Example and important formula in CT

Discrete Time Systems

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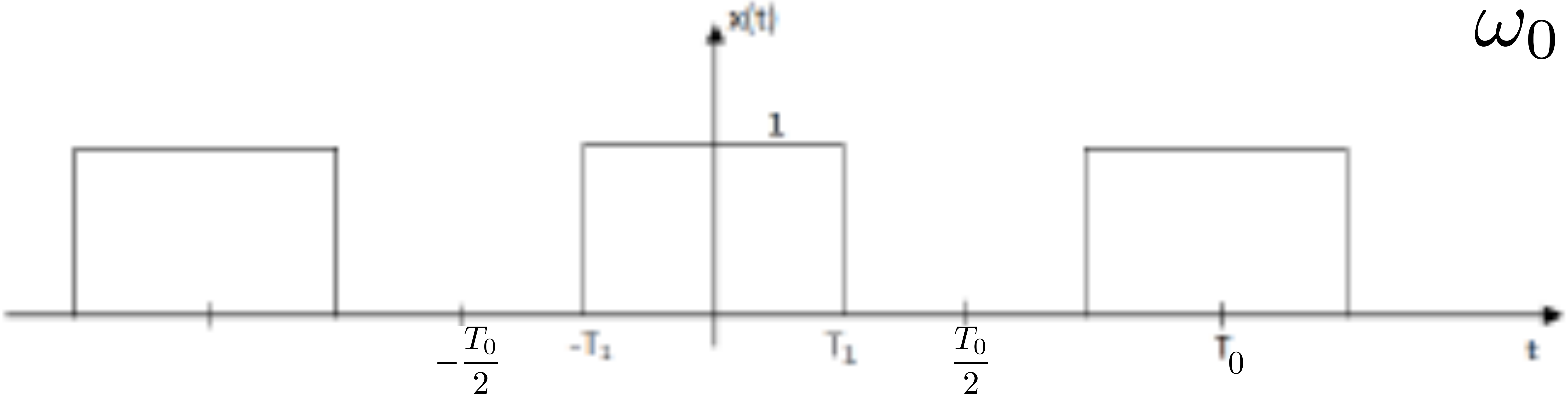
Important example

$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } T_1 < |t| < T_0/2 \end{cases}$$

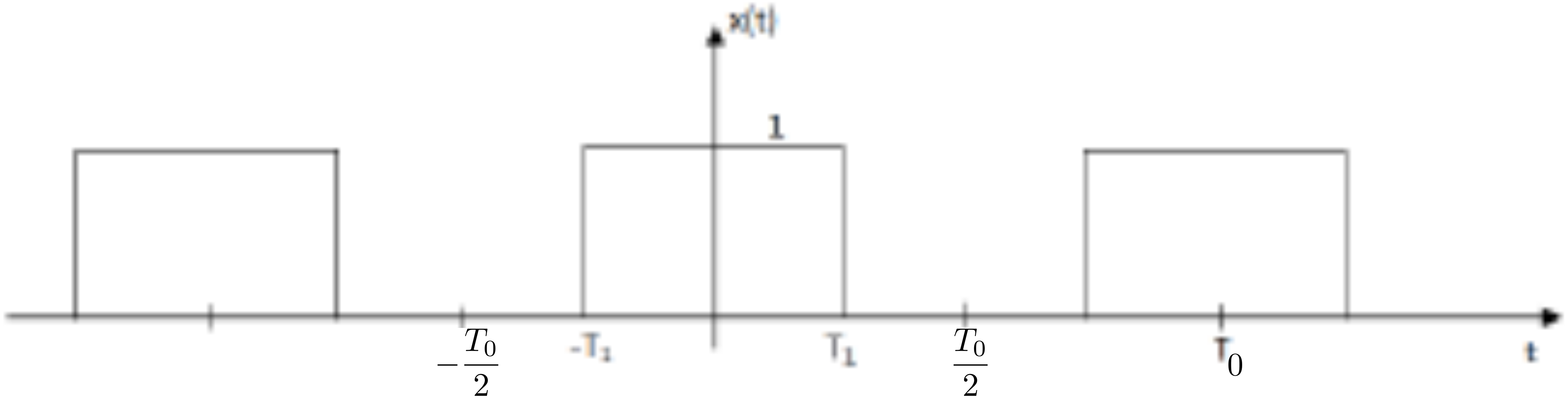
Periodic signal of period To

$$\omega_0 = \frac{2\pi}{T_0}$$



Use a proper mathematical tool for analyzing the Spectrum of this signal

Important example



- As $x(t)$ is periodic it can be represented using Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
- Coefficient calculation:

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}\int_{-T_1}^{T_1} 1 e^{-jk\omega_0 t} dt = (k \neq 0)$$

$$= \frac{1}{T_0}\frac{-1}{jk\omega_0}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} = \frac{-1}{jk\omega_0 T_0}\left[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right] = \cdots = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Important example

- we give more details
- we come back the notation for the period To

$$a_k = \frac{-1}{jk\omega_0 T_0}[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}]$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\sin(k\omega_0 T_1) = \frac{1}{2j}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}]$$

$$-2j\sin(k\omega_0 T_1) = -e^{jk\omega_0 T_1} + e^{-jk\omega_0 T_1}$$

Important example

- we give more details

$$a_k = \frac{-1}{jk\omega_0 T_0}[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}]$$

replacing inside:

$$a_k = \frac{-1}{jk2\pi}[-2j\sin(k\omega_0 T_1)]$$

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

Important example

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

- in this example, the a_k are real: this is due to x(t) is real and even…

- in k=0, we have an indeterminate form (0/0), we can solve in this way (again the notation for the period here is T, not To):

- For $k = 0$, we calculate the coefficient independently:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)e^{j0\omega_0 t}dt = \frac{1}{T_0}\int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T_0}$$

Important example: final solution

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi} \longrightarrow \omega^{(k)} = k\omega_0$$

$$a_0 = \frac{2T_1}{T_0} \longrightarrow \omega^{(0)} = 0$$

Considerations

- We have saw that the a_k are real (this is due to x(t) is real and even)
- Then, in this case we can represent directly a_k.
- For “normalization” reasons, we will plot a_kTo (a_k times To)

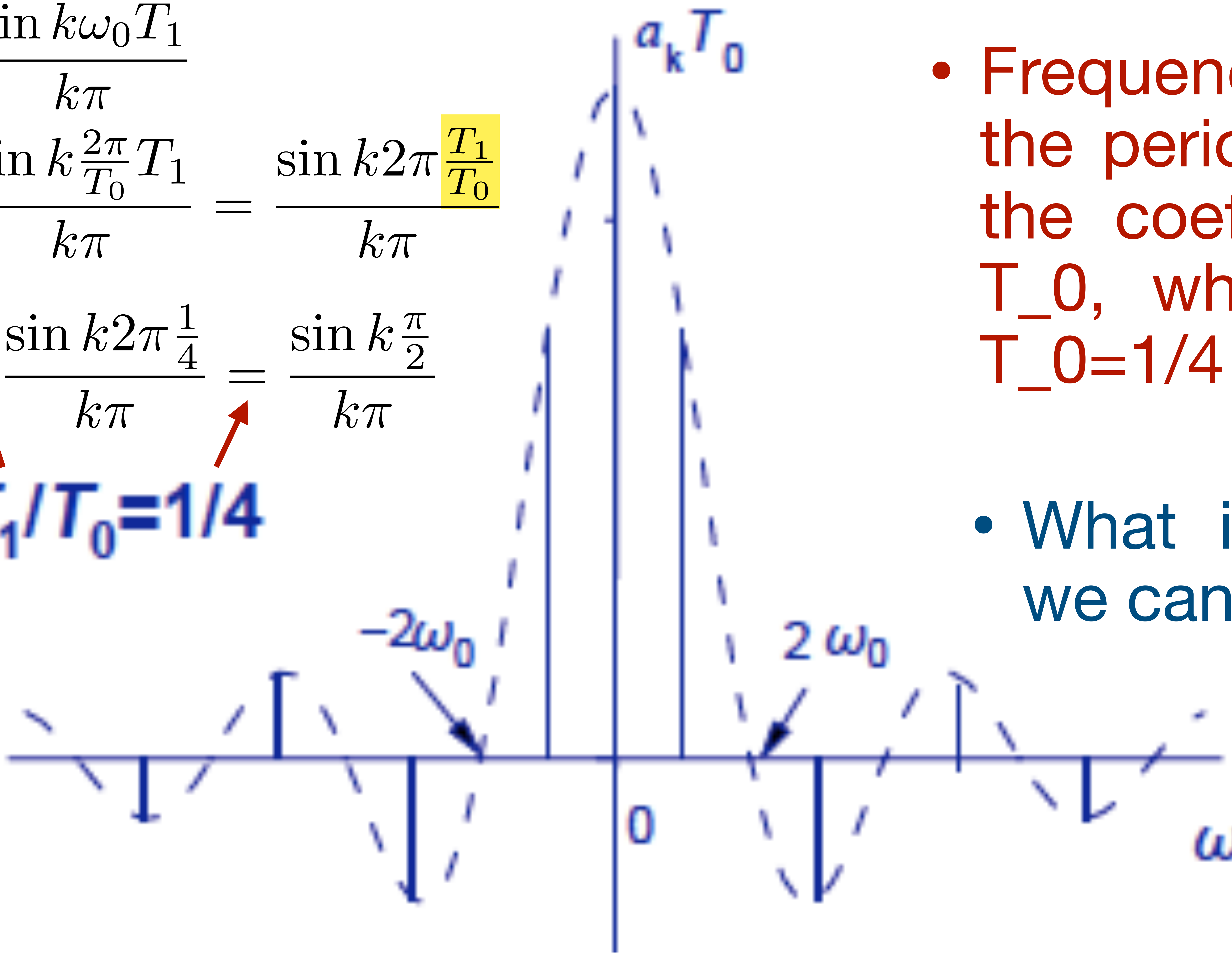
Again about the example of the rectangle

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_k = \frac{\sin k\frac{2\pi}{T_0}T_1}{k\pi} = \frac{\sin k2\pi\frac{T_1}{T_0}}{k\pi}$$

$$a_k = \frac{\sin k2\pi\frac{1}{4}}{k\pi} = \frac{\sin k\frac{\pi}{2}}{k\pi}$$

$T_1/T_0=1/4$



- Frequencies contained in the periodic signal x(t) and the coefficients a_k times T_0, when the ratio T_1/T_0=1/4

- What is the dashed line? we cannot answer now…

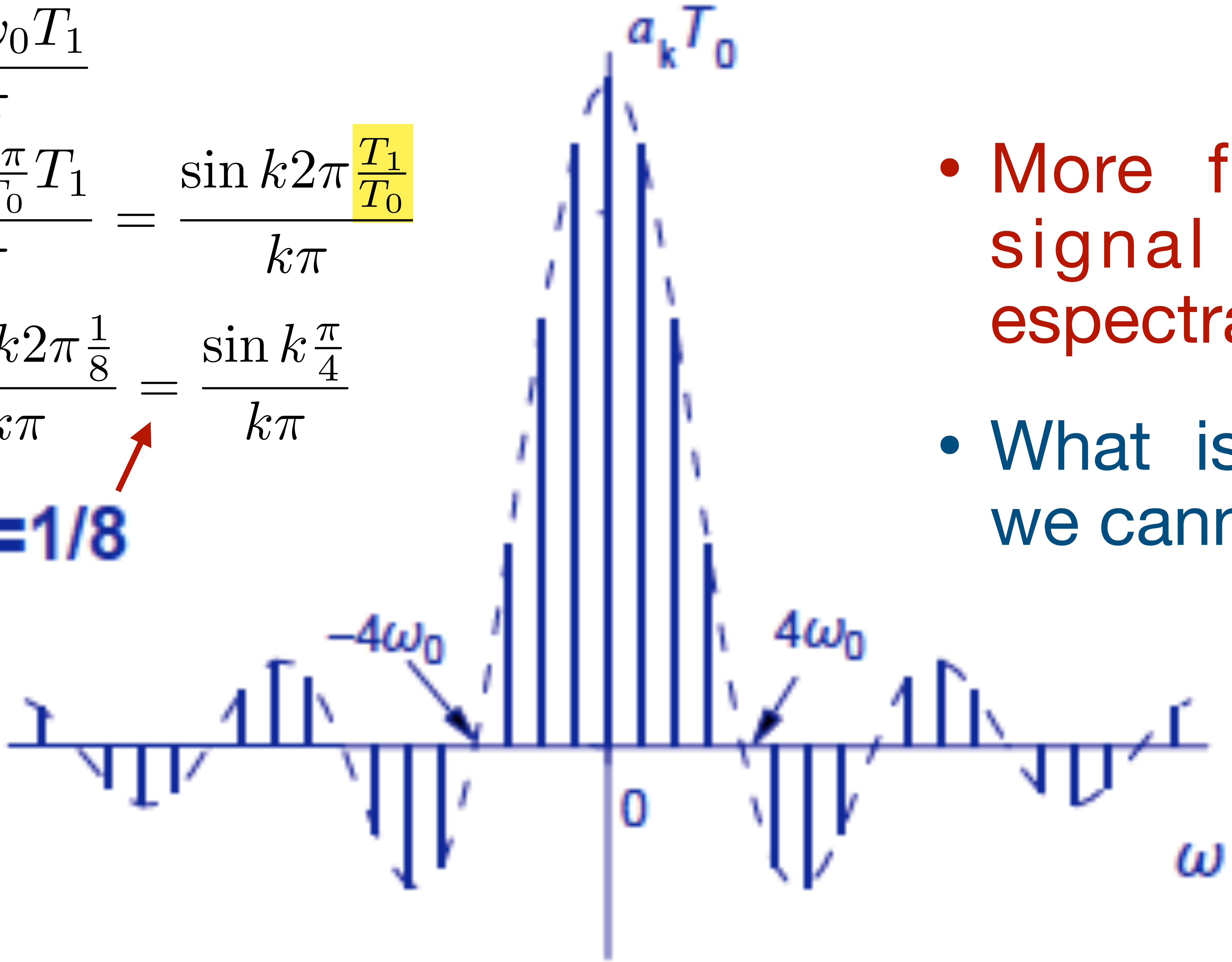
And increasing the period To?

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_k = \frac{\sin k\frac{2\pi}{T_0}T_1}{k\pi} = \frac{\sin k2\pi\frac{T_1}{T_0}}{k\pi}$$

$$a_k = \frac{\sin k2\pi\frac{1}{8}}{k\pi} = \frac{\sin k\frac{\pi}{4}}{k\pi}$$

$T_1/T_0=1/8$



- More frequencies in the signal !!! "mas banda espectrales"
- What is the dashed line? we cannot answer now…

And again increasing the period To?

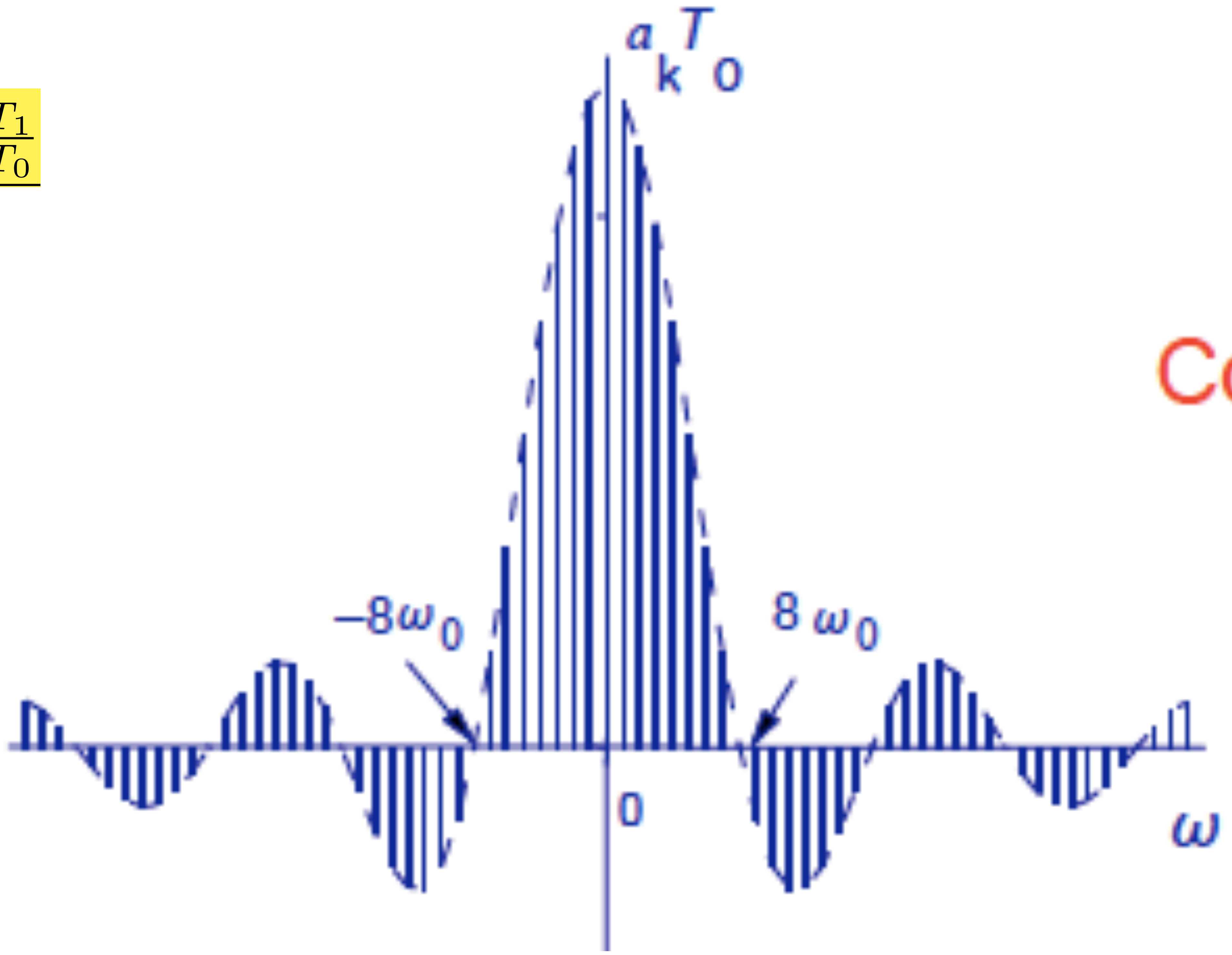
$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_k = \frac{\sin k\frac{2\pi}{T_0}T_1}{k\pi} = \frac{\sin k2\pi\frac{T_1}{T_0}}{k\pi}$$

$$a_k = \frac{\sin k2\pi\frac{1}{16}}{k\pi} = \frac{\sin k\frac{\pi}{8}}{k\pi}$$

$T_1/T_0=1/16$

Que pasa cuando To va a infinito?

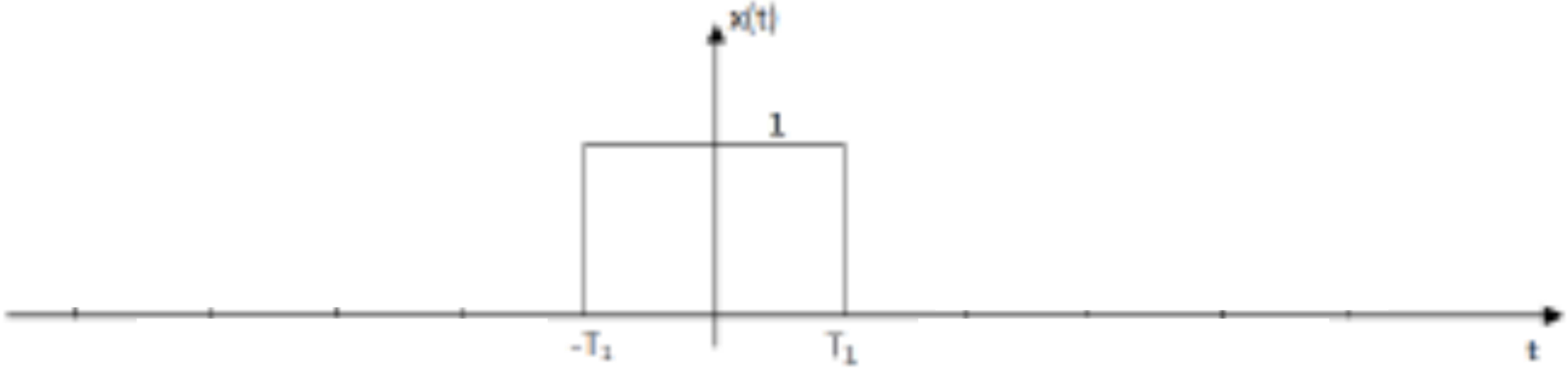


Conforme aumenta T_0, aumenta el número de componentes espectrales

- **More frequencies in the signal !!!**
- What is the dashed line? we cannot answer now…

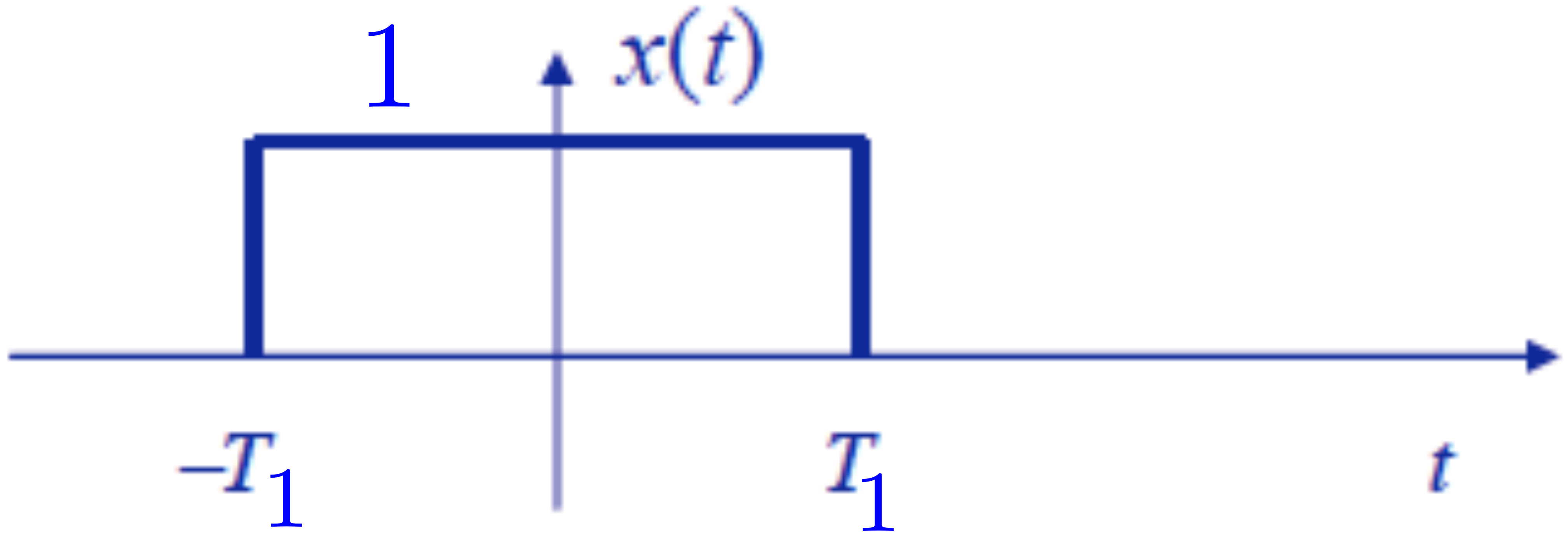
And again increasing the period To?

- what happens when To diverges? i.e., To goes to infinity?
- the signal becomes non-periodic….
- we will see it again, and we will see what is the dashed line…



Namely we saw and we will see:

Example: Stand. Fourier Transform (FT) of a rectangle



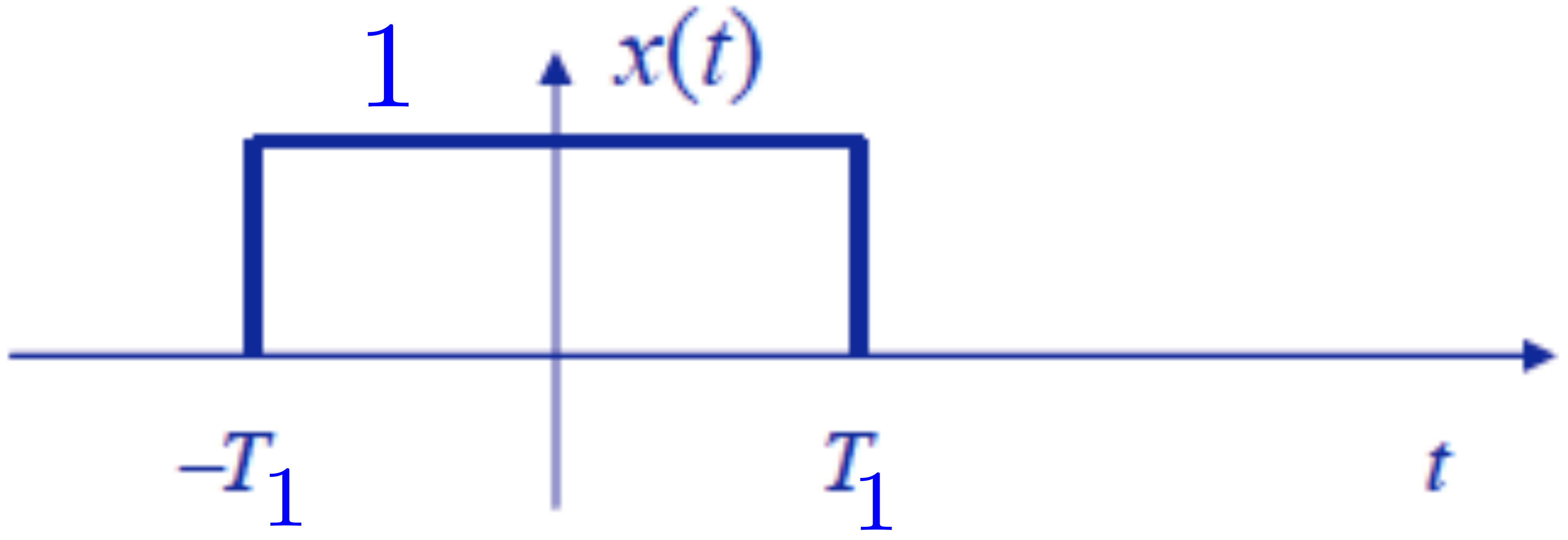
$$x(t) = \begin{cases} 1, & si\ |t| < T_1 \\ 0, & si\ |t| > T_1 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{+T_1} e^{-j\omega t} dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_{-T_1}^{+T_1}$$

$$= \frac{-1}{j\omega} \left[e^{-j\omega T_1} - e^{j\omega T_1} \right]$$

Example: Fourier Transform (FT) of a rectangle



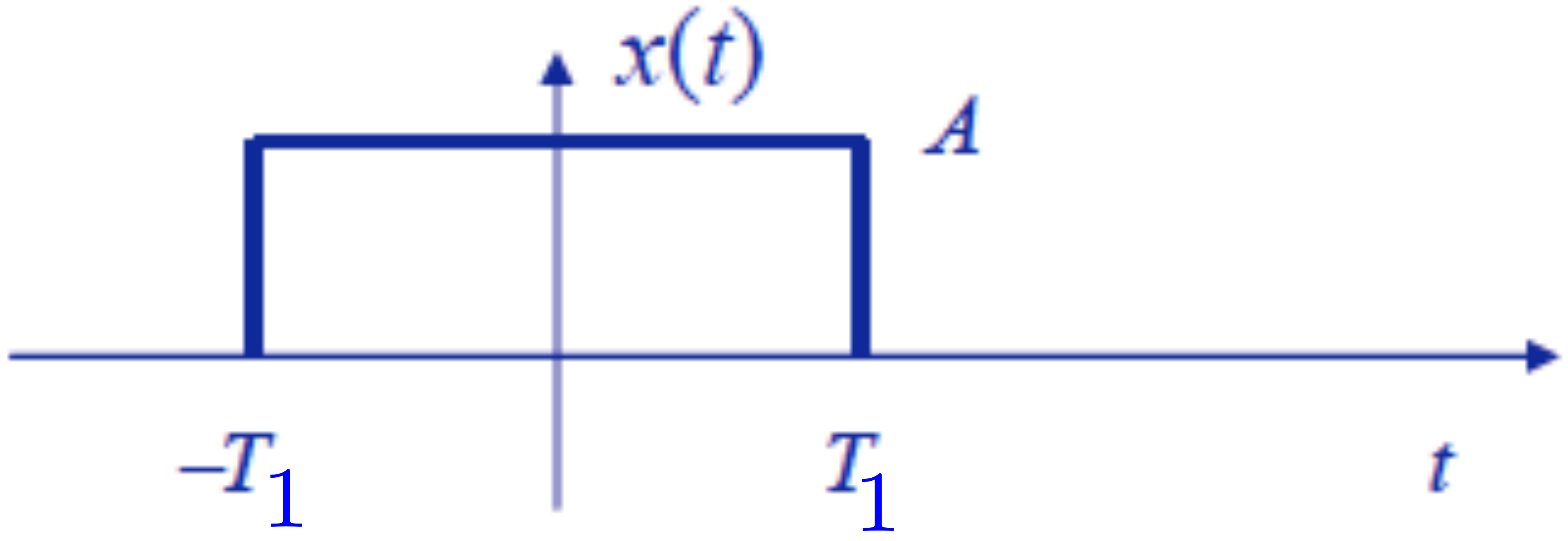
$$x(t) = \begin{cases} 1, & si\ |t| < T_1 \\ 0, & si\ |t| > T_1 \end{cases}$$

$$X(\omega) = \frac{-1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$= \frac{-1}{j\omega}\left[-2j\sin(\omega T_1)\right]$$

$$X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

Example: more general rectangle



$$x(t) = \begin{cases} A, & \text{si } |t| < T_1 \\ 0, & \text{si } |t| > T_1 \end{cases}$$

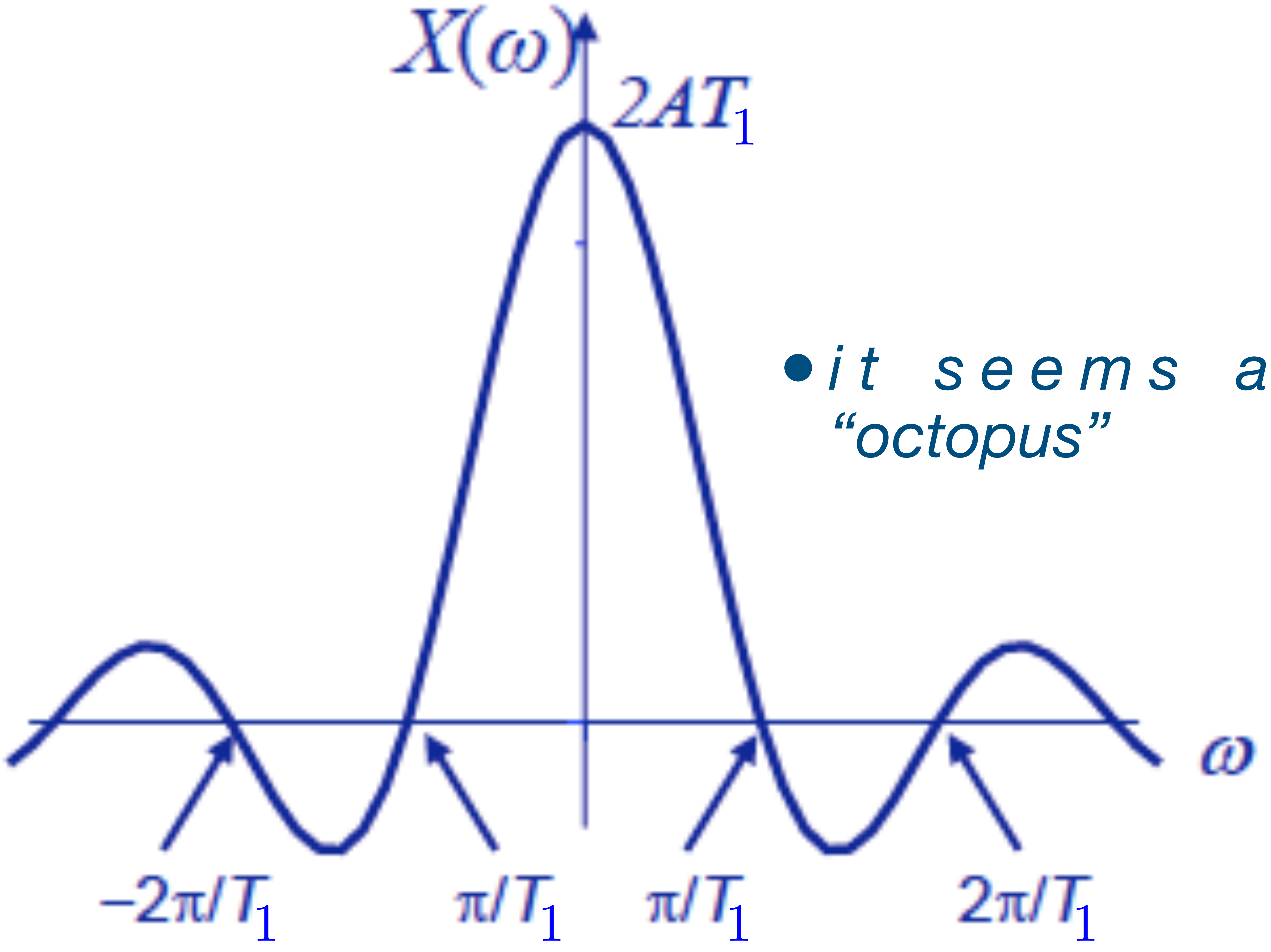
$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

- Let plot this function… *that in this case is real (since the rectangle is real and even)*

SINC FUNCTION: "the octopus"

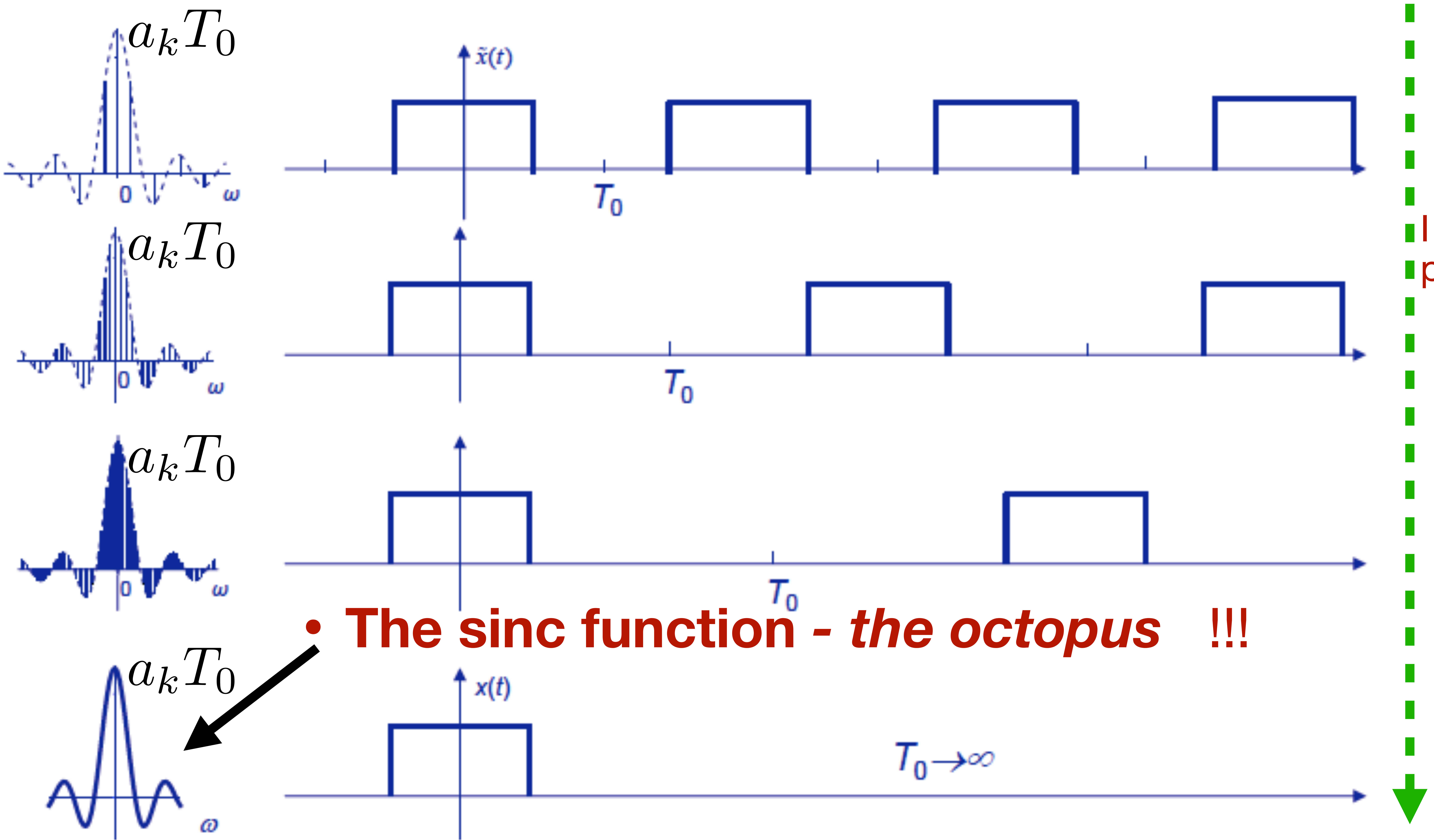
- Sinc function:

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$



- *it seems an "octopus"*

Recalling again this figure….



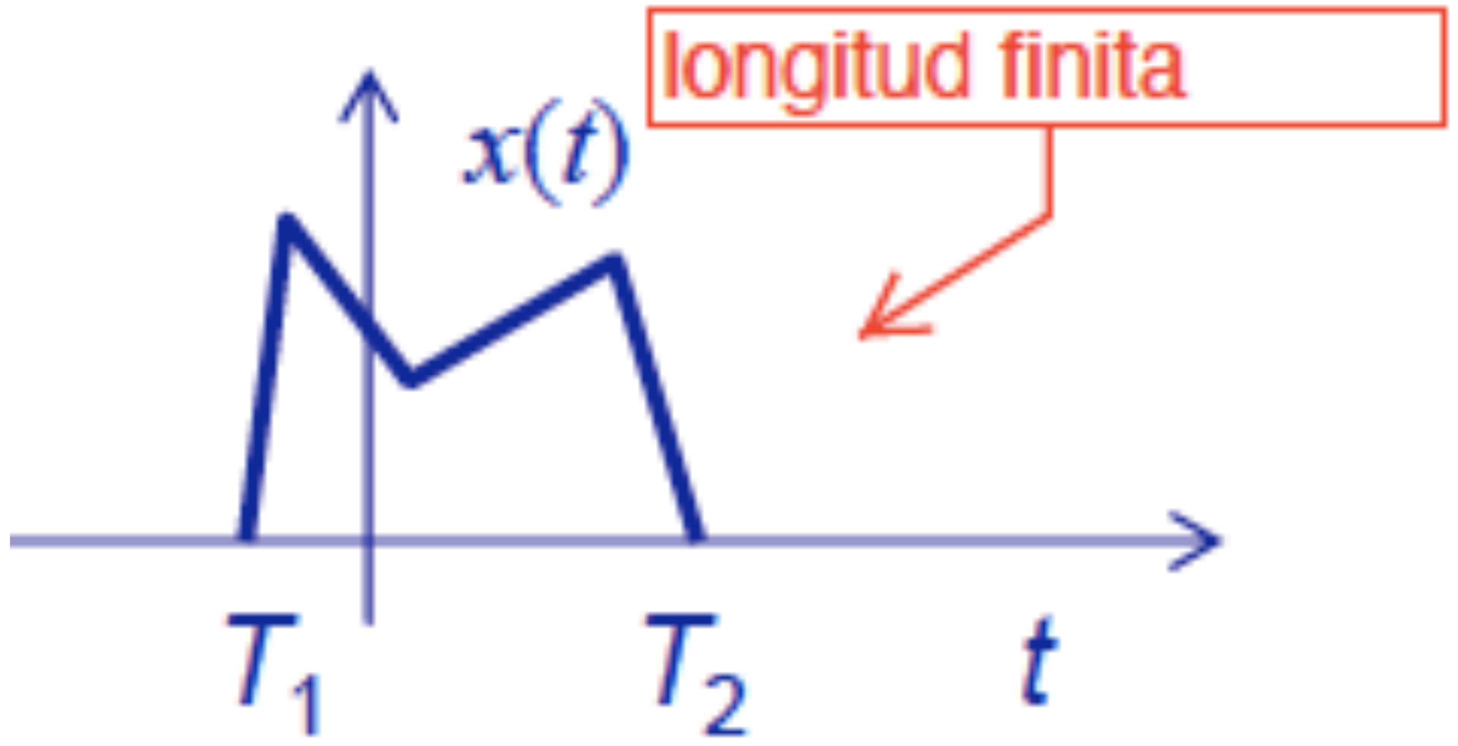
- **The sinc function - *the octopus* !!!**

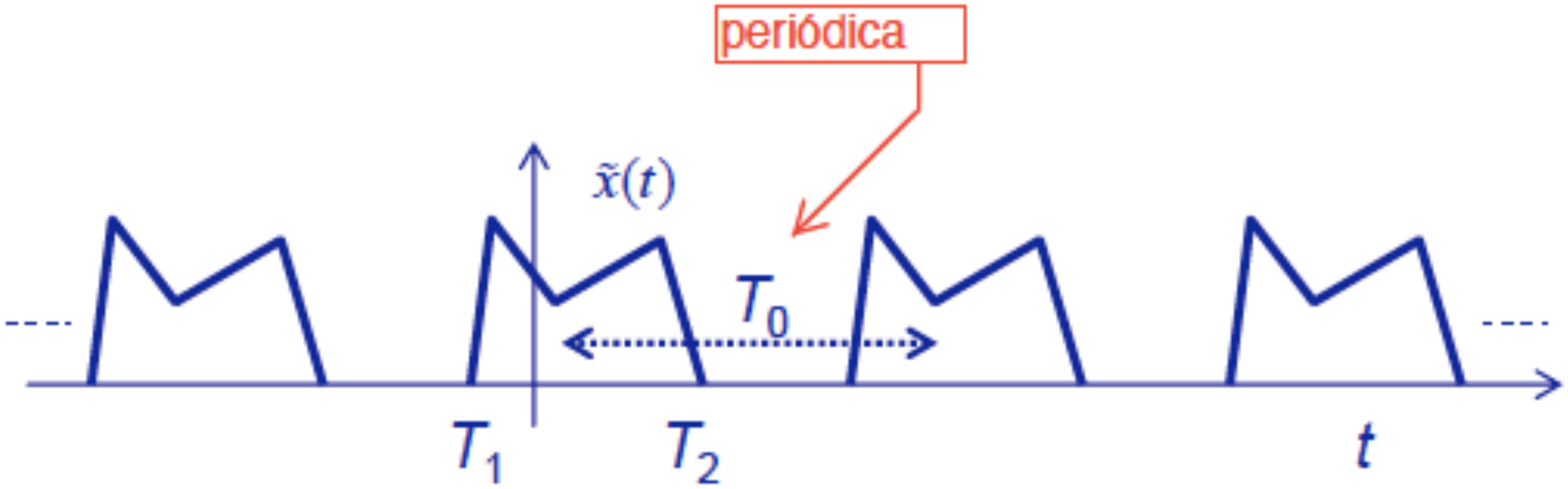
Increasing the period To

Generalization of this idea:
IMPORTANT RELATIONSHIP VALID FOR SIGNALS OF *FINITE LENGTH*

Signal with finite length and its periodic "brother"

Dada una señal de duración finita $x(t)$, realizamos una extensión periódica $\tilde{x}(t)$





$x(t)$

Signal with finite length

$\tilde{x}(t)$

Its periodic "brother"

Signal with finite length and its periodic "brother"

We can compute FT for

$$x(t)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

We can compute FS for

$$\widetilde{x}(t)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \quad \text{con}$$

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \quad y \quad \omega_0 = \frac{2\pi}{T_0}$$

Important result - relationship FT - FS

- En el intervalo $T_1 \le t \le T_2$, se cumple $\tilde{x}(t) = x(t)$ de modo que

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} \tilde{x}(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\,e^{-jk\omega_0 t}\,dt$$

Como

$$\left.\begin{aligned} a_k &= \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\,e^{-jk\omega_0 t}\,dt \\ X(\omega) &\equiv \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \Rightarrow \end{aligned}\right\} \Rightarrow \quad a_k = \frac{1}{T_0} X(\omega)\Big|_{\omega = k\omega_0}$$

- Los coeficientes a_k de la extensión periódica son **muestras equiespaciadas de la función $X(\omega)$**

more explications….

$\widetilde{x}(t) = x(t)$

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \tilde{x}(t)\mathrm{e}^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-jk\omega_0 t}dt + 0 + 0$$

$(2-2)$ $(2-2)$

$$\frac{1}{T_0}\int_{-\infty}^{-\frac{T_0}{2}} x(t)\mathrm{e}^{-jk\omega_0 t}dt + \frac{1}{T_0}\int_{\frac{T_0}{2}}^{\infty} x(t)\mathrm{e}^{-jk\omega_0 t}dt$$

more explications….

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-jk\omega_0 t}dt + \frac{1}{T_0}\int_{-\infty}^{-\frac{T_0}{2}} x(t)\mathrm{e}^{-jk\omega_0 t}dt + \frac{1}{T_0}\int_{\frac{T_0}{2}}^{\infty} x(t)\mathrm{e}^{-jk\omega_0 t}dt$$

$$\Downarrow$$

$$a_k = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-jk\omega_0 t}dt$$

$$\int_{-\infty}^{\infty} = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \ldots + \int_{-\infty}^{-\frac{T_0}{2}} \ldots + \int_{\frac{T_0}{2}}^{\infty} \ldots.$$

Important result - relationship FT - FS

- Valid only if x(t) has finite length:

$$a_k = \frac{1}{T_0}[X(\omega)]_{\omega=k\omega_0}$$

$$a_k = \frac{1}{T_0}X(k\omega_0)$$

Check in the rectangular example

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

$$X(k\omega_0) = \frac{2A\sin(k\omega_0 T_1)}{k\omega_0}$$

$$\frac{1}{T_0}X(k\omega_0) = \frac{1}{T_0}\frac{2A\sin(k\omega_0 T_1)}{k\omega_0} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\frac{1}{T_0}X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

Check in the rectangular example

$$X(\omega) = \frac{2A\sin(\omega T_1)}{\omega}$$

↓

$$\frac{1}{T_0}X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

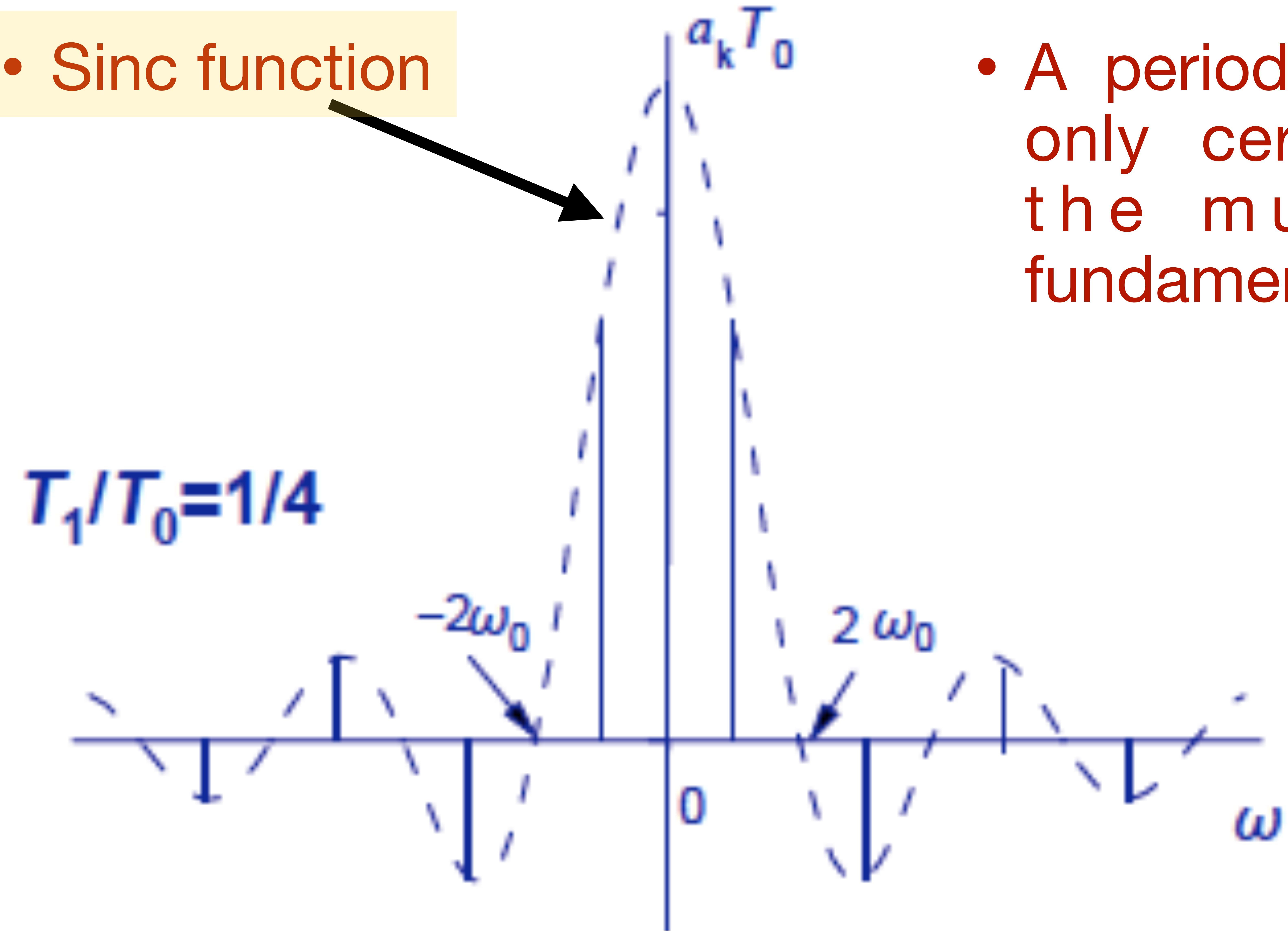
↓

$$a_k = \frac{1}{T_0}X(k\omega_0) = \frac{A\sin(k\omega_0 T_1)}{k\pi}$$

Exactly !!! (we obtain it with A=1, but it is very easy to re-do for a generic A)

For obtaining FS we are sampling of the FT

- Sinc function



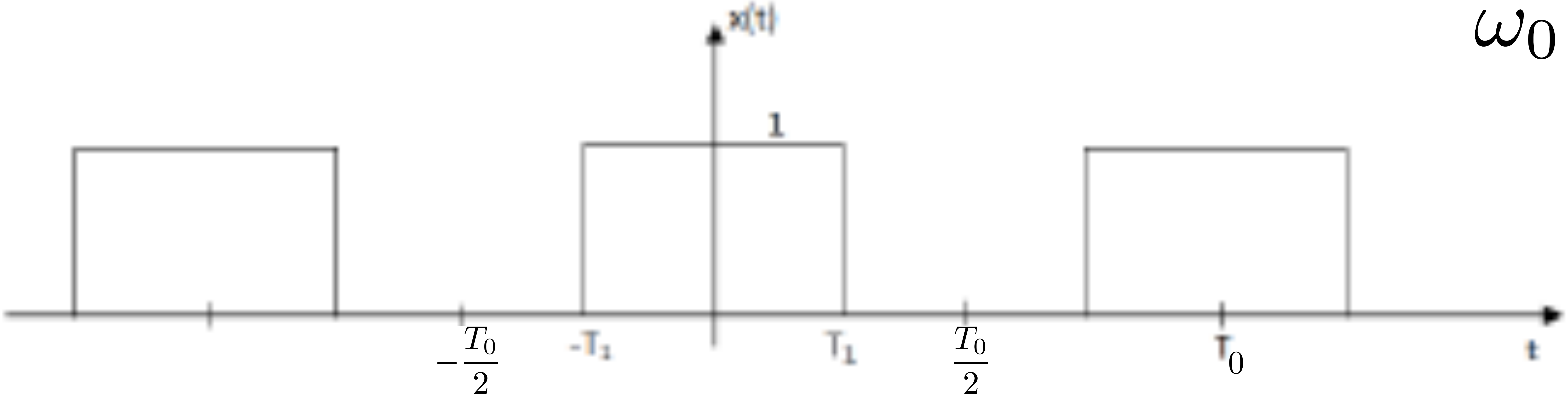
- A periodic signal contains only certain frequencies, the multiple of the fundamental frenquency.

Find the Generalized Fourier transform of…

$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } T_1 < |t| < T_0/2 \end{cases}$$

Periodic signal of period To

$$\omega_0 = \frac{2\pi}{T_0}$$



Use a proper mathematical tool for analyzing the Spectrum of this signal

Generalized Fourier Trasform (GFT) - for a periodic signal

For periodic signals

$$x(t) = x(t+T_0) \overbrace{- - - - >}^{FS} a_k$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

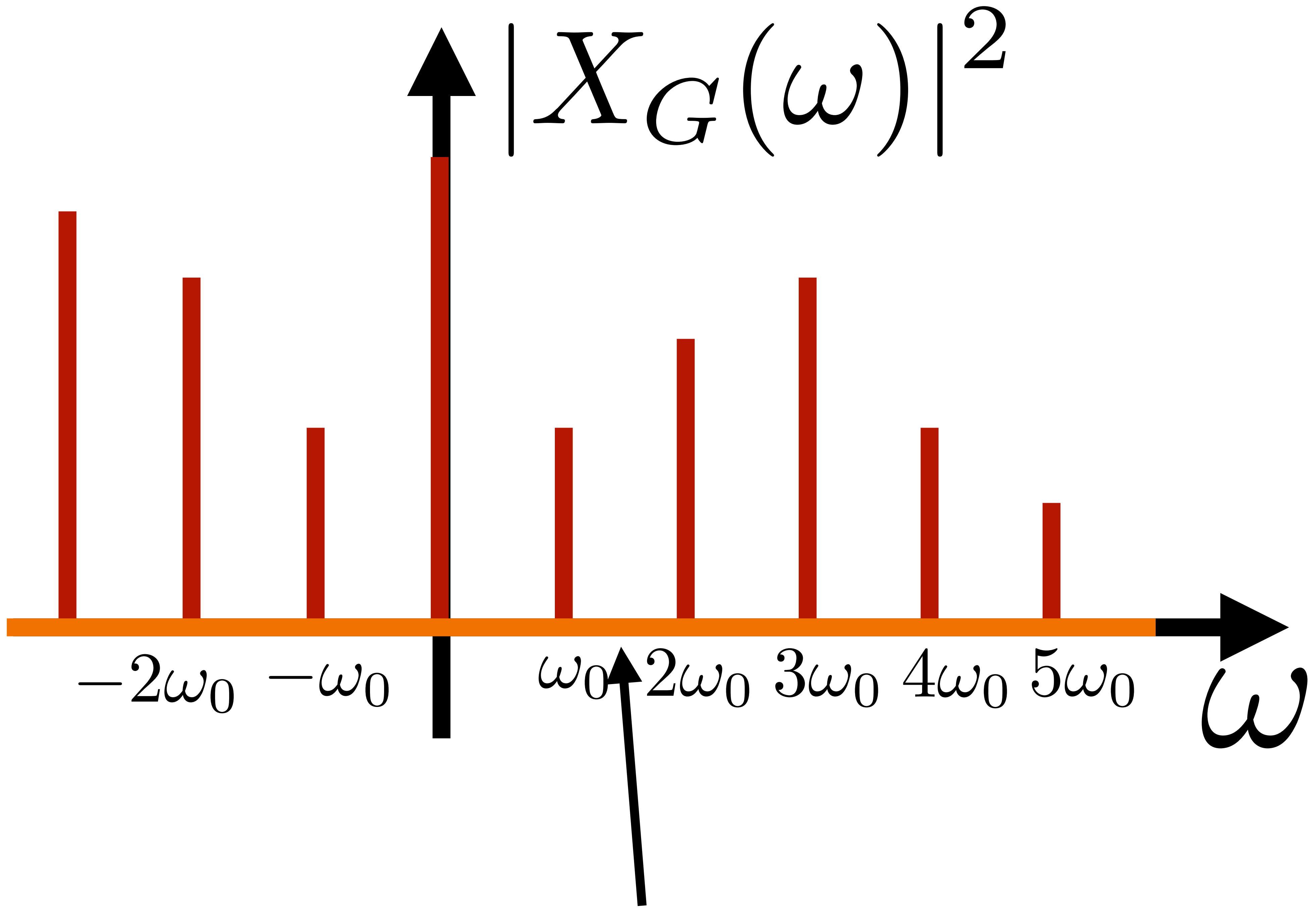
Generalized Fourier Trasform (GFT) in our example

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_0 = \frac{2T_1}{T_0}$$

$$X_G(\omega) = 4\pi \frac{T_1}{T_0}\delta(\omega) + 2\pi \sum_{k\neq 0} \frac{\sin k\omega_0 T_1}{k\pi}\delta\left(\omega - k\omega_0\right)$$

Recall something of the FS…



- If we consider a function that is equal to a_k at each kw_0 and “0” otherwise (there), then “almost nothing changes”…

Questions?