Topic 3 - Part 4: Standard Fourier Transform (en sentido ordinario)

Linear systems and circuit applications

Señales y Sistemas

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

Transformations for signal in continuous time

For Periodic signals

For non-periodic signals

Fourier Series (FS)

Stand. Fourier Transform (FT)

Laplace Transform (FT)

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Next?

- non-periodic signals in frequency domain
- some non-periodic signals…
- All the non-periodic signals, *with finite energy*, have *Fourier Transform*…

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- **Analysis equations:** from x(t) to the transformed domain

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\omega \in \mathbb{R}$$

- **Special case of the Laplace Transform when:**

$$\sigma = 0, \qquad s = 0 + j\omega$$

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\omega \in \mathbb{R}$$

- **But it takes complex values, i.e.,**

$$X(\omega) \in \mathbb{C}$$

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)dt$$

$$\omega \in \mathbb{R}$$

- **But it takes complex values, i.e.,**

$$X(\omega) \in \mathbb{C}$$

Stand. Fourier Transform

- But it takes complex values, i.e.,

$$X(\omega) \in \mathbb{C}$$

- then, in general, we can plot module/phase, real part and imaginary part…

Synthesis equation: Inverse Fourier Transform

- Definition (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

- **Synthesis equations:** from the transformed domain to x(t)

Brief summary

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

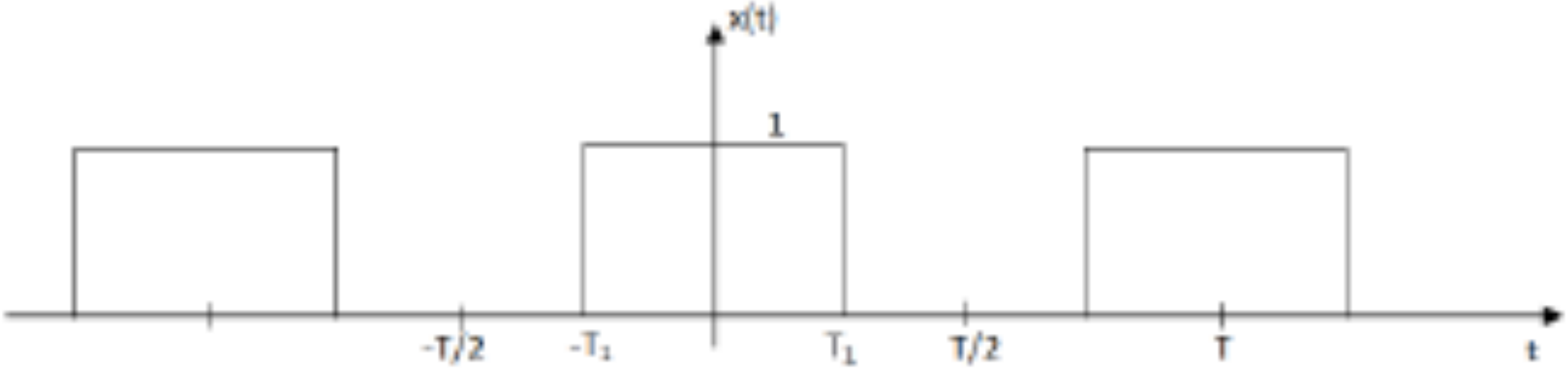
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- Inverse Fourier Transform
  Ec. de síntesis

- Stand. Fourier Transform
  Ec. de análisis,

Important example of Fourier Series (FS)

$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } T_1 < |t| < T/2 \end{cases}$$

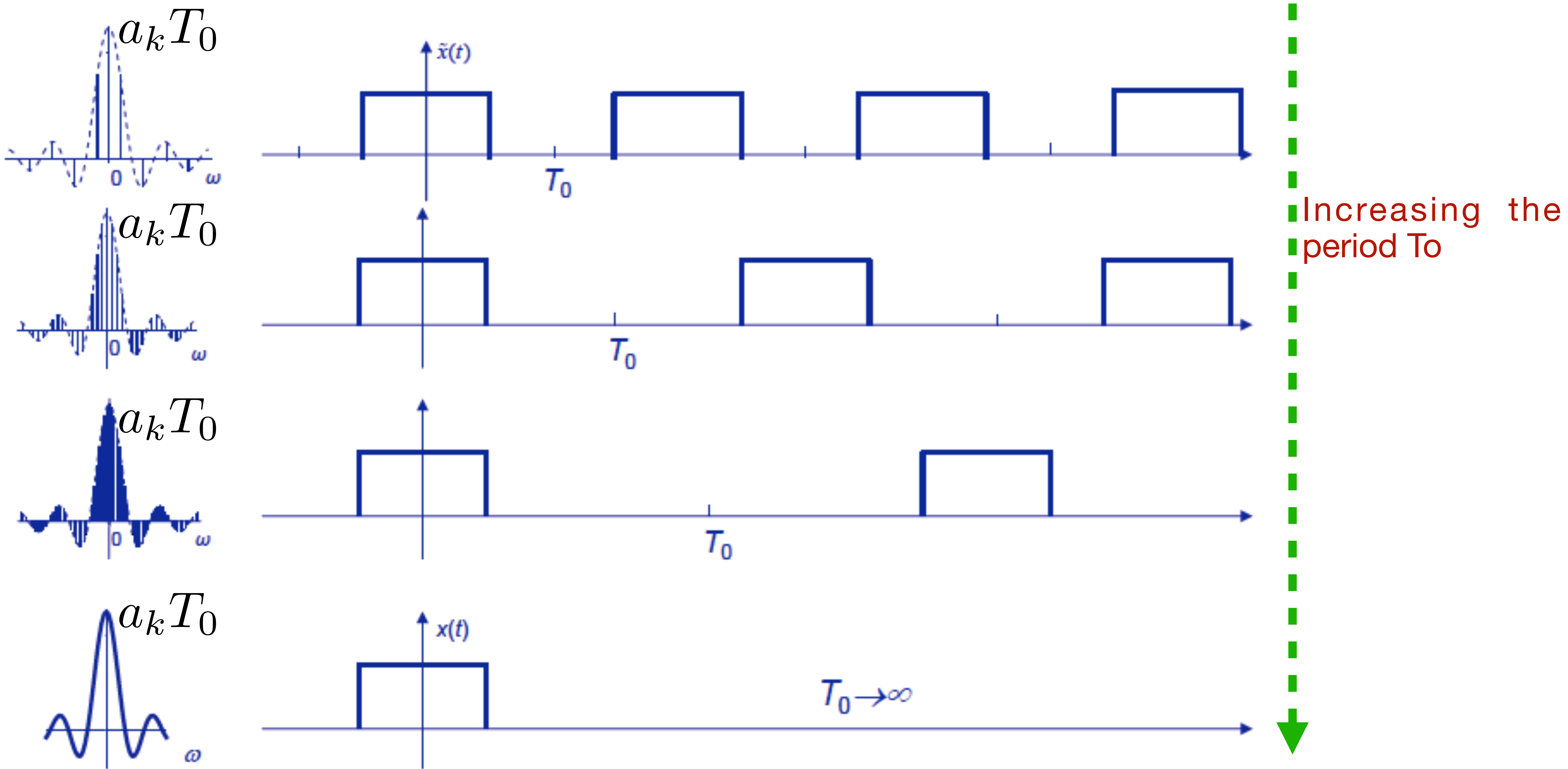


Important example: coef. of the FS

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_0 = \frac{2T_1}{T_0}$$

We saw…

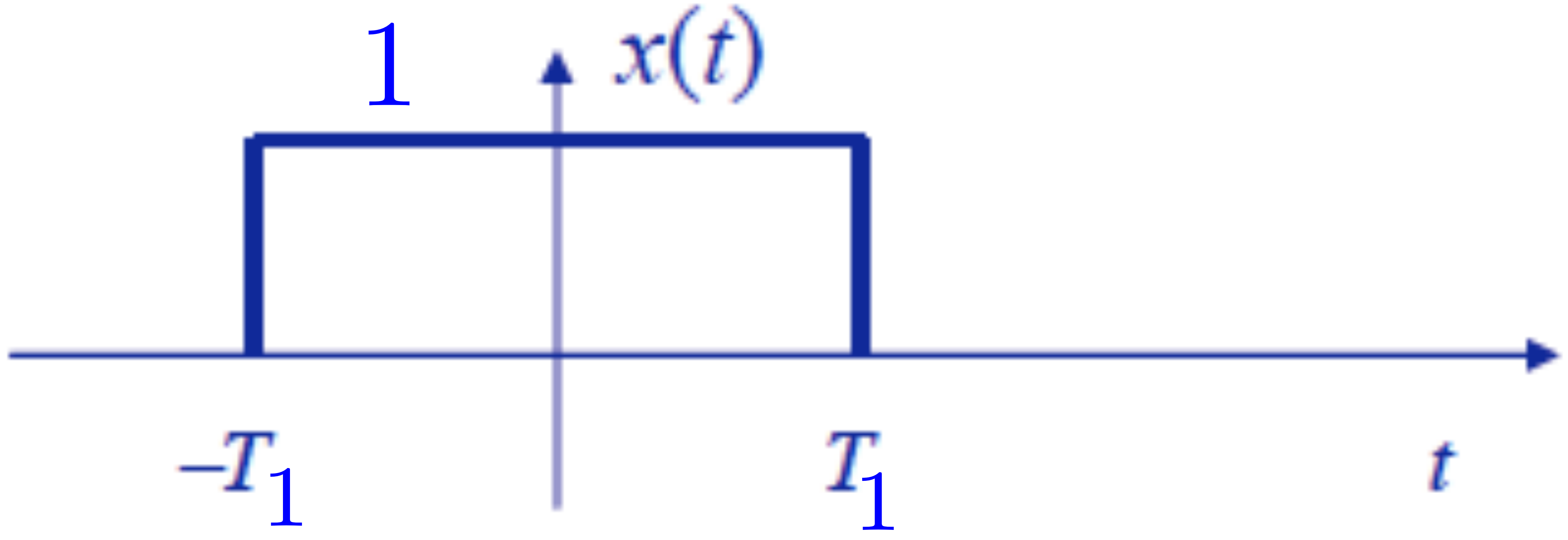


$a_k T_0$
0
ω
$\tilde{x}(t)$
T_0
$a_k T_0$
0
ω
T_0
$a_k T_0$
0
ω
T_0
$a_k T_0$
ω
x(t)
$T_0 \rightarrow \infty$
Increasing the period To

We saw…

- what happens when To diverges? i.e., To goes to infinity?
- the signal becomes non-periodic….
- **we will see it again, and we will see what is the dashed line…** that becomes solid line when To diverges**.**

Example: Fourier Transform (FT) of a rectangle



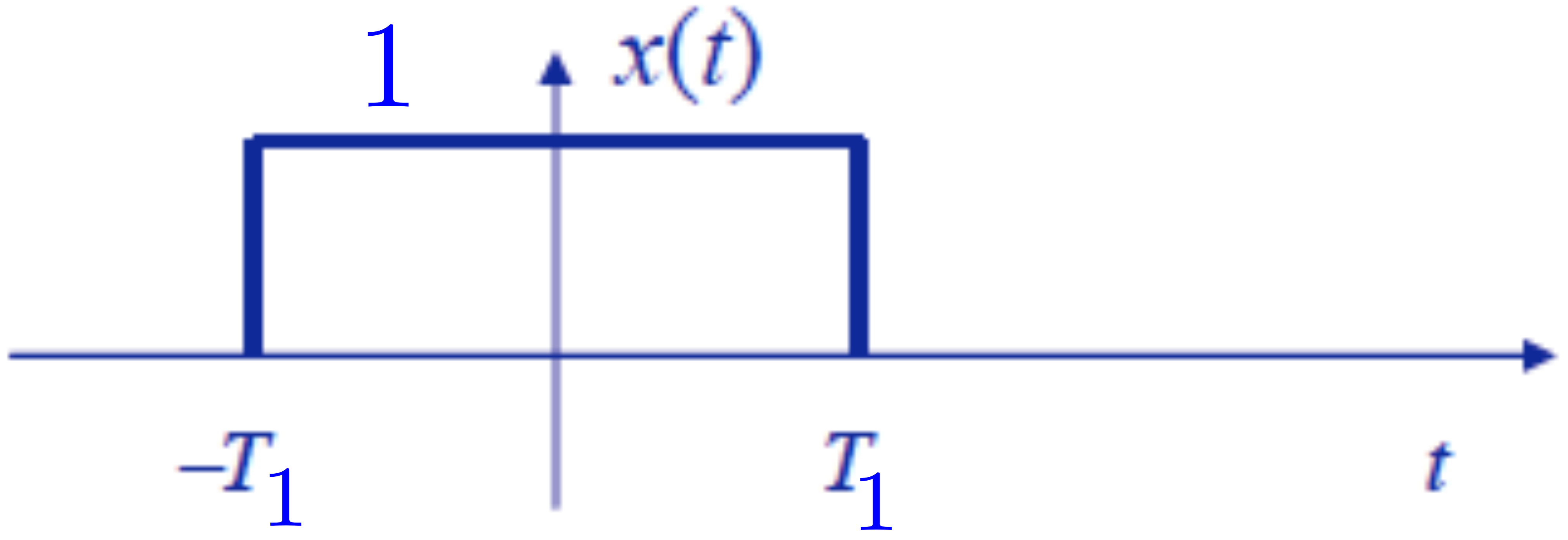
$$x(t) = \begin{cases} 1, & si\ |t| < T_1 \\ 0, & si\ |t| > T_1 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{+T_1} e^{-j\omega t} dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_{-T_1}^{+T_1}$$

$$= \frac{-1}{j\omega} \left[e^{-j\omega T_1} - e^{j\omega T_1} \right]$$

Example: Fourier Transform (FT) of a rectangle



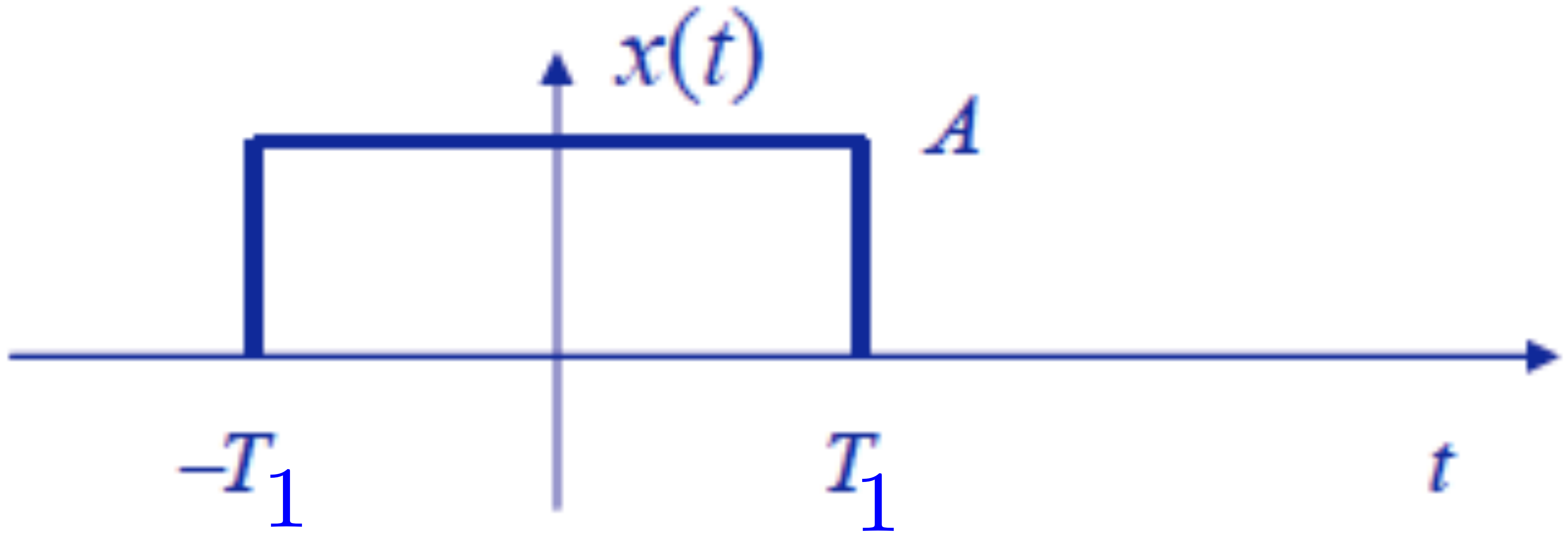
$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } |t| > T_1 \end{cases}$$

$$X(\omega) = \frac{-1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$= \frac{-1}{j\omega}\left[-2j\sin(\omega T_1)\right]$$

$$X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

Example: more general rectangle



$$x(t) = \begin{cases} A, & \text{si } |t| < T_1 \\ 0, & \text{si } |t| > T_1 \end{cases}$$

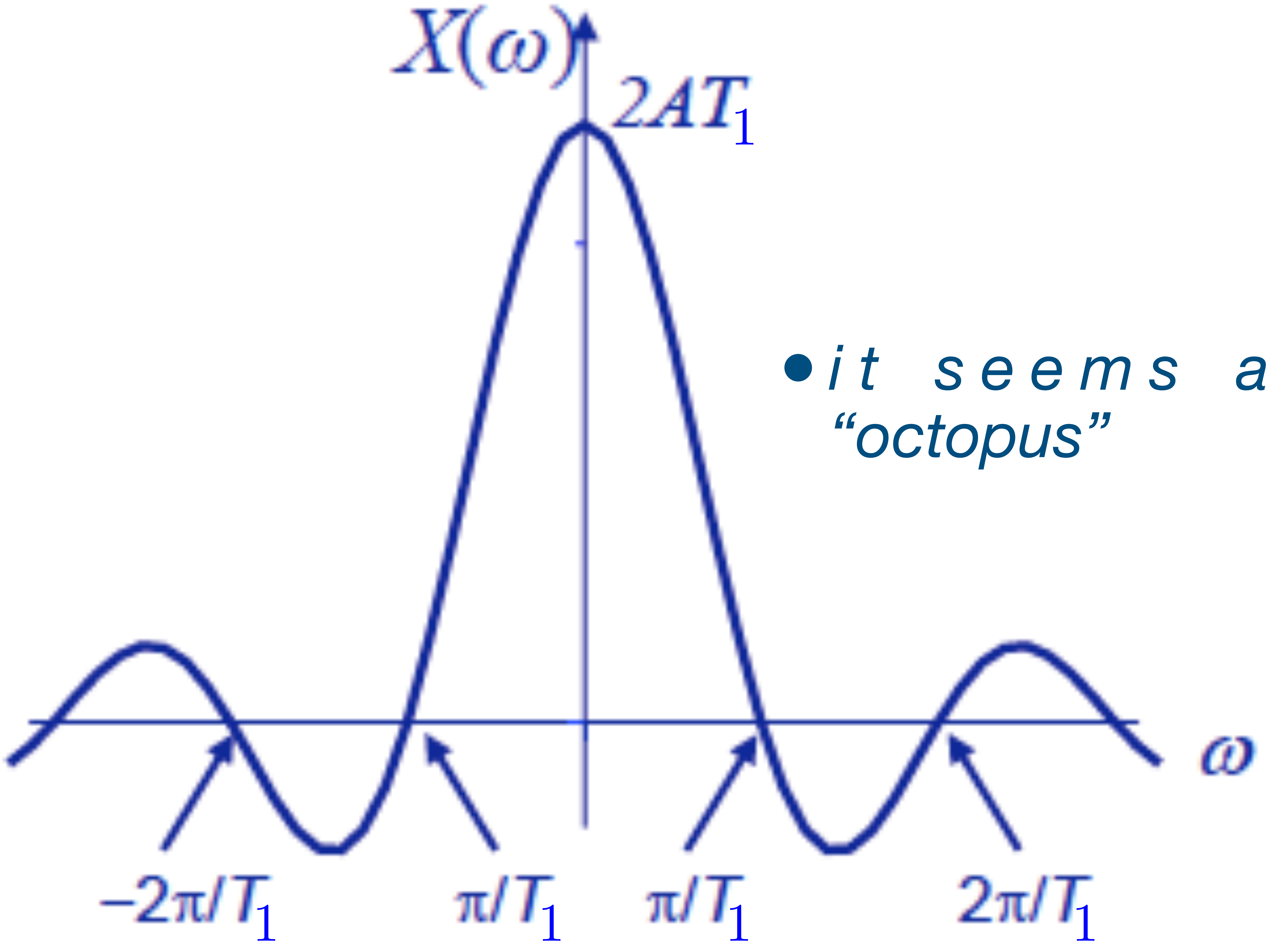
$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

- Let plot this function… *that in this case is real (since the rectangle is real and even)*

SINC FUNCTION: “the octopus”

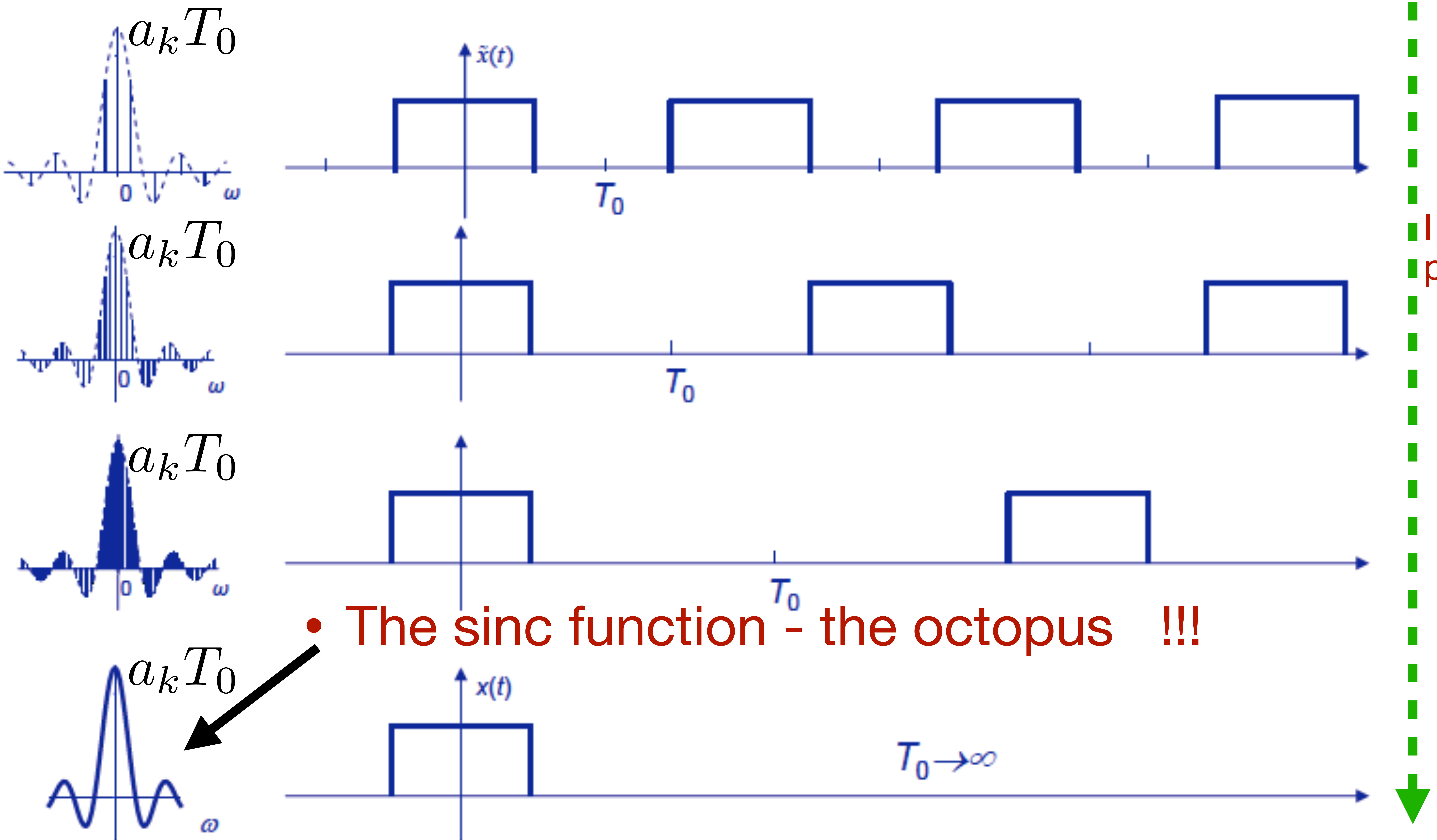
- Sinc function:

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$



- it seems an “octopus”

Recalling again this figure….

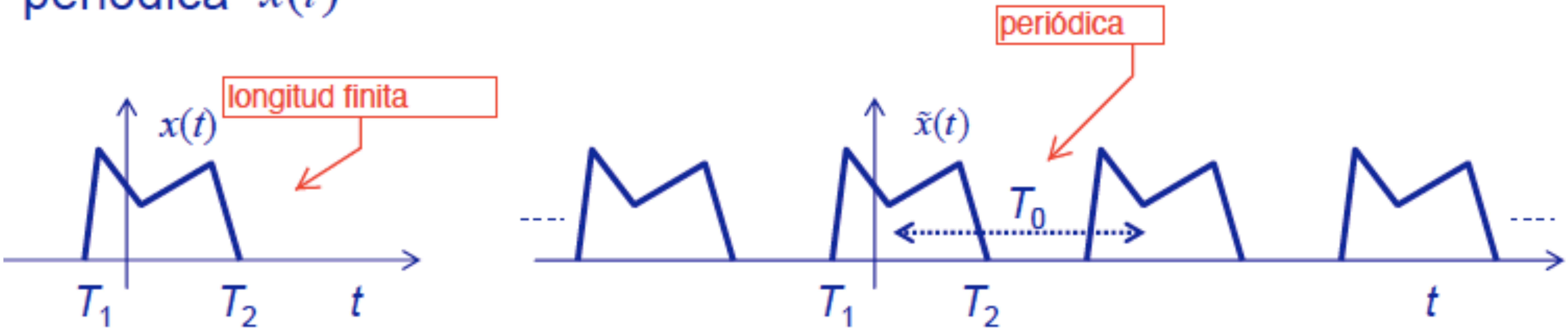


- The sinc function - the octopus !!!

Increasing the period To

Signal with finite length and its periodic “brother”

Dada una señal de duración finita $x(t)$, realizamos una extensión periódica $\tilde{x}(t)$



$$x(t)$$

Signal with finite length

$$\tilde{x}(t)$$

Its periodic “brother”

Signal with finite length and its periodic “brother”

We can compute FT for

$$x(t)$$

↓

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

We can compute FS for

$$\widetilde{x}(t)$$

↓

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k \, \mathrm{e}^{jk\omega_0 t} \quad \text{con}$$

$$a_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) \, \mathrm{e}^{-jk\omega_0 t} \, dt \quad \text{y} \quad \omega_0 = \frac{2\pi}{T_0}$$

Important result - relationship FT - FS

- En el intervalo $T_1 \leq t \leq T_2$, se cumple $\tilde{x}(t) = x(t)$ de modo que

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

Como

$$\left.\begin{aligned} a_k &= \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt \\ X(\omega) &\equiv \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow \end{aligned}\right\} \Rightarrow \quad \boxed{a_k = \frac{1}{T_0} X(\omega)\Big|_{\omega = k\omega_0}}$$

- Los coeficientes a_k de la extensión periódica son **muestras equiespaciadas de la función $X(\omega)$**

Important result - relationship FT - FS

- Valid only if x(t) has finite length:

$$a_k = \frac{1}{T_0}[X(\omega)]_{\omega=k\omega_0}$$

$$a_k = \frac{1}{T_0}X(k\omega_0)$$

Check in the rectangular example

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

$$X(k\omega_0) = \frac{2A\sin(k\omega_0 T_1)}{k\omega_0}$$

$$\frac{1}{T_0}X(k\omega_0) = \frac{1}{T_0}\frac{2A\sin(k\omega_0 T_1)}{k\omega_0}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\frac{1}{T_0}X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

Check in the rectangular example

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

$$\frac{1}{T_0} X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

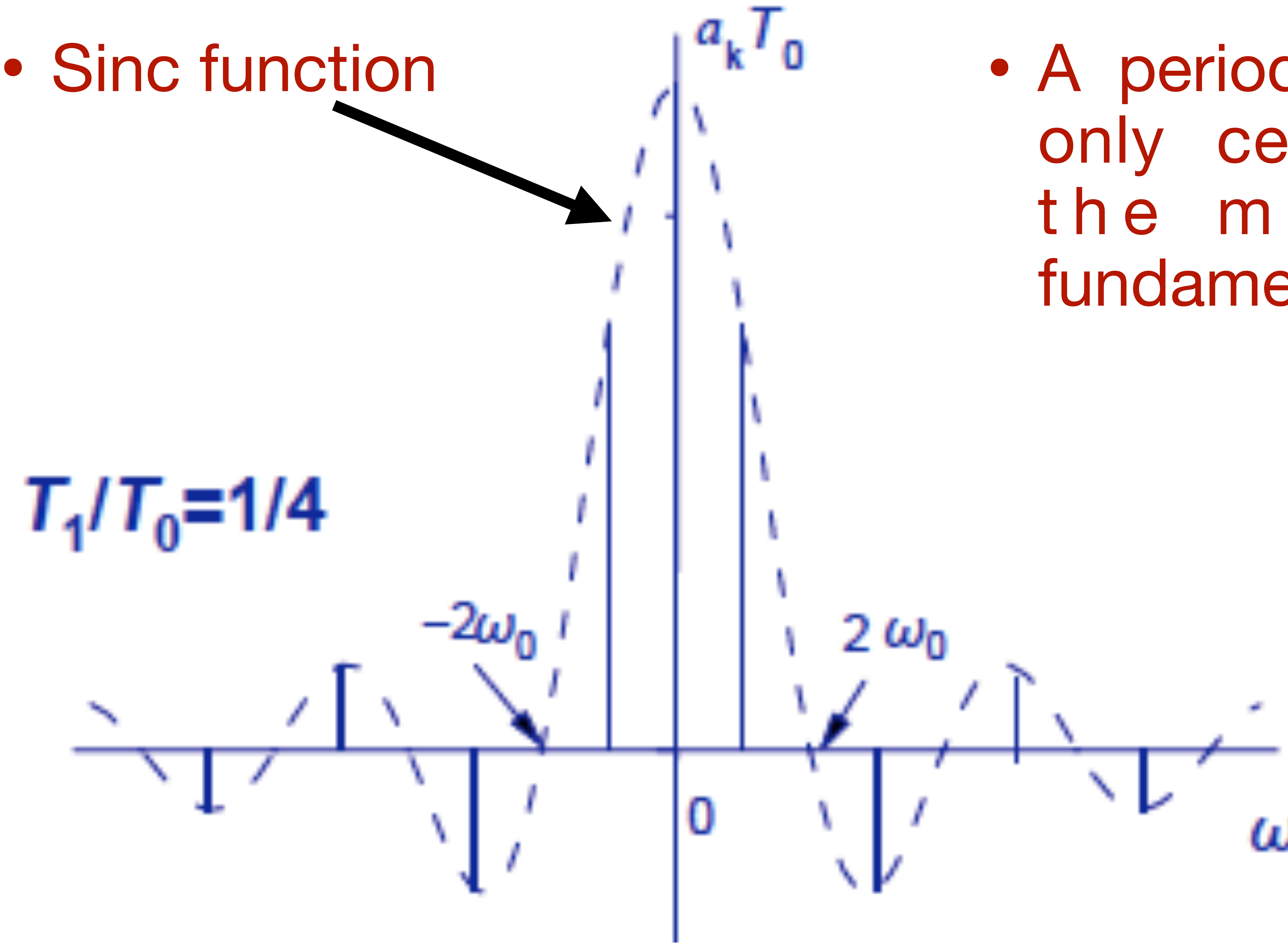
$$\omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{1}{T_0} X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

Exactly !!! (we obtain it with A=1, but it is very easy to re-do for a generic A)

For obtaining FS we are sampling of the FT

- Sinc function



- A periodic signal contains only certain frequencies, the multiple of the fundamental frenquency.

Example of FT

Example: Calculation of the Fourier Transform of a positive exponential function

- Calculate the Fourier Transform of $x(t) = e^{-at}u(t)$, being $a > 0$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \cdots = \frac{1}{a + j\omega}$$

- Higher values are localized at low frequencies.

Example of FT

Example: Calculation of the Fourier Transform of the unit impulse

- Calculate the Fourier Transform of $x(t) = \delta(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

- The unit impulse has a Fourier Transform consisting of equal contributions at all frequencies.

Homework

Calculate the Fourier Transform of $x(t) = e^{-a|t|}$.

Properties

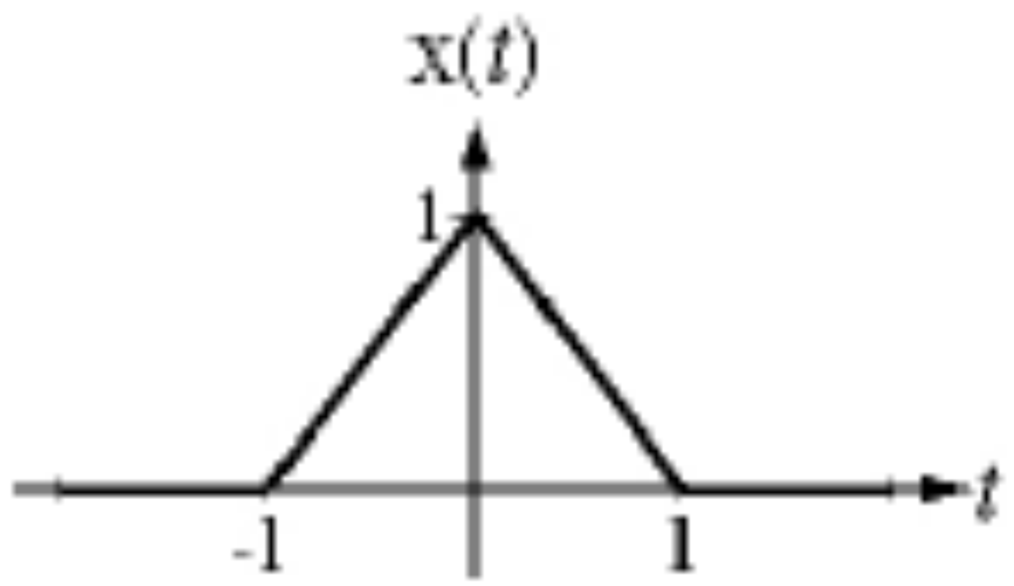
Señal	Transformada
$x(t)$	$X(\omega)$
$ax(t)+by(t)$	$aX(\omega)+bY(\omega)$
$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
$x^*(t)$	$X^*(-\omega)$
$x(-t)$	$X(-\omega)$
$x(at)$	$\frac{1}{\lvert a \rvert}X(\omega/a)$
$x(t)*y(t)$	$X(\omega)Y(\omega)$
$x(t)y(t)$	$X(\omega)*Y(\omega)\frac{1}{2\pi}$
$dx(t)/dt$	$j\omega X(\omega)$

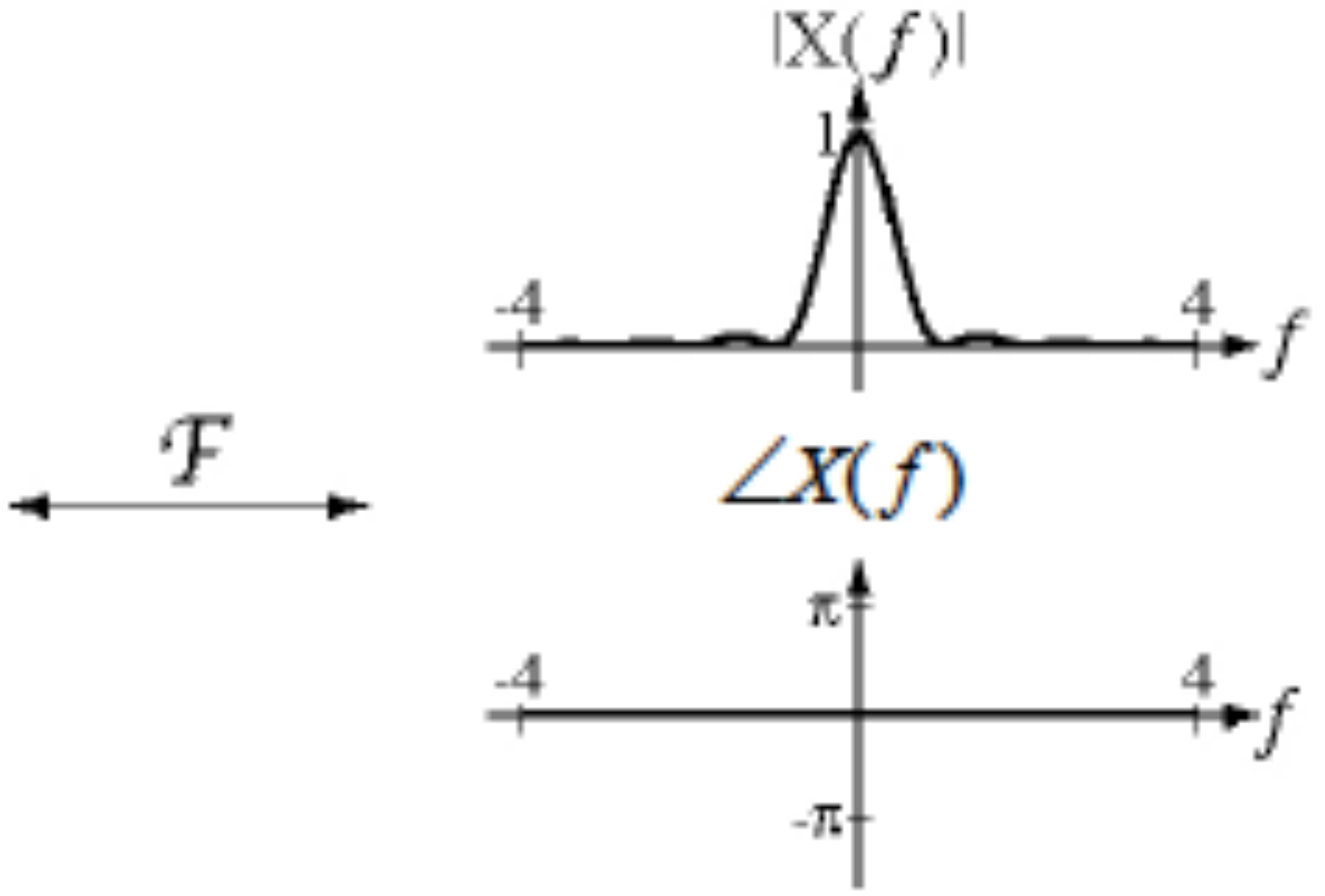
Properties

Señal	Transformada
$\int_{-\infty}^{t} x(\tau)d\tau$	$\dfrac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
$tx(t)$	$j\, dX(\omega)/d\omega$
$x(t)$ *real*	$\begin{cases} X(\omega) = X^*(-\omega) \\ \mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\} \\ \mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\} \\ \lvert X(\omega)\rvert = \lvert X(-\omega)\rvert \\ \angle X(\omega) = -\angle X(-\omega) \end{cases}$
Relación de Parseval para señales no periódicas	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \dfrac{1}{2\pi}\int_{-\infty}^{-\infty} \lvert X(\omega)\rvert^2 d\omega$

Dualidad : $\begin{cases} g(t) \xleftrightarrow{TF} f(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \\ f(t) \xleftrightarrow{TF} 2\pi g(-\omega) \end{cases}$

Shifting in time

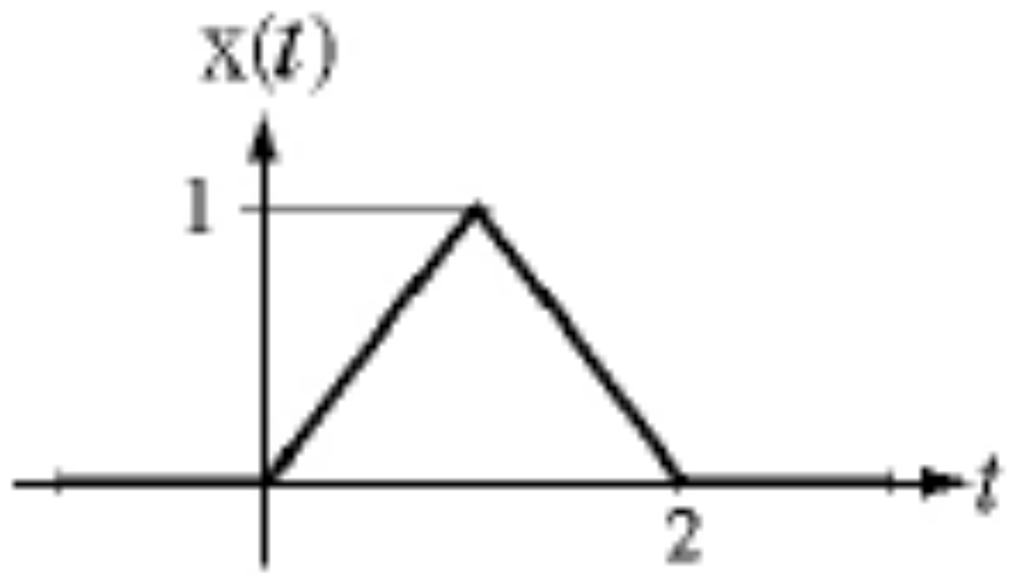


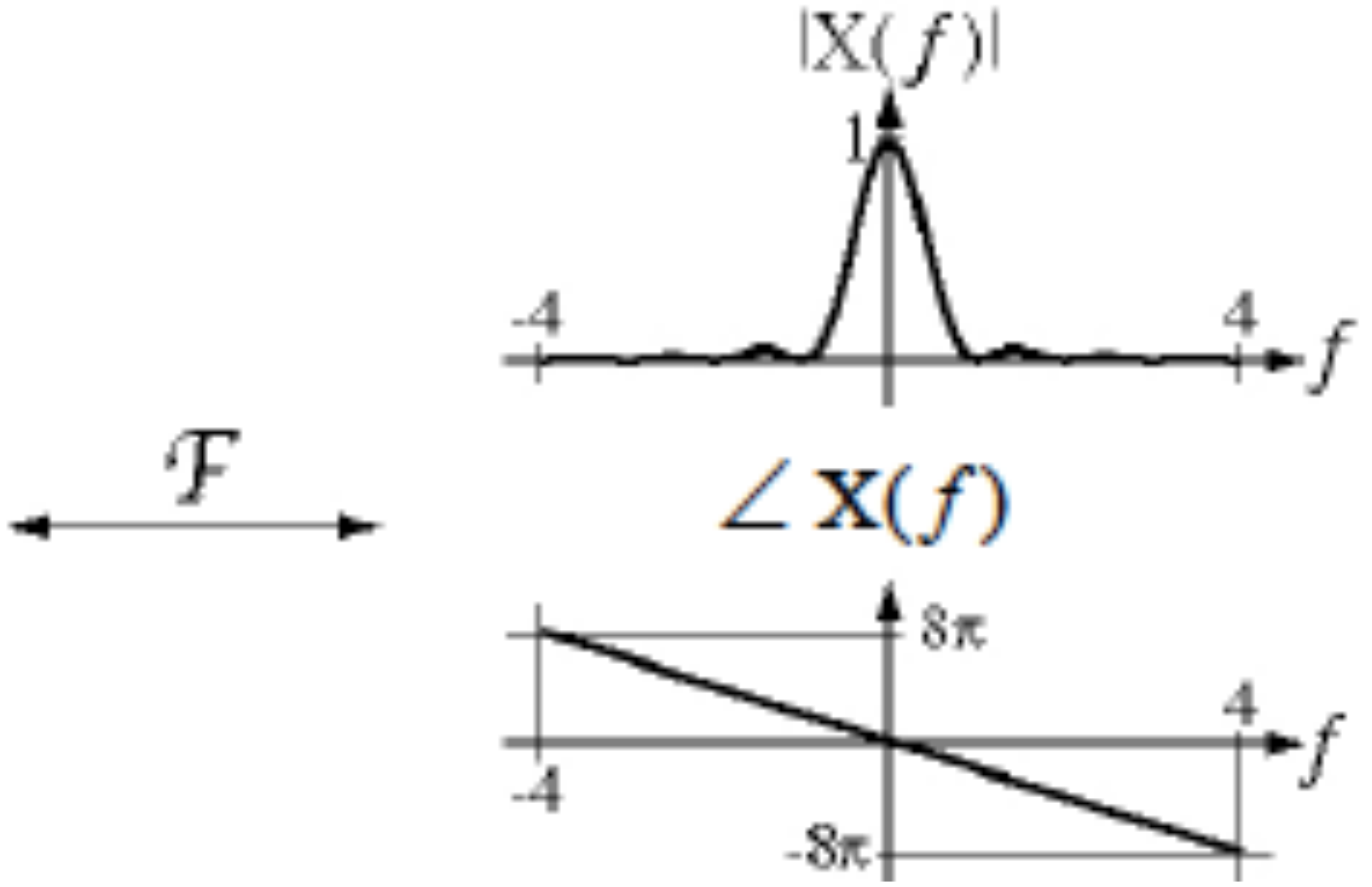


Desplazamiento en el tiempo

$$x(t-t_0) \overset{TF}{\longleftrightarrow} X(f)\,e^{-j2\pi f t_0}$$

$$x(t-t_0) \overset{TF}{\longleftrightarrow} X(\omega)\,e^{-j\omega t_0}$$



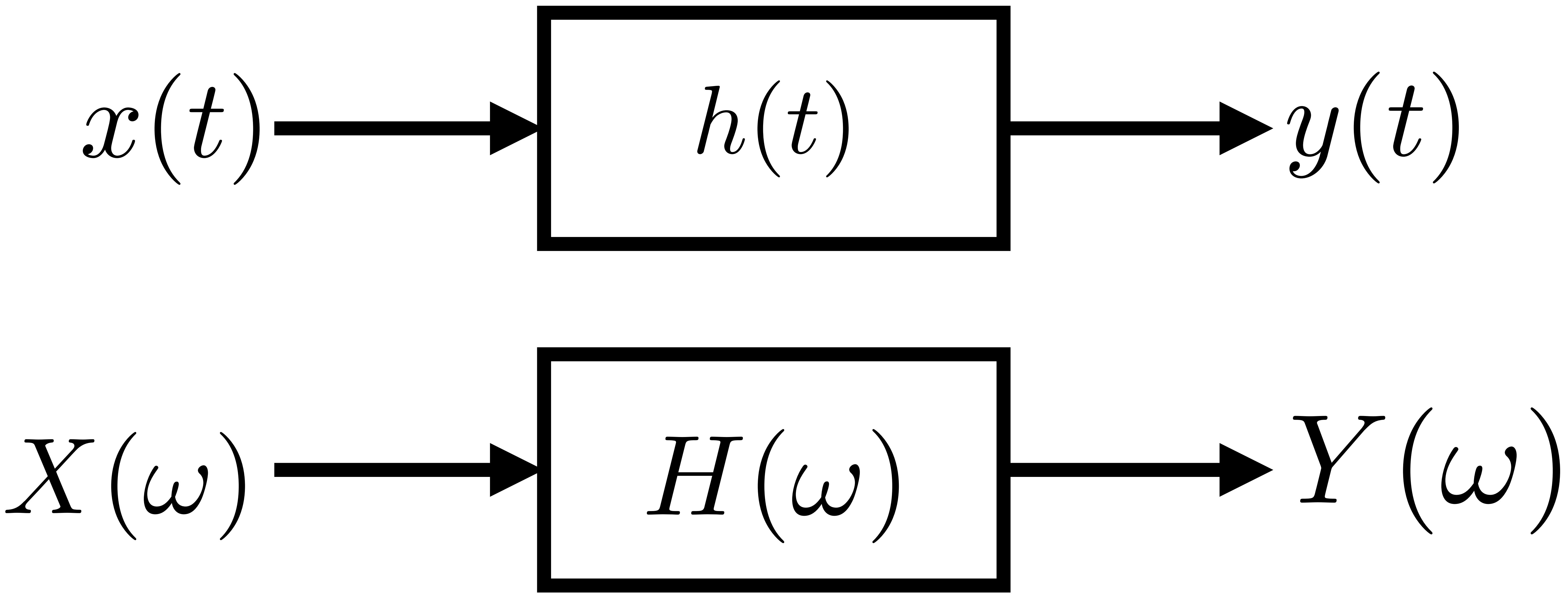


Shifting in time and frequency

$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$

Convolution in time = multiplication in frequency

$$y(t) = x(t) * h(t) \implies Y(\omega) = X(\omega)H(\omega)$$



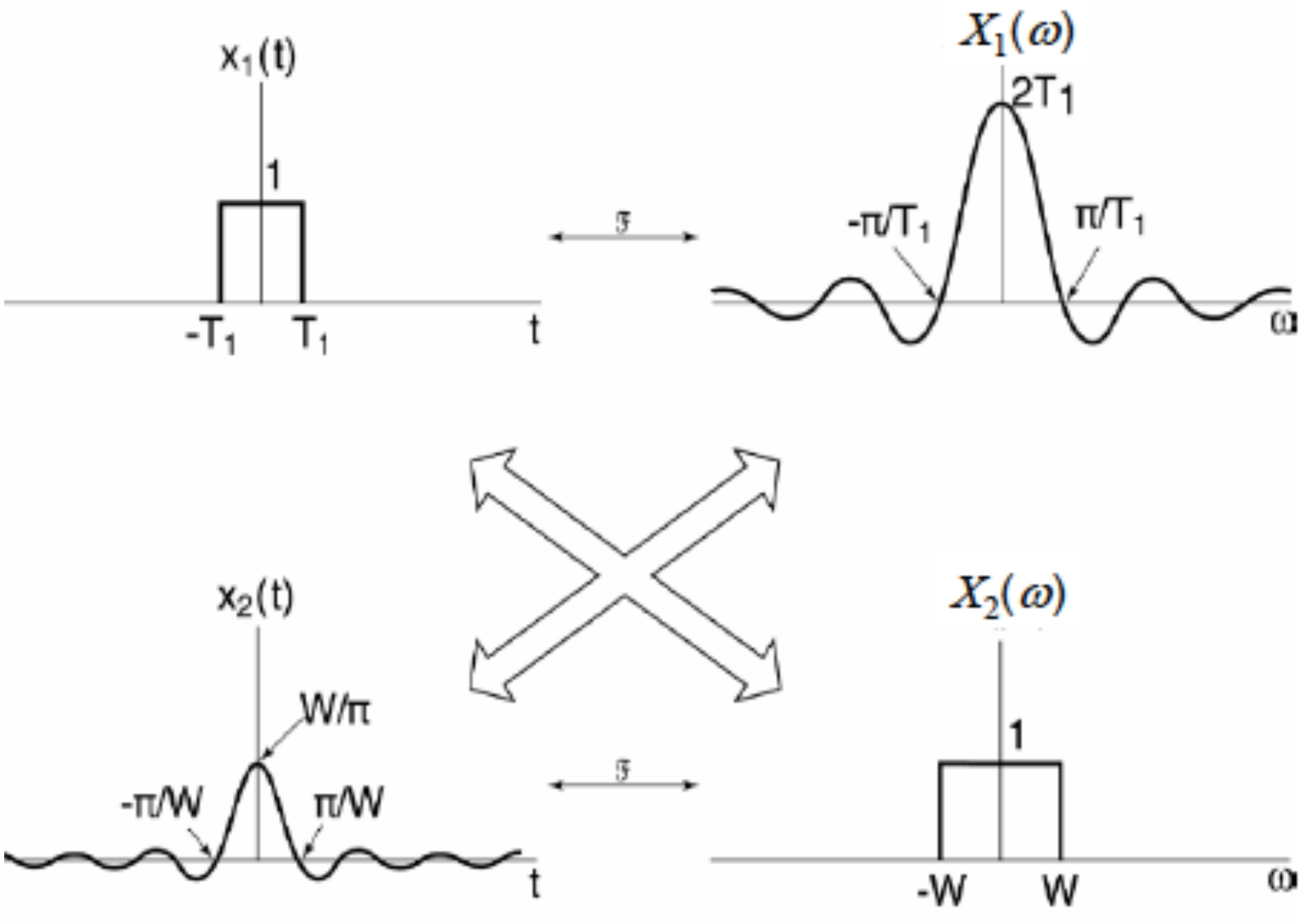
Derivative in time…

$$\frac{dx(t)}{dt} \Longrightarrow j\omega X(\omega)$$

with Laplace

$$\frac{dx(t)}{dt} \Longrightarrow sX(s)$$

Duality

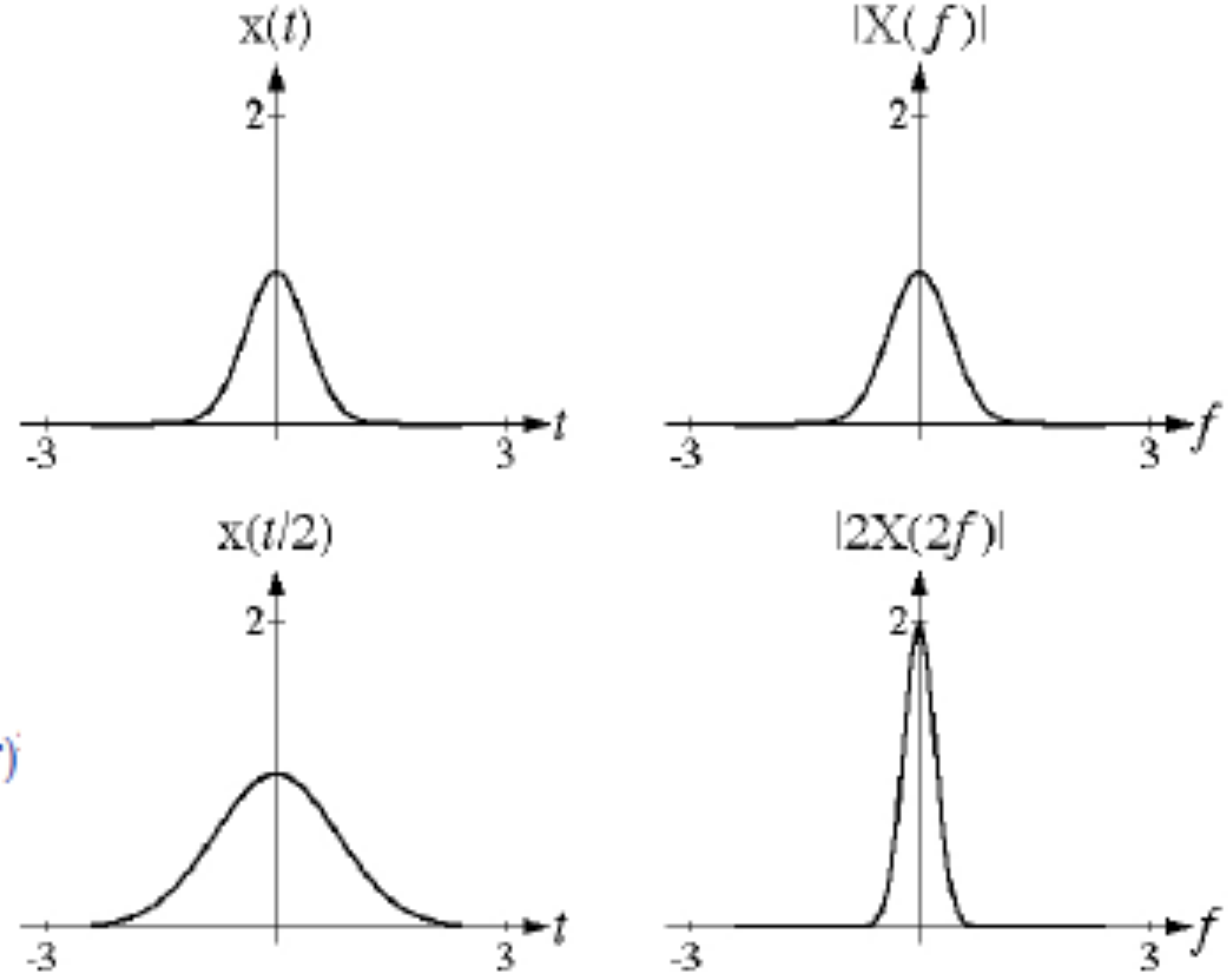


Uncertainty principle

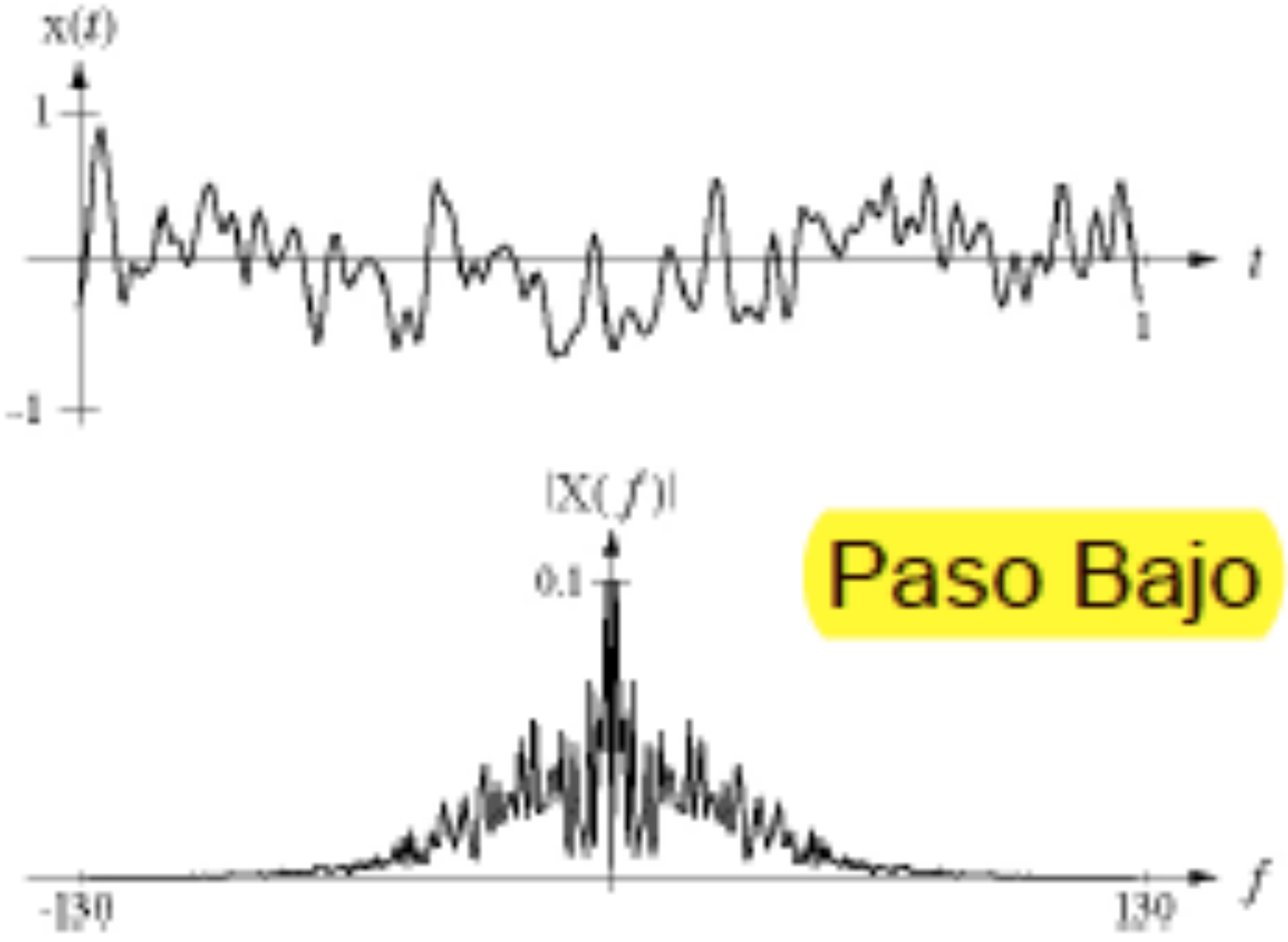
Las propiedades de escalado en el tiempo y en frecuencia nos indican que **si una señal se expande en uno cualquiera de los dominios**, *t* o *ω* (*f*), **inevitablemente se comprime en el dominio complementario**.

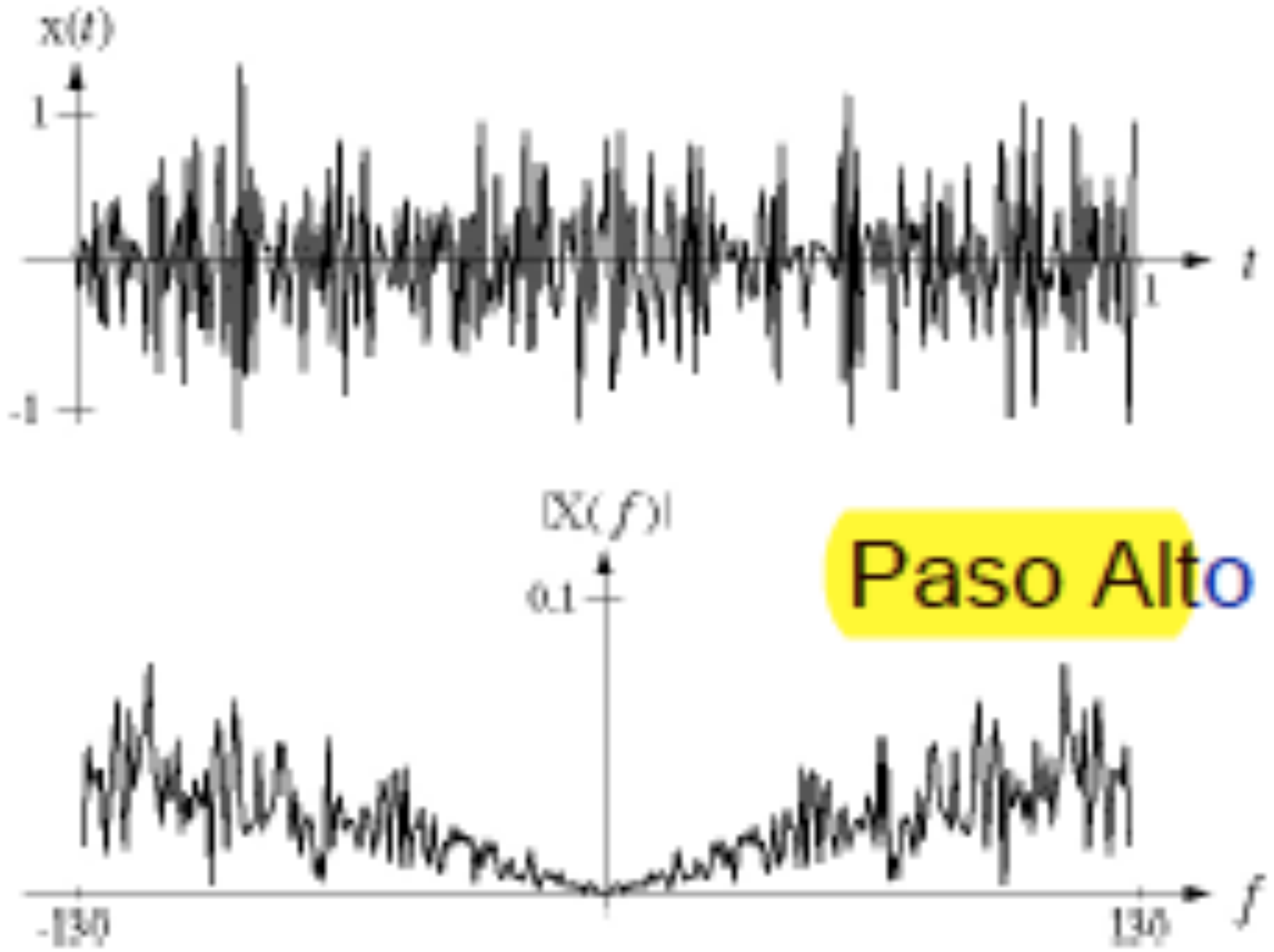
$$e^{-\pi t^2} \xleftrightarrow{\text{TF}} e^{-\pi f^2}$$

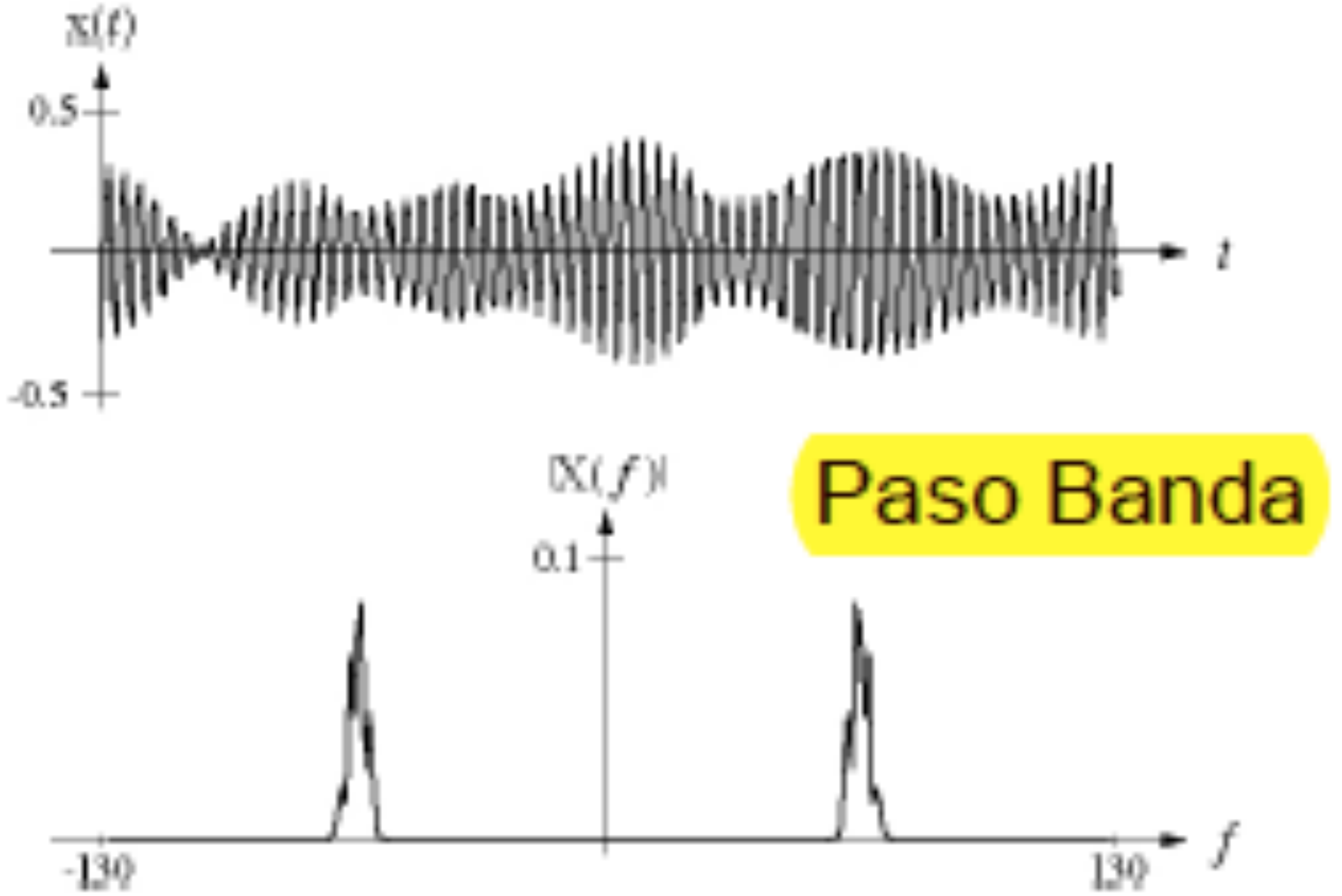
$$e^{-\pi\left(\frac{t}{2}\right)^2} \xleftrightarrow{\text{TF}} 2e^{-\pi(2f)}$$



Low-pass, High-pass, Band-pass filters







Existence of the Fourier Transform

- The integral below must exist:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- **Sufficient condition (signal with finite energy):**

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

Questions?