

Topic 3 - Part 5: Generalized Fourier Transform

Linear systems and circuit applications

Señales y Sistemas

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

Transformations for signal in continuous time

For Periodic signals

For non-periodic signals

Fourier Series (FS)

Stand. Fourier Transform (FT)

Laplace Transform (FT)

also for some
Signals with
Infinite Energy

Generalized
Fourier Transform
(GFT)

*Mathematically, it is not
completely valid... or we need
other definition of Fourier
Transformation....*

Existence of the Fourier Transform

- The integral below must exist:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- **Sufficient conditions (signal with finite energy):**

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty$$

it ensures the existence of the direct transformation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

it ensures the existence of the direct and inverse transformations in L_2 sense

Fourier Transform: other form

- Other way of writing the FT:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{+\infty} x(t) \sin(\omega t) dt$$

- Then the FT can be seen as these two integrals
- These two integrals must exist, to have - $x(t)$ - a FT

Periodic signals have not FT

- Periodic signals have infinite energy !!!
- so they have not FT
- You can also think in this way: the integral of the type below does not exist when $x(t)$ is periodic...

$$\int_{-\infty}^{+\infty} x(t) \cos(\omega t) dt$$

does not exist or infinity
in a standard sense

- think to the limits $-\infty$ o $+\infty$

Huge confusion

- Some books say that exists FT of periodic signal, it is not true.... mathematically, it is a nightmare...
- **For a periodic signal, the only true/proper mathematical tool is the Fourier Series (FS).**
- We will describe something that contains the same information of the FS - for periodic signals - (actually, it needs the information of the FS) that can be considered a generalization of the FT, but mathematically it is not a true-FT in a standard sense...

Huge confusion

- The **Generalized Fourier Transform** that will present it a properly defined only under the **Distribution Theory**.
- This requires more theory and other “calculus rules” (in an extended sense)
- Technically speaking, all the transformation involving the Dirac Deltas are properly valid under the **Distribution Theory**.

Huge confusion

- For the Laplace Transform, we consider standard calculus rules of integrals...and for Zeta Transform, standard sum rules

Other example

- Let us consider: $x(t) = e^{j\omega_0 t}$
- (this signal is periodic, then with infinite energy)
- A **good student** that have studied standard calculus (and does not know *Distribution Theory*) **MUST** write

$$X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \begin{cases} \nexists & \text{if } \omega \neq \omega_0, \\ \infty & \text{if } \omega = \omega_0. \end{cases}$$

Other example

- Indeed:

$$\int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{+\infty} e^{j(\omega_0 - \omega)t} dt$$

$$= \left[\frac{1}{j(\omega_0 - \omega)} e^{j(\omega_0 - \omega)t} \right]_{-\infty}^{+\infty} = \not\exists \quad \text{when } \omega_0 \neq \omega$$

Other example

- A bad student could write

$$X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \begin{cases} 0 & \text{if } \omega \neq \omega_0, \\ \infty & \text{if } \omega = \omega_0. \end{cases}$$

- In an standard sense, we cannot write

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \delta(\omega - \omega_0)$$

Other example

- Someone believes that the solution (or just makes confusion) is the *Cauchy Principal Value*, that is another “thing”:

$$P.V. \implies \lim_{R \rightarrow \infty} \int_{-R}^{+R} e^{j\omega_0 t} e^{-j\omega t} dt = 0$$

- **BUT:**

$$\lim_{R \rightarrow \infty} \int_{-R}^{+R} e^{j\omega_0 t} e^{-j\omega t} dt \neq \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt$$

Other example

- ...and, more generally, the FT is not defined as a Cauchy P.V.
- Other example of P.V.:

$$P.V. \implies \lim_{R \rightarrow \infty} \int_{-R}^{+R} t \, dt = 0$$

$$\int_{-\infty}^{+\infty} t \, dt = \nexists$$

....in the Laplace's world: "come back to normality"

- All these considerations are confirmed in *all books*, when they talk regarding Laplace Transform.
- Indeed the (Bilateral) Laplace Transform of the following signals does not exist:

$$\left. \begin{array}{l} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{array} \right\} e^{j\omega_0 t}$$

They have not Laplace Transform

Some possible “generalized definition”

- Some partial “patch” (“parche”), is to consider the following definition: we can Generalized Fourier Transform of $x(t)$ the function $X(\omega)$ such that:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_G(\omega) e^{j\omega t} d\omega$$

- i.e., we use the inverse Fourier Transform. The direct formula cannot work but maybe the inverse formula one can work...

Some possible “generalized definition”

- Then, if we accept the Dirac Delta as a mathematical operator with some special rule, as an example we can see that

$$X_G(\omega) = 2\pi\delta(\omega - \omega_0)$$

- is the Generalized Fourier Transform of $x(t) = e^{j\omega_0 t}$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

Some possible “generalized definition”

- Then, we can write

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{GF}} 2\pi\delta(\omega - \omega_0)$$

$$1 \xleftrightarrow{\mathcal{GF}} 2\pi\delta(\omega) \quad (\omega_0 = 0)$$

Some possible “generalized definition”

- Then, if a periodic signal $x(t)$ has a Fourier Series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\mathcal{GF}\{x(t)\} = \mathcal{GF}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\}$$

$$\mathcal{GF}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{GF}\{e^{jk\omega_0 t}\}$$

Some possible “generalized definition”

$$\mathcal{GF}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{GF}\{e^{jk\omega_0 t}\}$$

$$X_G(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0)$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Generalized Fourier Transform (GFT) - for a periodic signal

For periodic signals

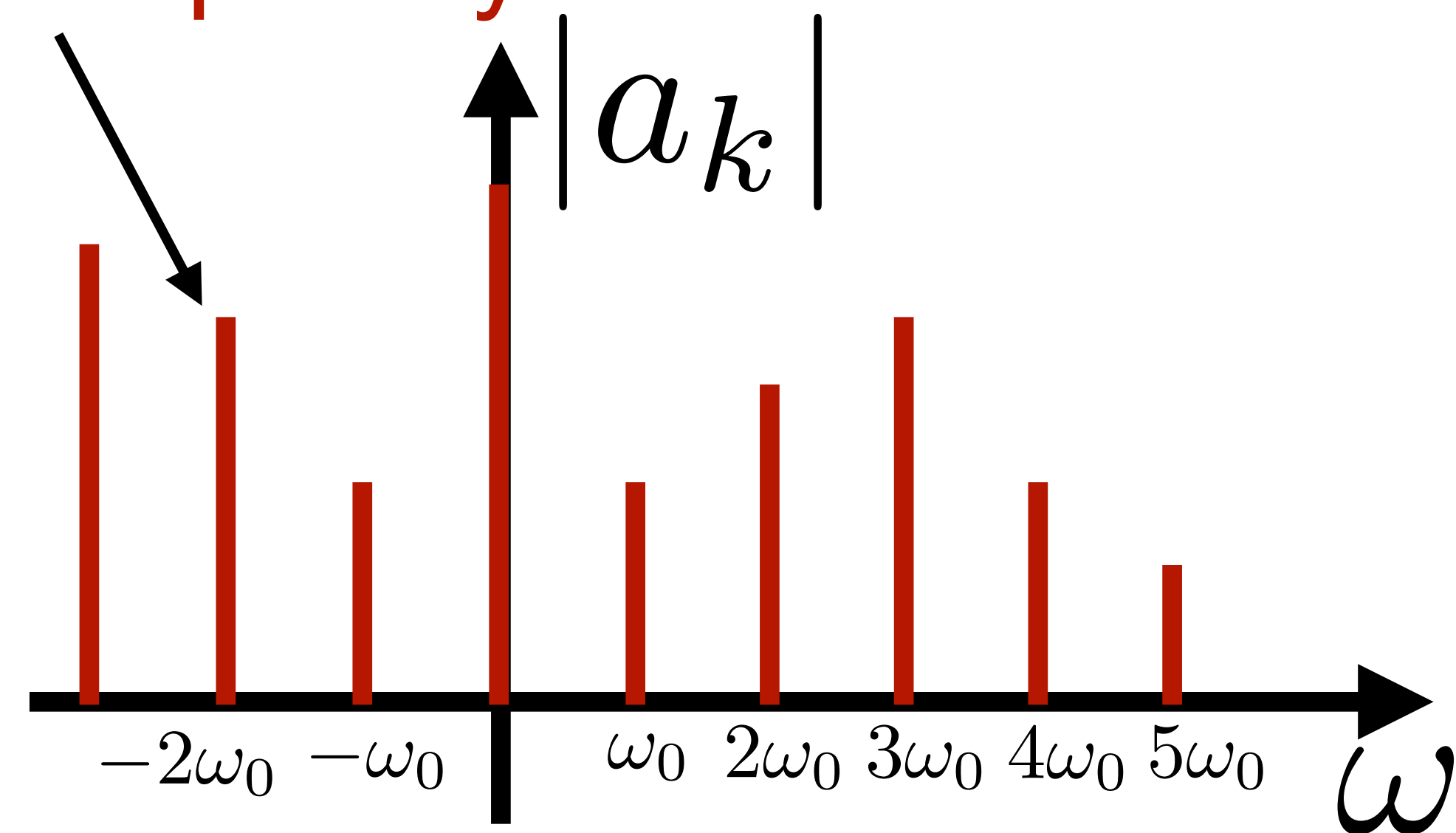
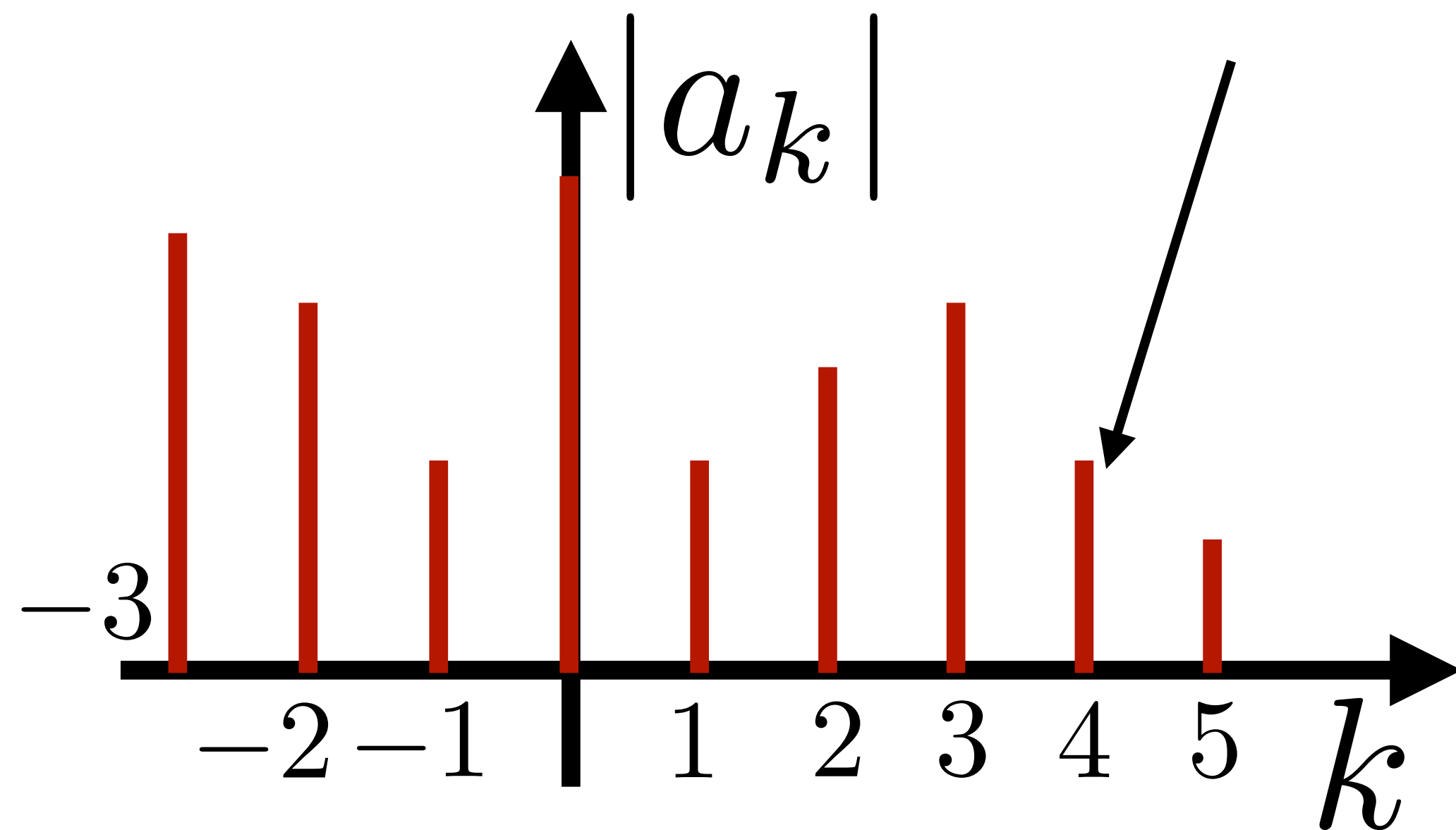
$$x(t) = x(t + T_0) \overbrace{\text{---}}^{FS} > a_k$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

Recall something of the FS...

- The coefficients a_k 's are complex numbers

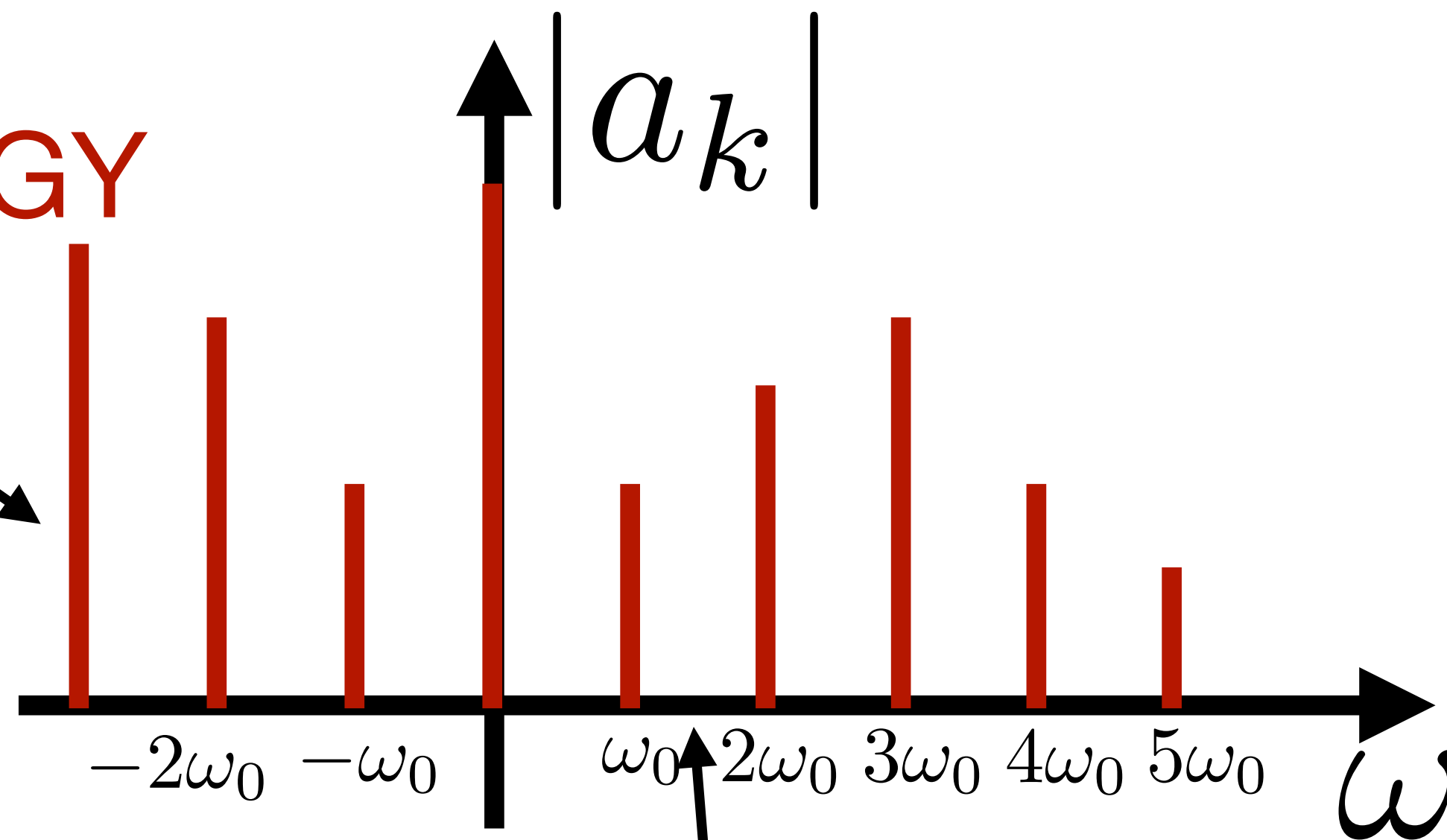
- (square root of) ENERGY in each frequency



Recall something of the FS...

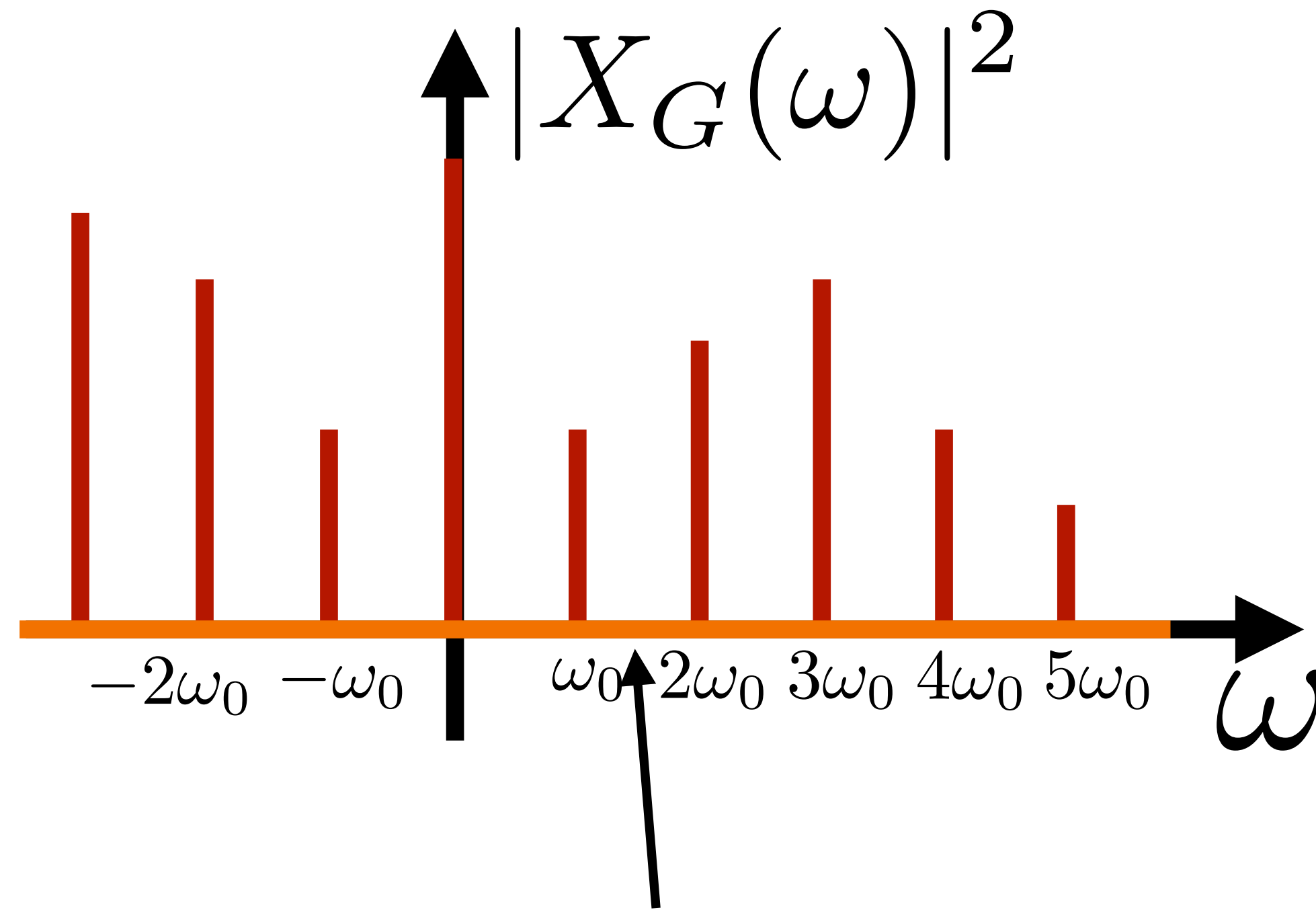
- Important observation:

- (square root of) ENERGY in each frequency



- In the middle is not defined... these frequencies are not contained

Recall something of the FS...



- If we consider a function that is equal to a_k at each $k\omega_0$ and “0” otherwise (there), then “almost nothing changes”...

Generalized Fourier Transform (GFT)

For periodic signals

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

- It contains only the information of the FS
- we need to do the FS, to have the GFT

Example: GFT of the cosine

$$x(t) = \cos(\omega_0 t) \rightarrow \begin{array}{l} a_1 = a_{-1} = \frac{1}{2} \\ a_k = 0, \quad k \neq 1, -1 \end{array}$$

$$X_G(\omega) = 2\pi \left[\frac{1}{2} \delta(\omega - \omega_0) + \frac{1}{2} \delta(\omega + \omega_0) \right]$$

$$X_G(\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

Example: GFT of the sine

$$x(t) = \sin(\omega_0 t)$$

$$X_G(\omega) = 2\pi \left[\frac{1}{2j} \delta(\omega - \omega_0) - \frac{1}{2j} \delta(\omega + \omega_0) \right]$$

$$X_G(\omega) = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$

$$X_G(\omega) = -j\pi \delta(\omega - \omega_0) + j\pi \delta(\omega + \omega_0)$$

The GFT for other “non-periodic” signals

- Given its definition, GTF can be used for other signals that have not the FT.
- For instance, a constant has FS (it is like periodic with infinite period...)
- or a sum of two periodic signal that is not always periodic...

Example: The GFT of a constant

$x(t) = 1 \implies a_0 = 1$ the rest are zeros...

$$X_G(\omega) = 2\pi\delta(\omega)$$

The GFT of a sum of two periodic signals

$$z(t) = x_1(t) + x_2(t)$$

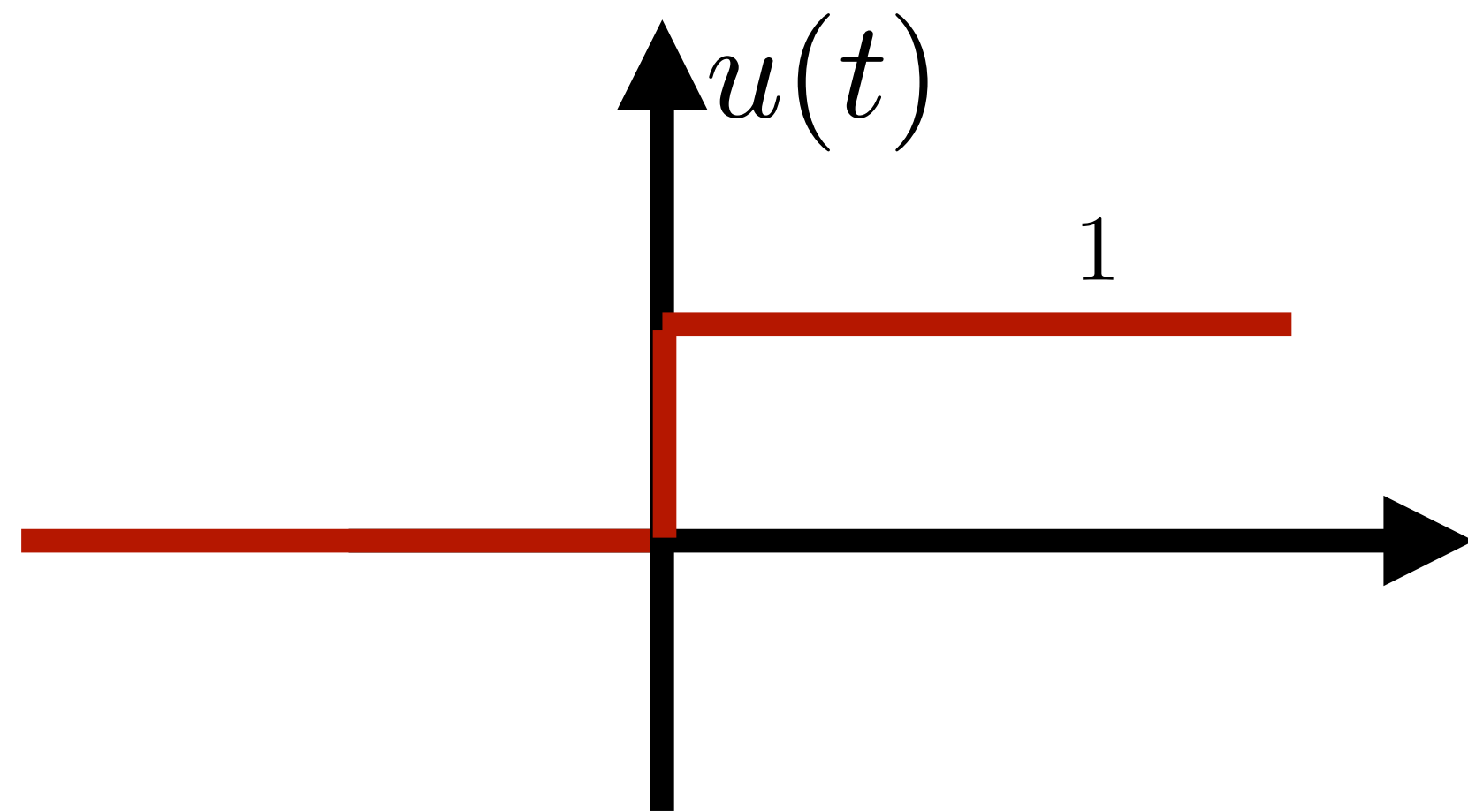
- it could be non-periodic

- periodic
- periodic

$$Z_G(\omega) = X_{1,G}(\omega) + X_{2,G}(\omega)$$

Even more confusion....and more difficult

- Consider the step function (escalón) - infinite energy:

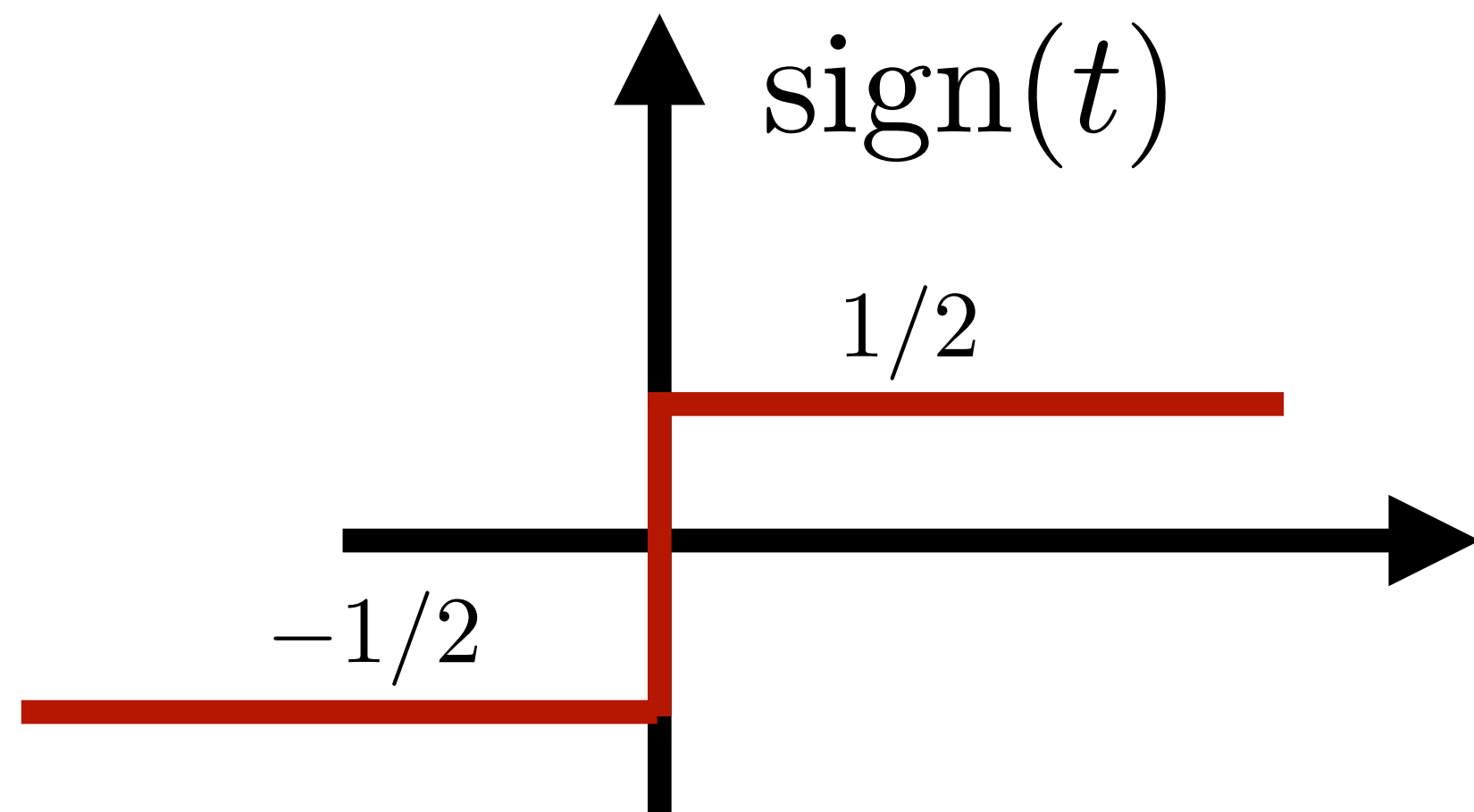


$$u(t) \xleftrightarrow{\mathcal{G}\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega)$$

- Many people prove it using FT properties, but the properties are valid only if FT exists !!

Even more confusion....

- Sign function - infinite energy:



$$\text{sign}(t) \xleftrightarrow{\mathcal{GF}} \frac{1}{j\omega}$$

- **The Sign function has not (Bilateral) Laplace Transform**

Avoid confusion and recall

- Even if people says the opposite, the GTF is not coming from the standard definition of the FT. It requires the DISTRIBUTION THEORY for a proper mathematical definition.
- Usually people call it simply “FT” generating a lot of confusion.
- The LAPLACE TRANSFORM extends the Standard FT.

Questions?