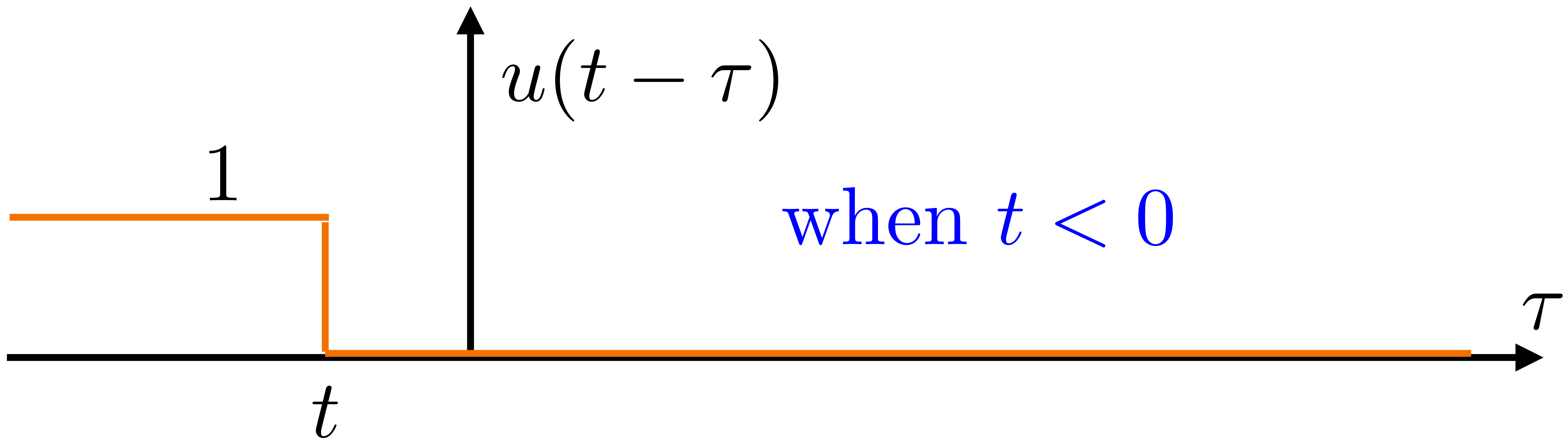
$$= \int_{-\infty}^{+\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{0}^{+\infty} e^{-\alpha\tau}e^{-\beta(t-\tau)}u(t-\tau)d\tau$$

Example 2

$$y(t) = x(t) * h(t) = \int_{0}^{+\infty} e^{-\alpha\tau} e^{-\beta(t-\tau)} u(t-\tau) d\tau$$





Example 2

- Then (first part of the solution):

$$y(t) = 0, \text{ for all } t < 0$$

Example 2

$$x(t) * h(t) = \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)} d\tau \qquad t > 0$$

$$= \int_0^t e^{-\alpha\tau+\beta\tau-\beta t} d\tau \qquad t > 0$$

$$= e^{-\beta t} \int_0^t e^{-(\alpha-\beta)\tau} d\tau \qquad t > 0$$

Example 2

$$y(t) = x(t) * h(t) = e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau} d\tau \qquad t > 0$$

$$= e^{-\beta t} \left[\frac{1}{\beta - \alpha} e^{(\beta-\alpha)\tau} \right]_0^t \qquad t > 0$$

$$= e^{-\beta t} \left[\frac{1}{\beta - \alpha} e^{(\beta-\alpha)t} - \frac{1}{\beta - \alpha} \right] \qquad t > 0$$

$$= \left[\frac{1}{\beta - \alpha} e^{-\alpha t} - \frac{1}{\beta - \alpha} e^{-\beta t} \right] \qquad t > 0$$

Example 2

$$y(t) = x(t) * h(t) = \left[\frac{1}{\beta - \alpha} e^{-\alpha t} - \frac{1}{\beta - \alpha} e^{-\beta t}\right] \qquad t > 0$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \qquad t > 0$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t)$$

- since y(t)=0 for t<0

$$y(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t)$$

Example 3

$$x(t) = u(t),$$

$$h(t) = u(t),$$

$$y(t) = x(t) * h(t) = ?$$

Example 3

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} u(\tau)u(t-\tau)d\tau$$

Example 3

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} u(\tau)u(t-\tau)d\tau$$

- As in example 2 - see previously slides:

$$y(t) = 0, \quad t < 0$$

$$y(t) = \int_{0}^{t} 1 \cdot d\tau, \quad t > 0$$

Example 3

$$y(t) = \int_0^t 1 \cdot d\tau, \quad t > 0$$

$$y(t) = [\tau]_0^t \, , \quad t > 0$$

$$y(t) = t - 0, \quad t > 0$$

$$y(t) = t, \quad t > 0$$

- finally, since y(t)=0 for t<0:

$$y(t) = tu(t)$$

Example 3

- For a more graphical derivation, see:

https://www.youtube.com/watch?v=iAuVYJLjsII

Example 4

$$x(t) = 6e^{-\alpha t}u(t) \quad \beta \neq 0,$$

$$h(t) = \delta(t) - \delta(t-2) + e^{-\beta t}u(t), \quad \beta \neq 0$$

$$y(t) = x(t) * h(t) = ?$$

Example 4

$$y(t) = 6e^{-\alpha t}u(t) * \delta(t) - 6e^{-\alpha t}u(t) * \delta(t-2) + 6e^{-\alpha t}u(t) * e^{-\beta t}u(t)$$

- **See Example 2**

$$y(t) = 6e^{-\alpha t}u(t) - 6e^{-\alpha(t-2)}u(t-2) + 6\frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}u(t)$$

Example 5

$$x(t) = rect(t), \quad \text{is 1 when } t \in [0, 1], \text{and 0 otherwise}$$

$$h(t) = \sin(t)$$

$$y(t) = x(t) * h(t) = ?$$

Example 5

- We can choose one of the two *equivalent* possibilities:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} \sin(\tau) rect(t-\tau) d\tau$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} rect(\tau) \sin(t-\tau) d\tau$$

- this one is better in this scenario….

Example 5

- Replacing *the meaning of* rect(t):

$$y(t) = x(t) * h(t) = \int_0^1 1 \cdot \sin(t-\tau)d\tau$$

$$= \Big[-\big(-\cos(t-\tau)\big)\Big]_0^1 = \cos(t-1) - \cos(t)$$

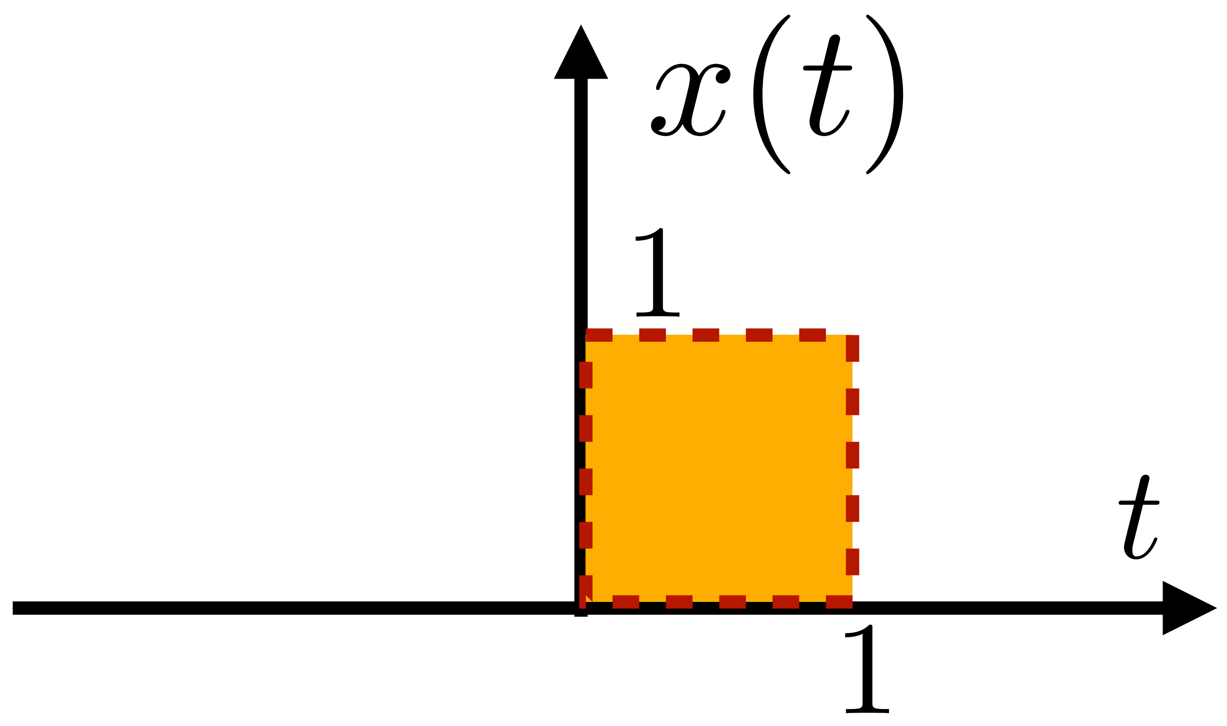
$$y(t) = \cos(t-1) - \cos(t)$$

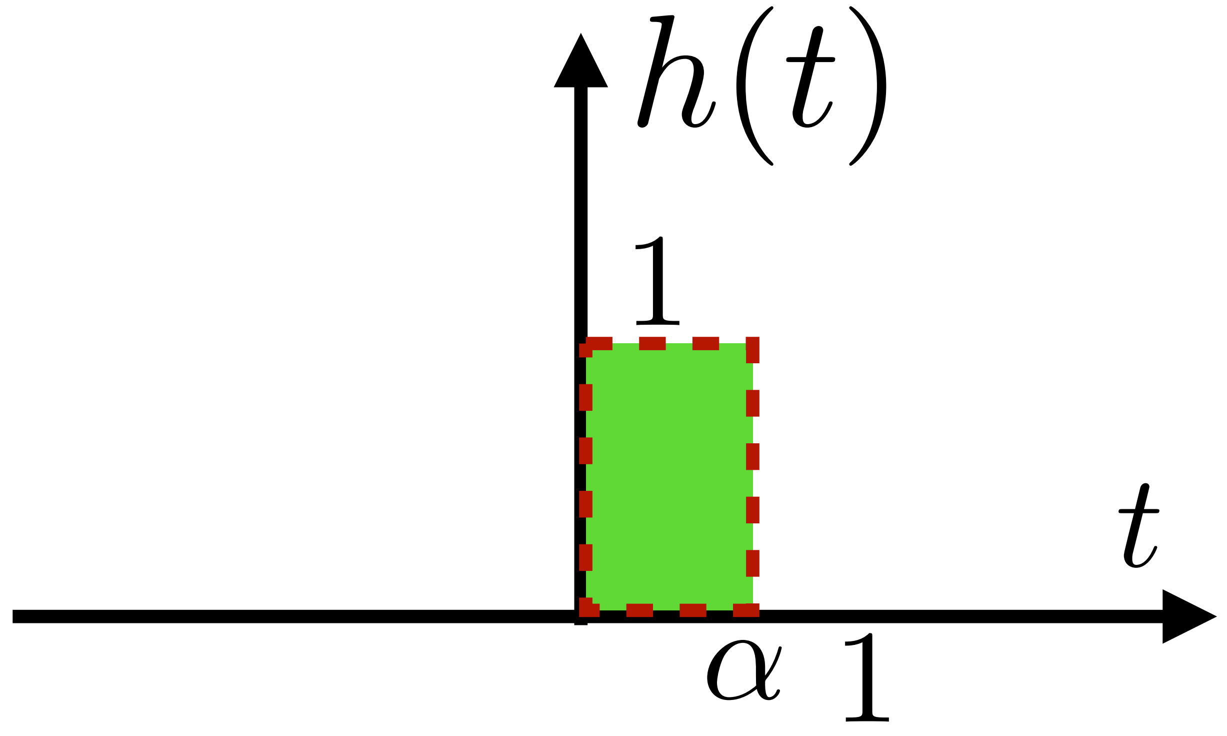
Example 6

$x(t) = 1$ if $t \in [0, 1]$, otherwise $x(t) = 0$

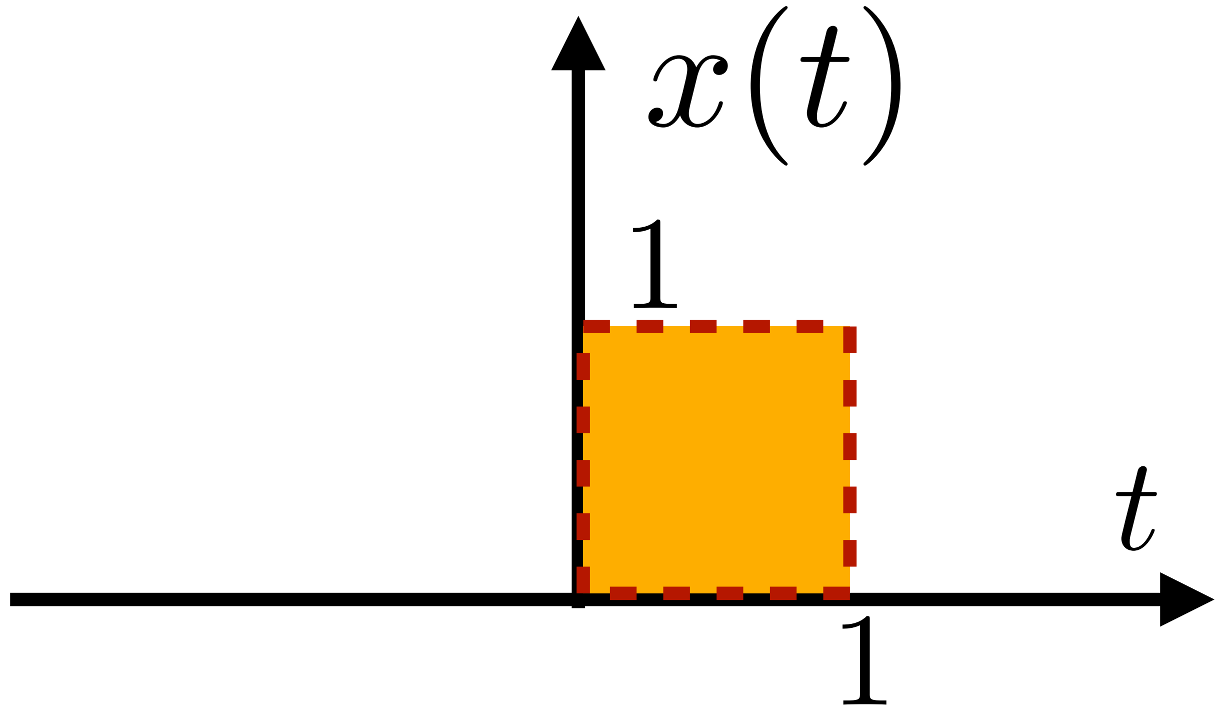
$h(t) = 1$ if $t \in [0, \alpha]$, otherwise $h(t) = 0$ with $\alpha < 1$

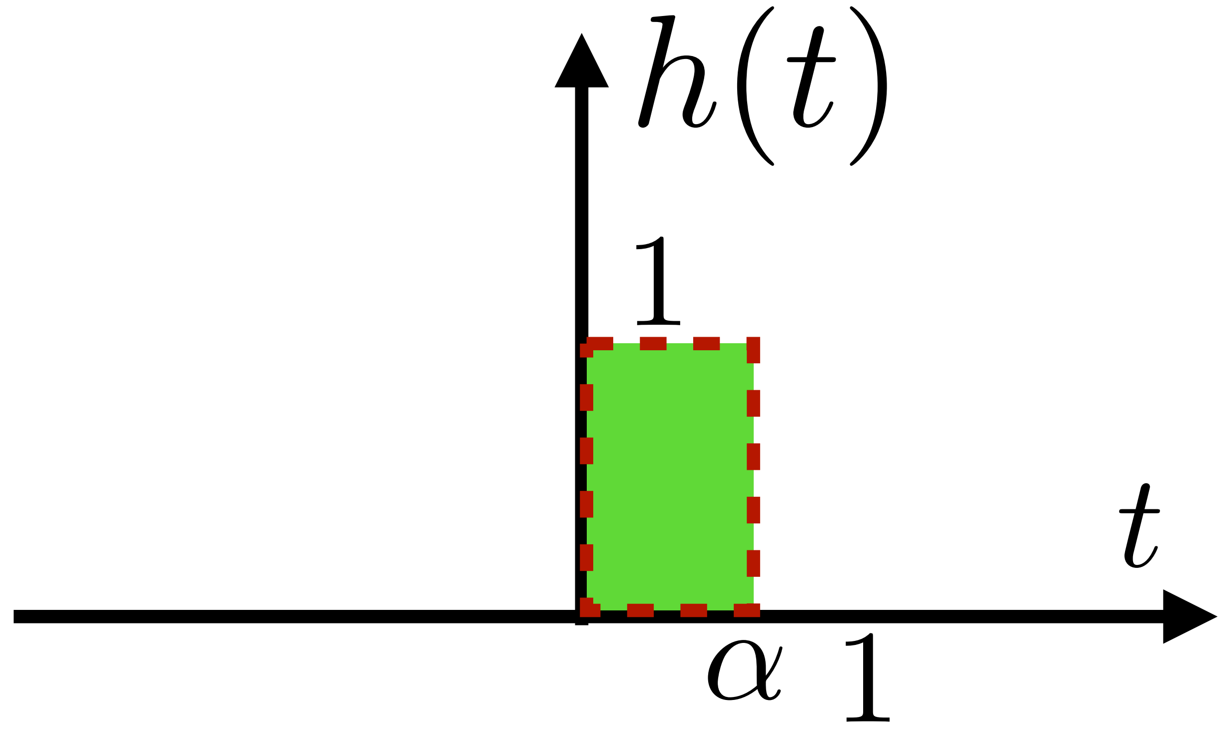
$$y(t) = x(t) * h(t) = ?$$



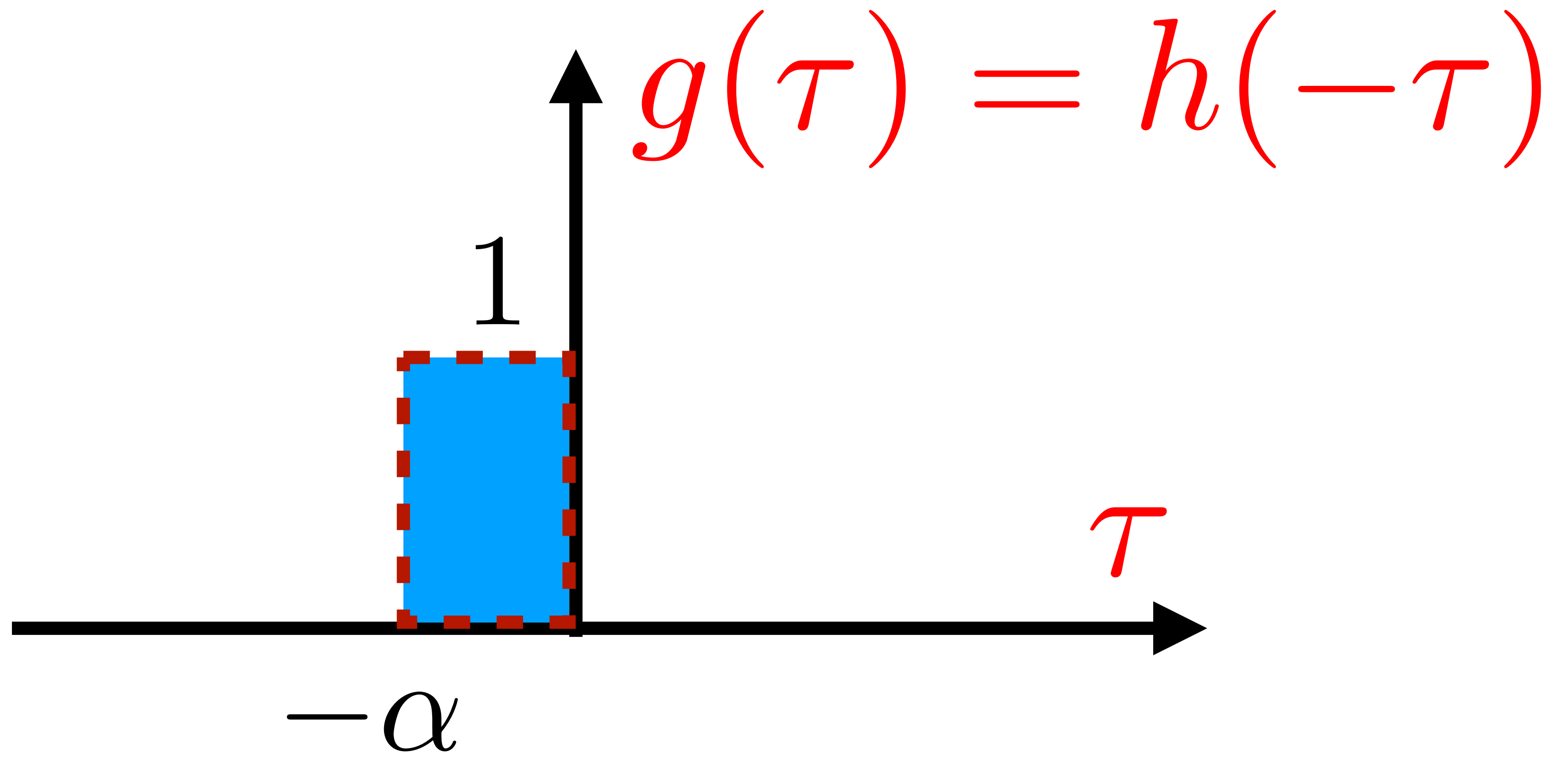


Example 6

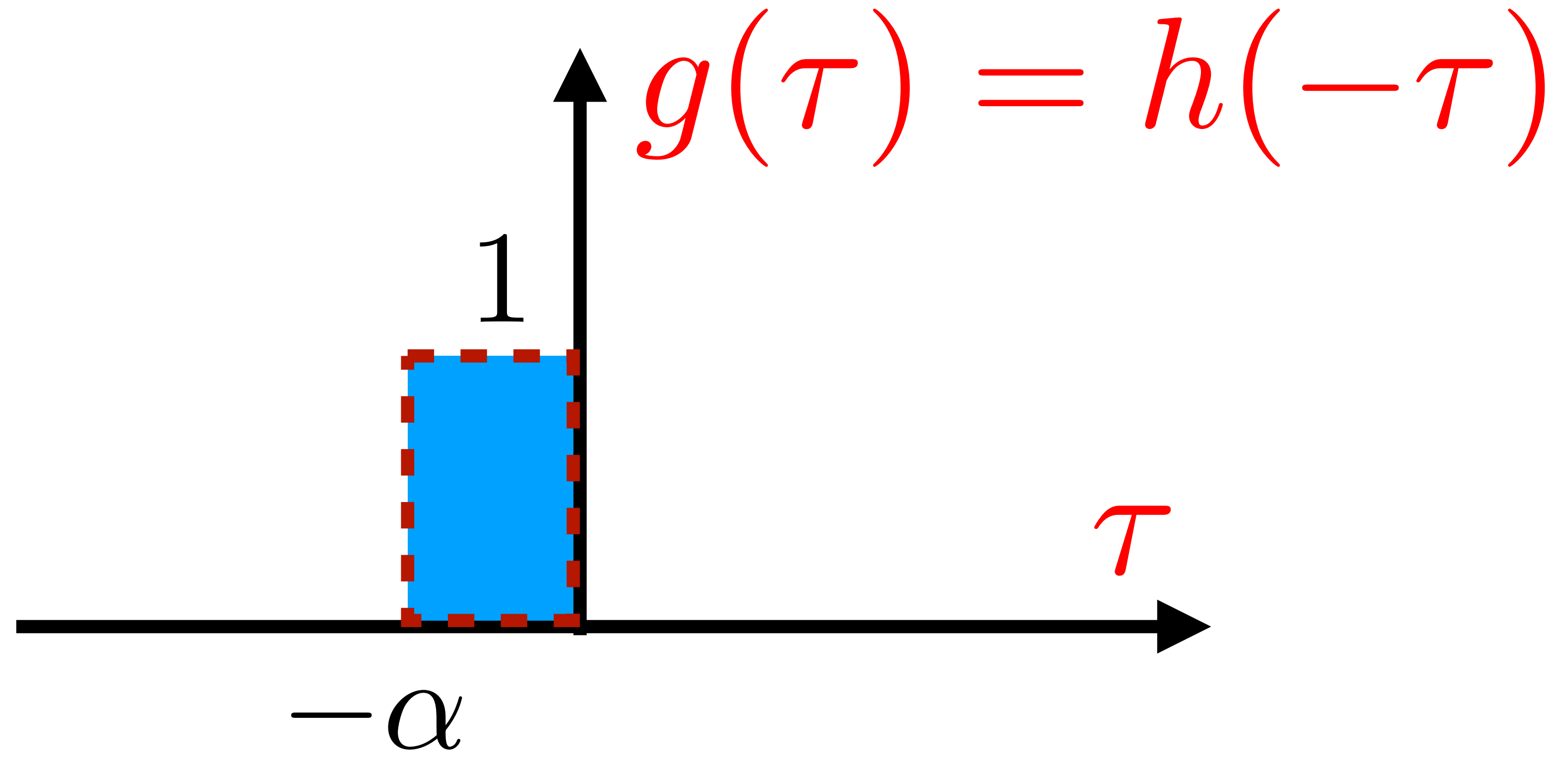




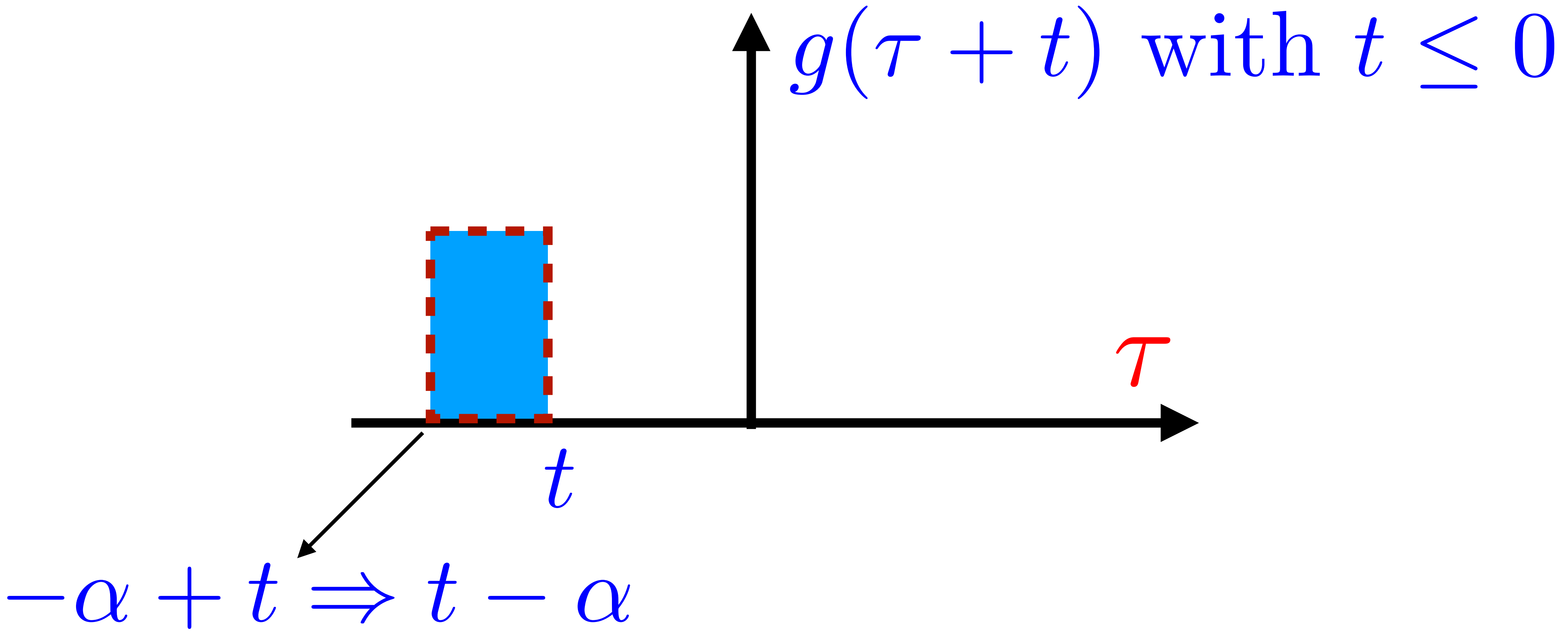
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

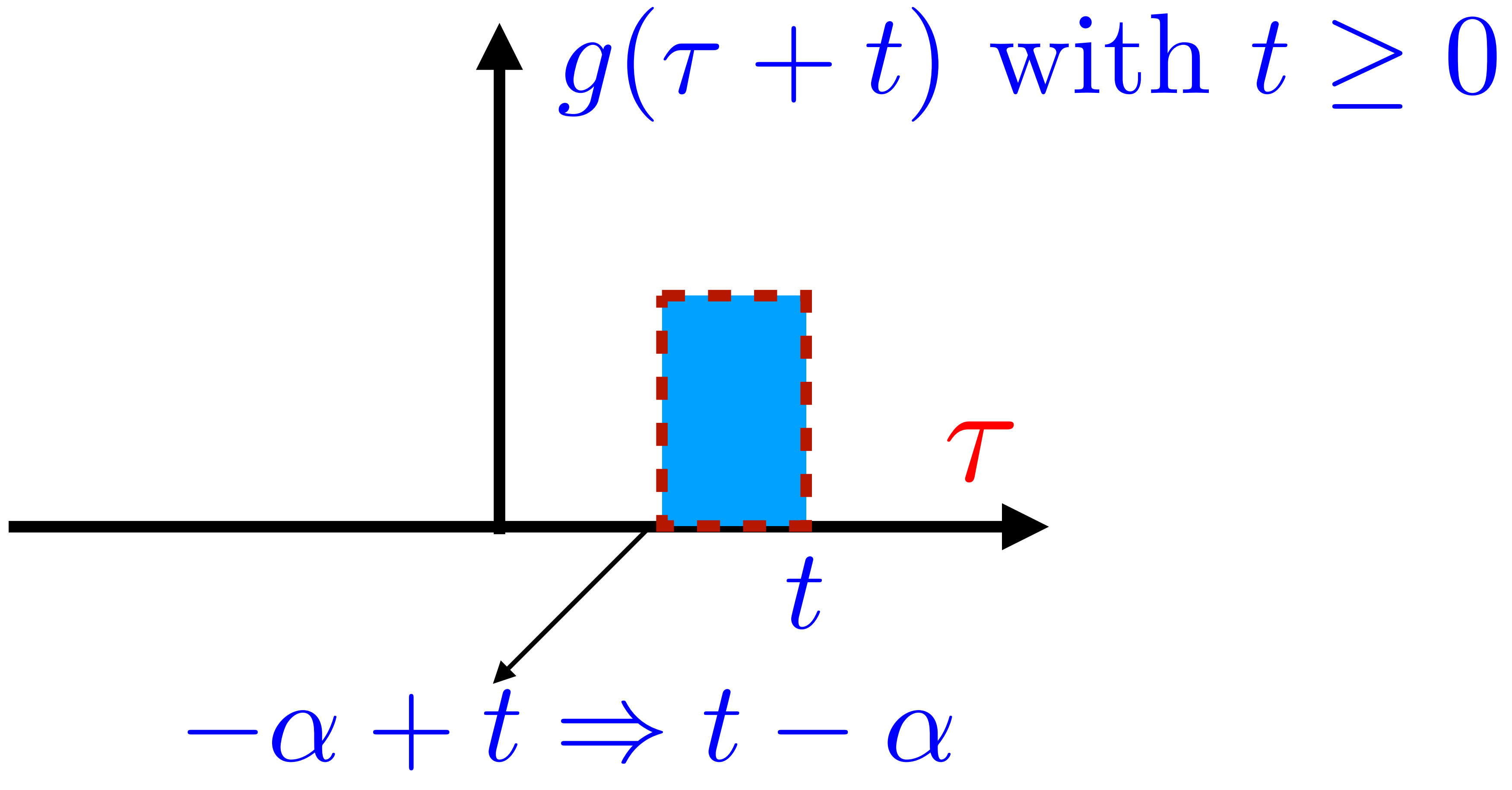


Example 6



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

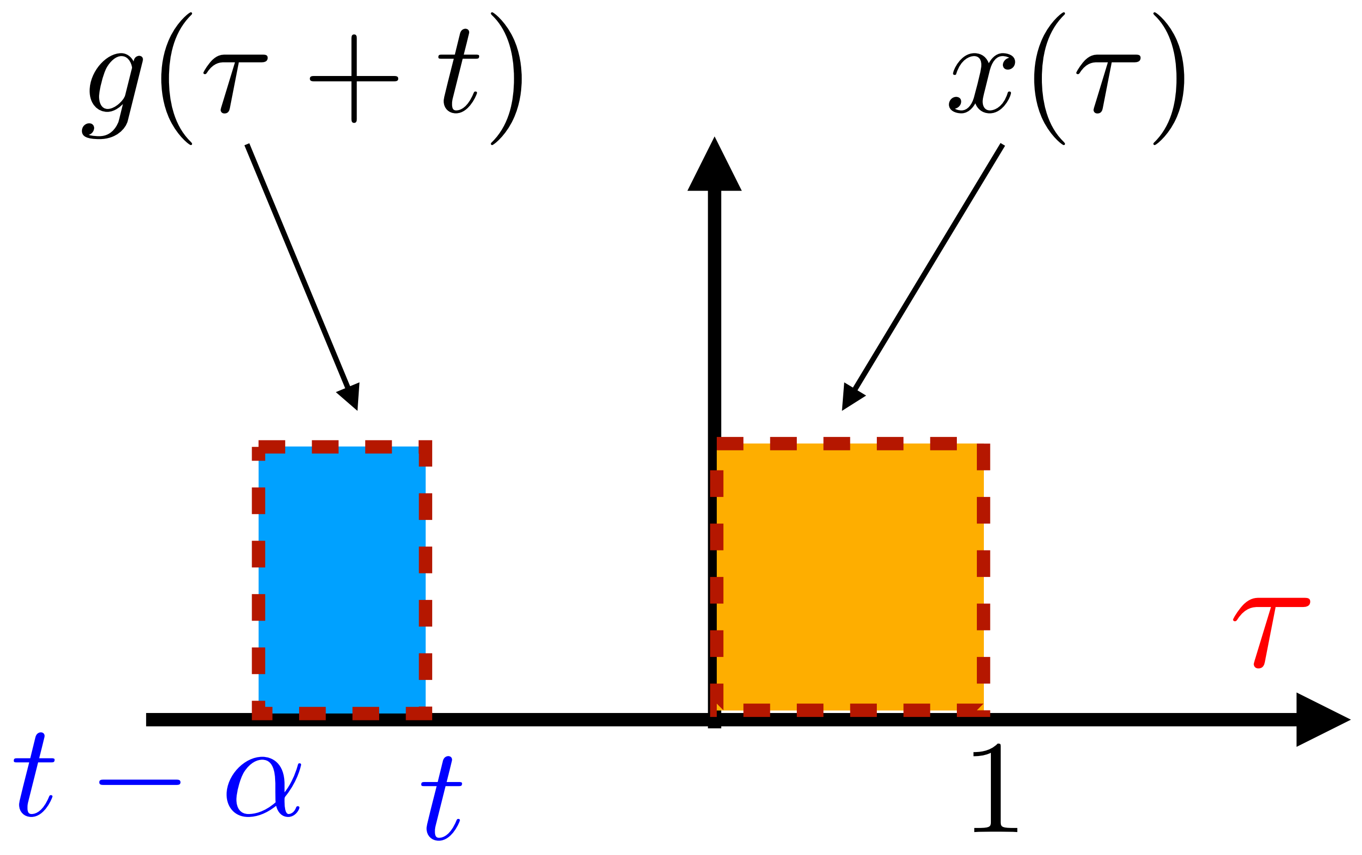




Example 6

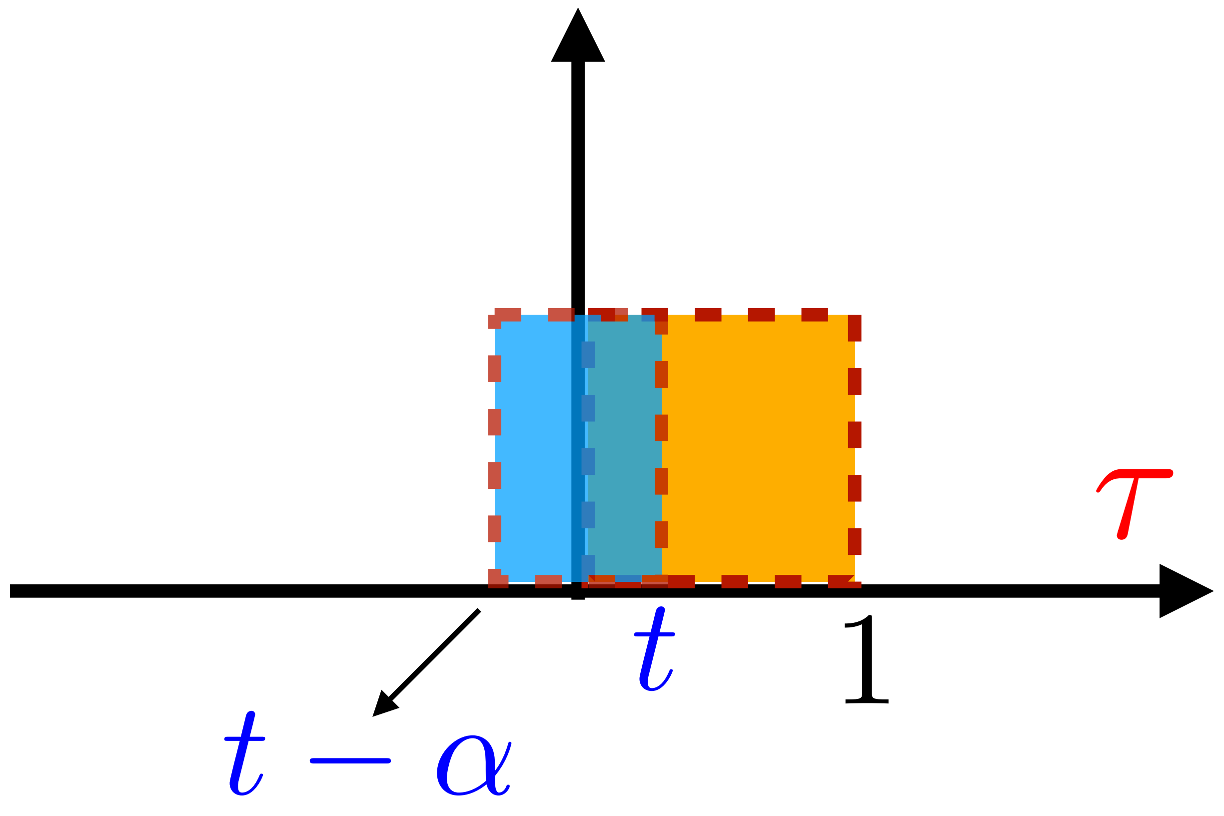
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- **Five - 5 -** Scenarios:

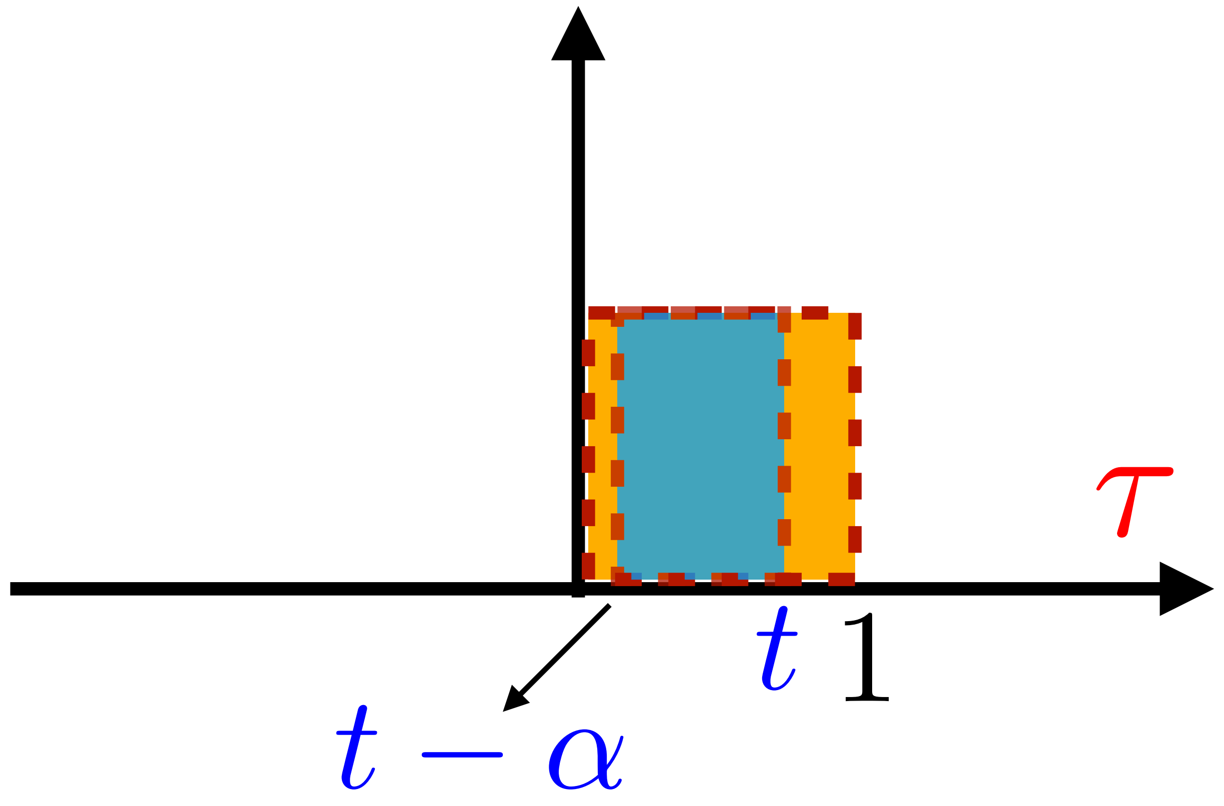


- Completely Out - left

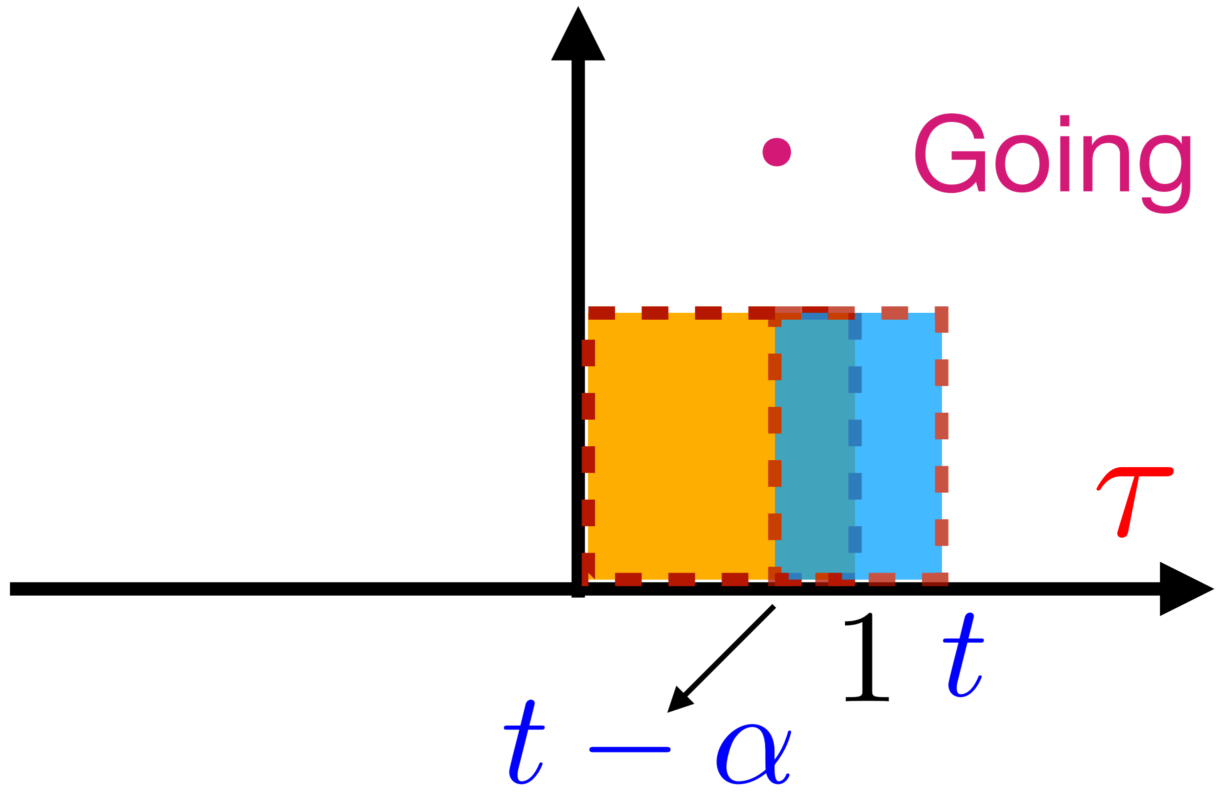
- Going “inside"



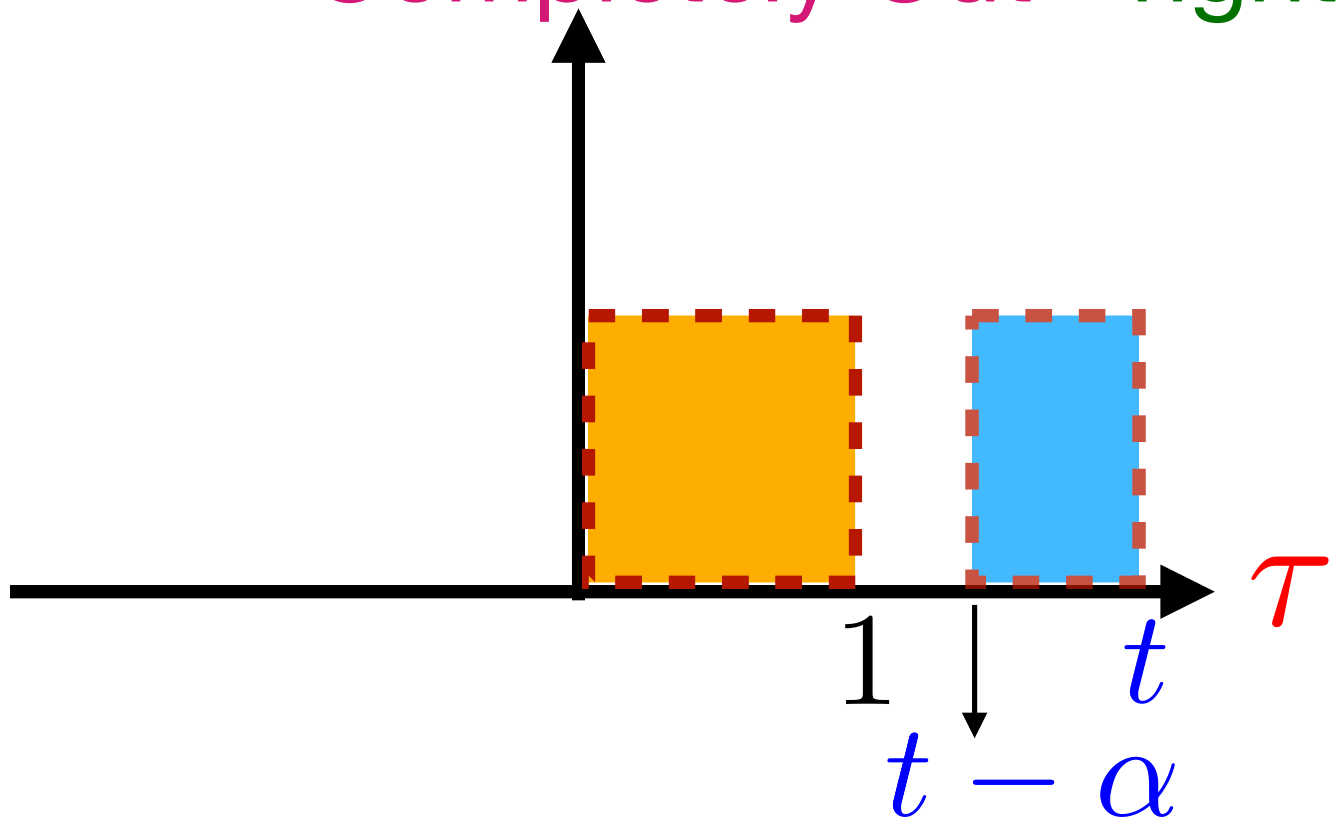
- completely “inside”



- Going “out”



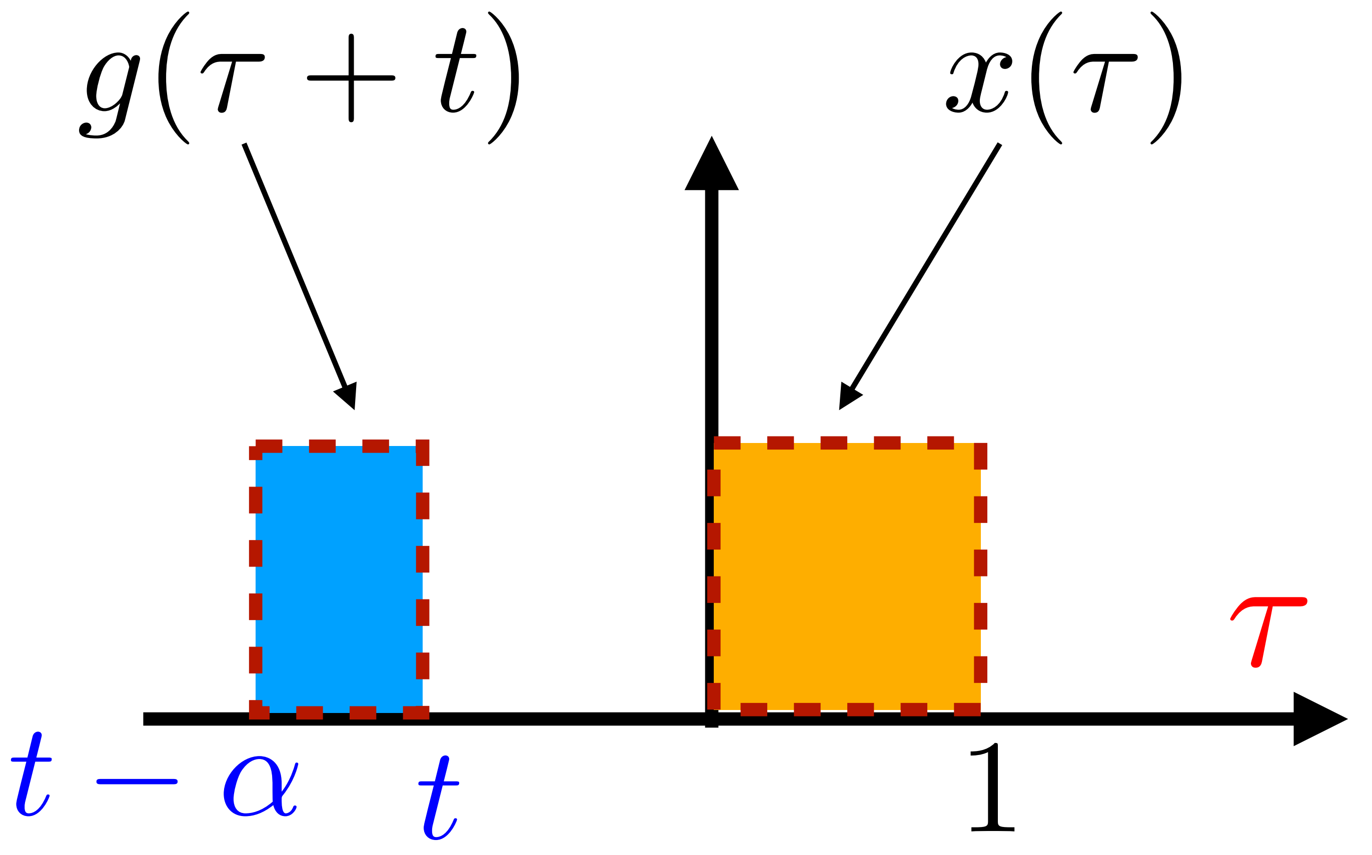
- Completely Out - right



Example 6

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- Scenario 1:



- Completely Out - left

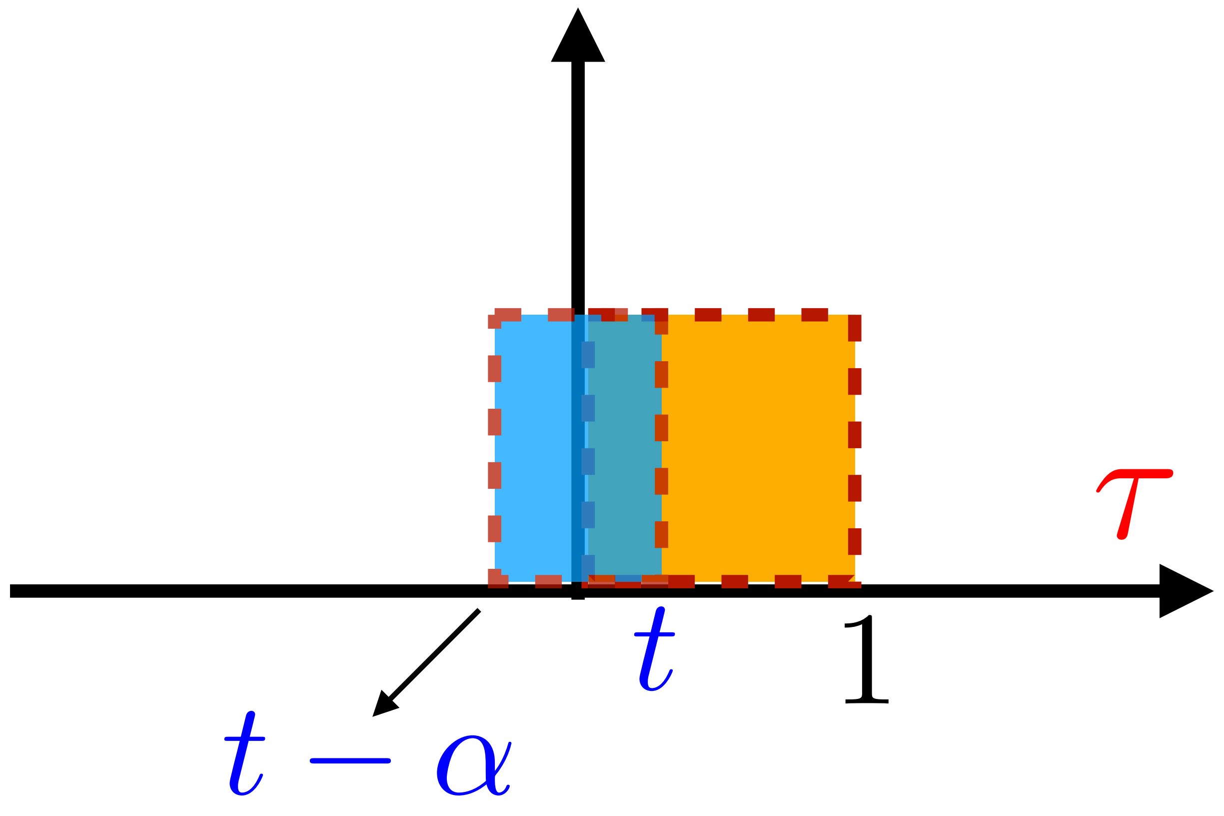
$$y(t) = 0 \qquad \text{for } t \leq 0$$

Example 6

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- Going "inside"

- Scenario 2:

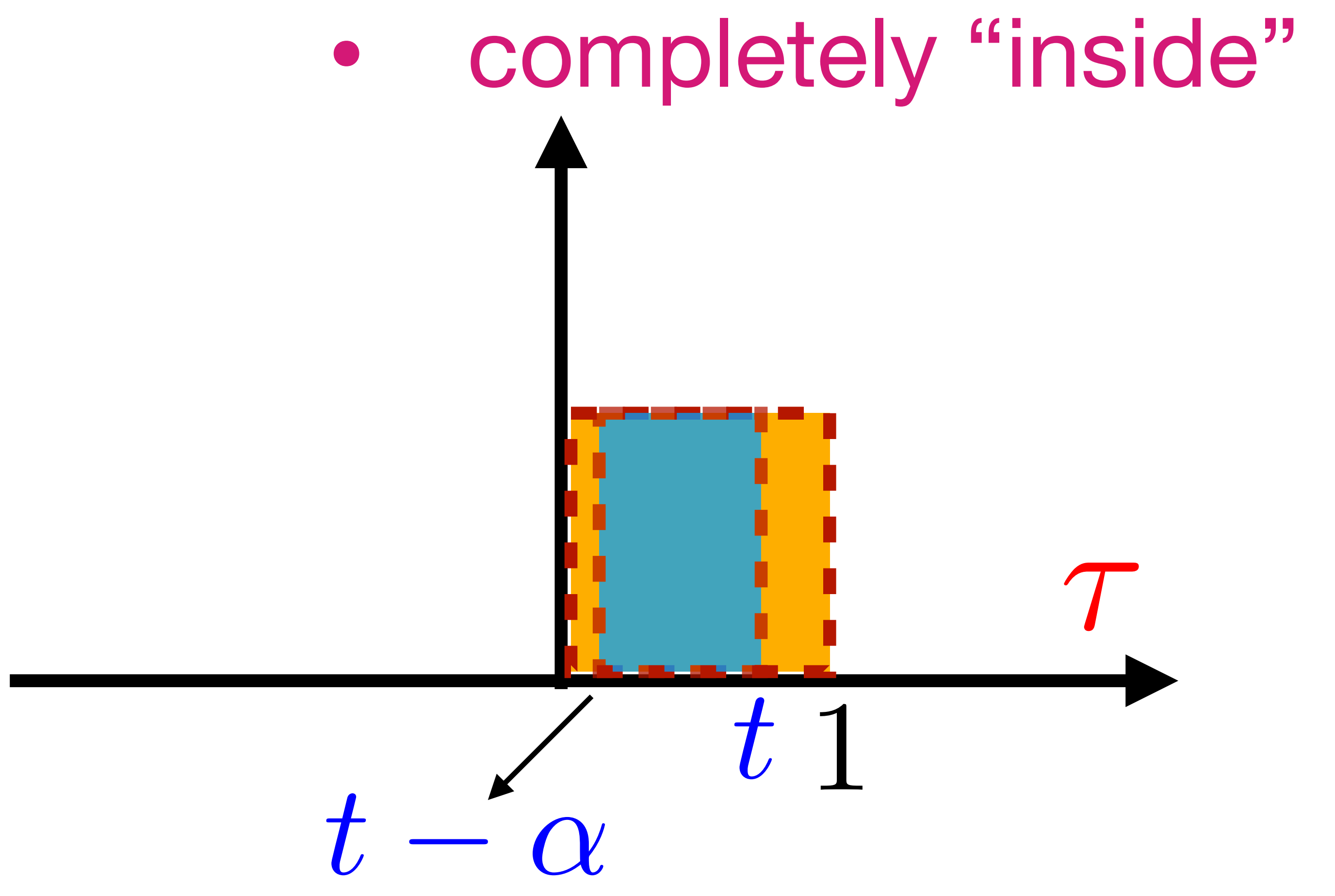


$$y(t) = t \cdot 1 = t \qquad \text{for } 0 \leq t \leq \alpha$$

Example 6

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- Scenario 3:

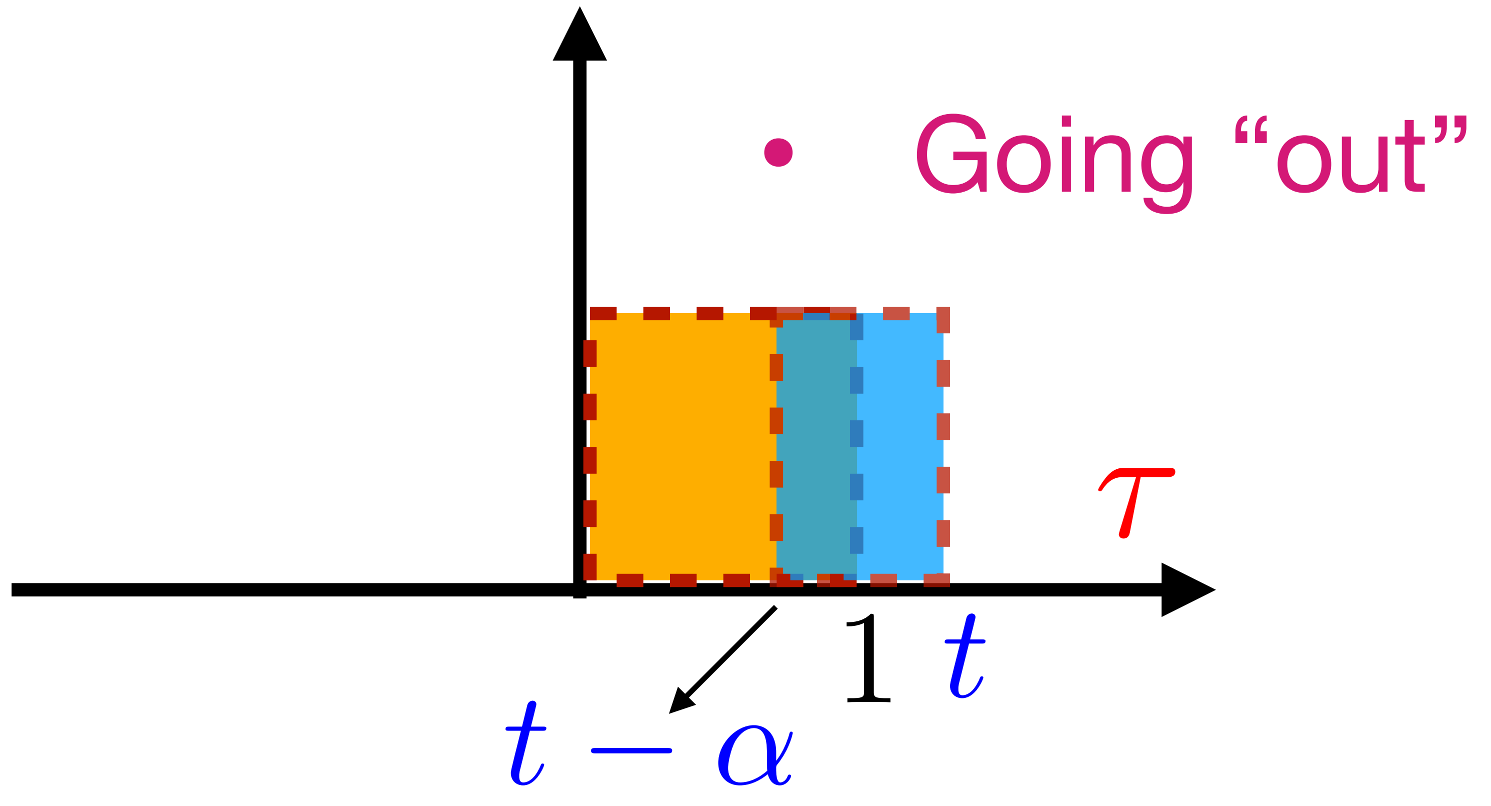


$$y(t) = (t - t + \alpha) \cdot 1 = \alpha \qquad \text{for } \alpha \leq t \leq 1$$

Example 6

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- Scenario 4:



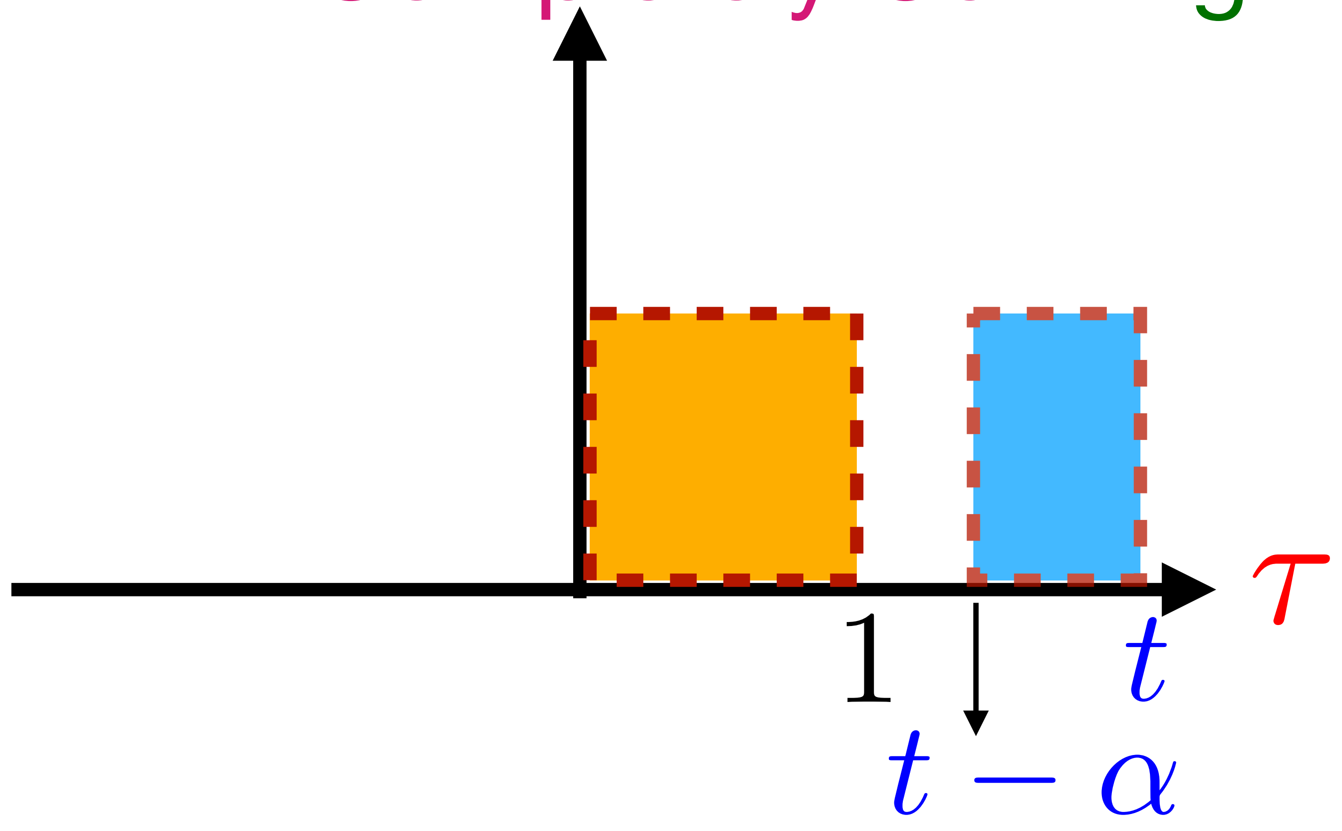
$$y(t) = (1 - t + \alpha) \cdot 1 = 1 + \alpha - t \qquad \text{for } 1 \leq t \leq 1 + \alpha$$

Example 6

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) g(\tau + t) d\tau$$

- Scenario 5:

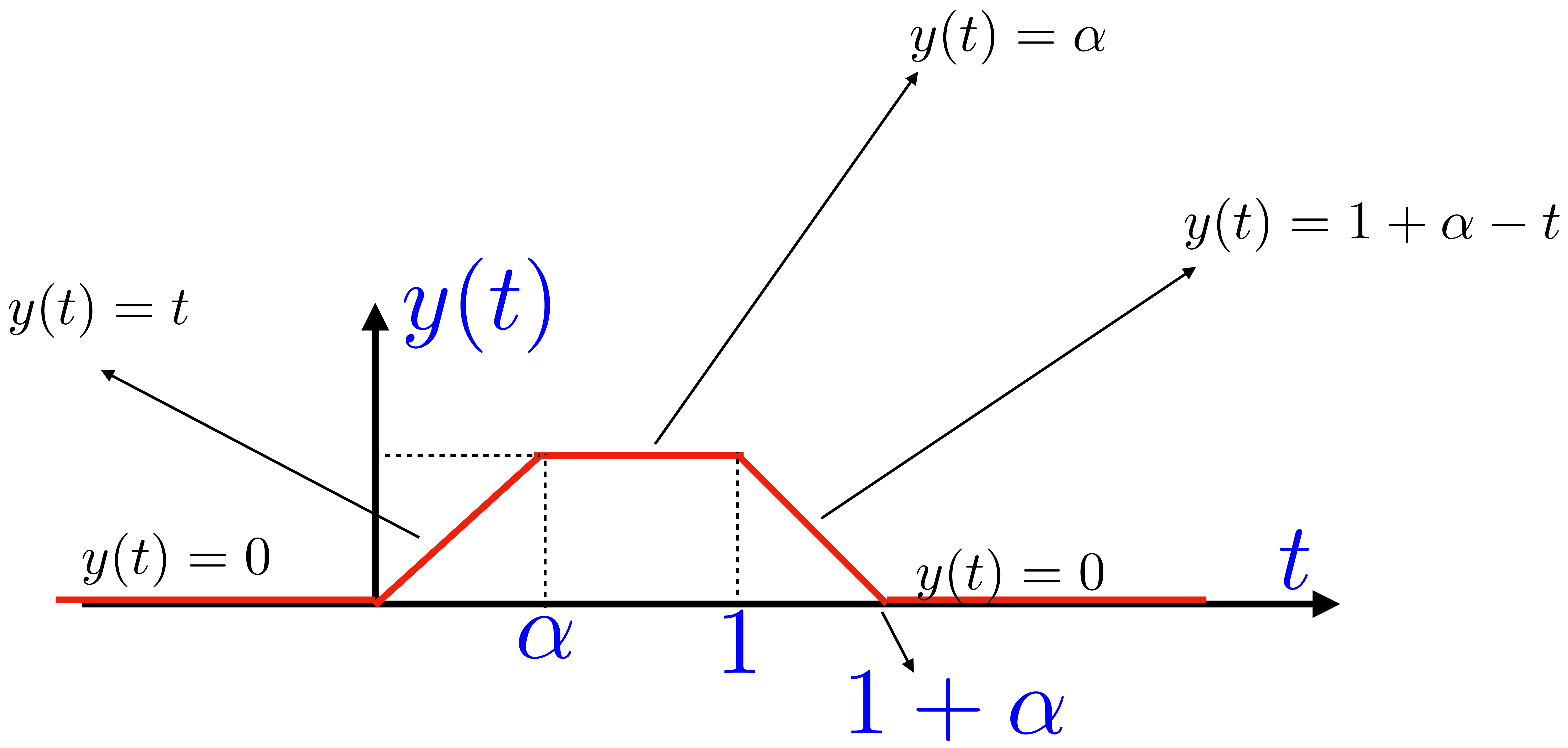
- Completely Out - right



$$y(t) = 0 \qquad \text{for } t \geq 1 + \alpha$$

Example 6

- **Final Solution:**



Example 6

- A similar solved problem, with a nice animation can be found at:

https://www.youtube.com/watch?v=_HATc2zAhcY

- Nice animations can be found also in wikipedia's page:

https://en.wikipedia.org/wiki/Convolution

Other examples

- https://www.youtube.com/watch?v=e0Sy_l5boh0
- https://www.youtube.com/watch?v=35gc3GE4Ddo

Questions?