Generalized Fourier Transform for signals in discrete time

Discrete Time Systems

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Transformations for signal in discrete time

For Periodic signals	For non-periodic signals	
Fourier Series (FS)	Stand. Fourier Transform (FT)	Zeta Transform (ZT)

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Standard Fourier Transform

DEFINITIONS: (x[n] NO-periodic)

Analysis Equation:

periodic with period 2π

Direct
time ==> freq.

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

Fourier Transform

Syntesis Equation:

Inverse
freq. ==> time

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Inverse Fourier Transform

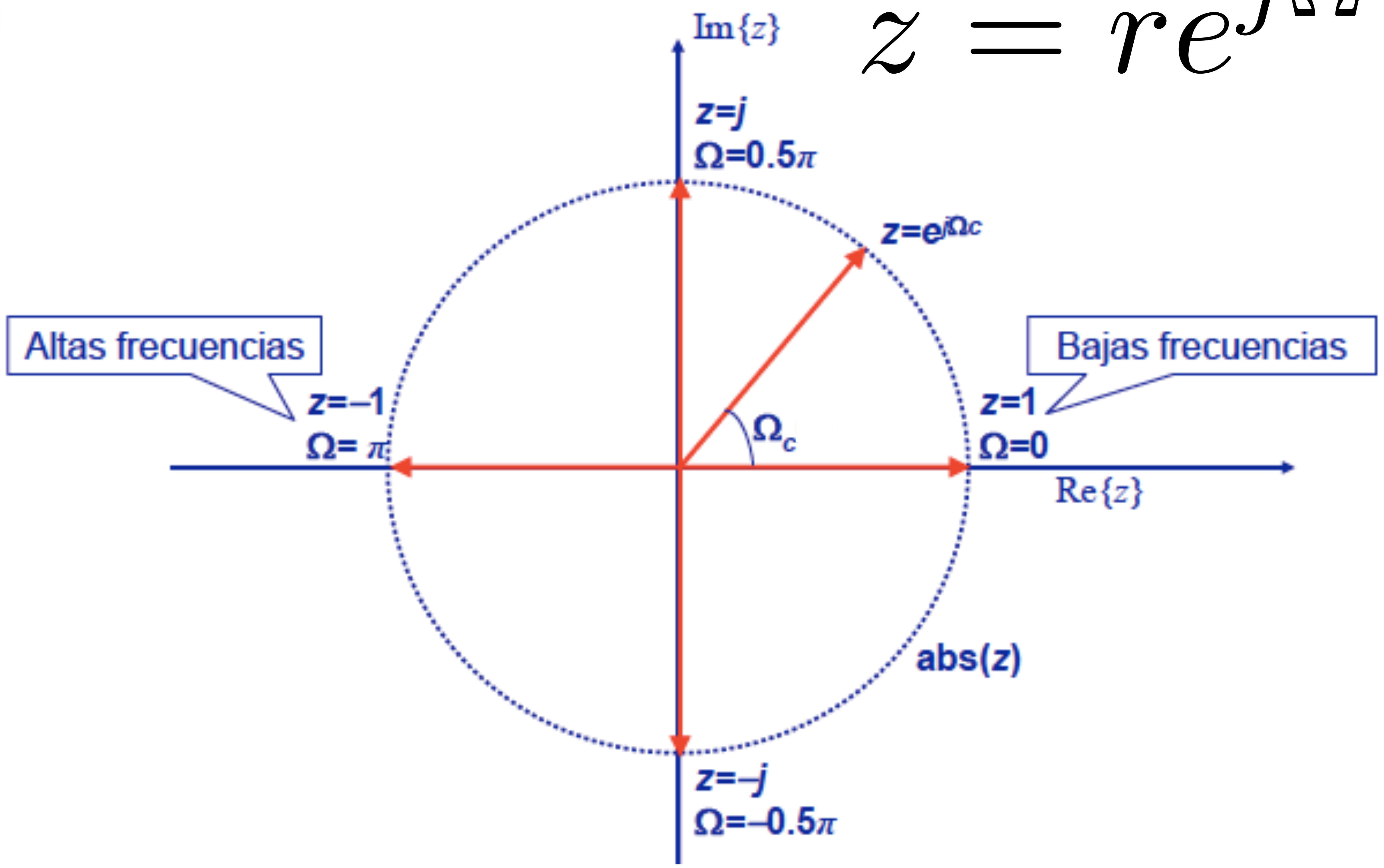
Standard Fourier Transform

Important property:

$$X(\Omega) = X(\Omega + 2\pi)$$

periodic with period 2π

$$z = re^{j\Omega}$$



Existence (convergence of series of stand. FT)

- Ec. síntesis: $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$
  - Integramos sobre un intervalo finito → no hay problemas de convergencia
  - Si los valores de la TF son finitos, la integral (el área) es finita
- Ec. análisis: $$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$
  - Tenemos una suma infinita, esto es "peligroso" porque aunque los valores de la secuencia sean finitos, sumamos infinitos términos, con lo cual la suma sí puede dar infinito
  - Conclusión, no todas las señales discretas tienen TF
  - Para poder garantizar que la TF existe, necesitamos exigir a la señal unas condiciones análogas a las que pedíamos en CT, por ejemplo:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \text{ -Energía finita}$$

o

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Existence (convergence of series of stand. FT)

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

with the calculus rules you know for Series…. we need:

Both are sufficient conditions !!

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \text{ -Energía finita}$$

FINITE ENERGY

o

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

…the for the convergence of the analysis equation (direct Fourier transformation)

Extended theories for Series (in order to divergent series become convergent in some "world"…)

- Cesaro convergence, Cesaro summation
- Abel summation
- Borel summation
- Euler summation, Srinivasa Ramanujan formulas…

Extended theories for Series (in order to divergent series become convergent in some "world"…)

- In Srinivasa Ramanujan's "world":

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + ... = -\frac{1}{12}$$

- but only in Ramanujan's "world" !!! ("*Analytic continuation of the Zeta function*")

Extended theories for Series (in order to divergent series become convergent in some “world”…)

In the following text, $(\mathfrak{R})$ indicates "Ramanujan summation". This formula originally appeared in one of Ramanujan's notebooks, without any notation to indicate that it exemplified a novel method of summation.

For example, the $(\mathfrak{R})$ of 1 − 1 + 1 − ⋯ is:

$$1 - 1 + 1 - \cdots = \frac{1}{2}\ (\mathfrak{R}).$$

Ramanujan had calculated "sums" of known divergent series. It is important to mention that the Ramanujan sums are not the sums of the series in the usual sense,[2][3] i.e. the partial sums do not converge to this value, which is denoted by the symbol $(\mathfrak{R})$. In particular, the $(\mathfrak{R})$ sum of 1 + 2 + 3 + 4 + ⋯ was calculated as:

$$1 + 2 + 3 + \cdots = -\frac{1}{12} \quad (\mathfrak{R})$$

Extended theories for Series (in order to divergent series become convergent in some "world"…)

- **a GOOD STUDENT MUST SAY:**

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + \ldots = \infty$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + \ldots = -\frac{1}{12}$$

Extended theories for Series (in order to divergent series become convergent in some "world"…)

- very good, and proper/correct videos (from Mathologer):
- https://www.youtube.com/watch?v=YuIIjLr6vUA
- https://www.youtube.com/watch?v=jcKRGpMiVTw&t=1s

Generalized Fourier Transform

Discrete Time Systems

Generalized Fourier Transform
(for some signals with infinite energy)

- We should use an extended theory for series… as we saw….
- The next formulas cannot be obtained with the calculus rules that you know so far (in this sense, they are *not true, and they cannot be proved* in “our world").

“Inverse” definition for GFTs

- “Inverse” definition: We could consider a *generalized* FT as a function in the transformed domain (Omega domain) such that *inverting it* with the inverse FT transformation (the synthesis equation) we get the signal x[n] we desire (in the time domain).

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X_G(\Omega)e^{j\Omega n}d\Omega$$

Example

- For instance:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega-\Omega_0)e^{j\Omega n}d\Omega = e^{j\Omega_0 n} = x[n]$$

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} X_G(\Omega) = 2\pi\delta(\Omega-\Omega_0)$$

periodic with period 2π

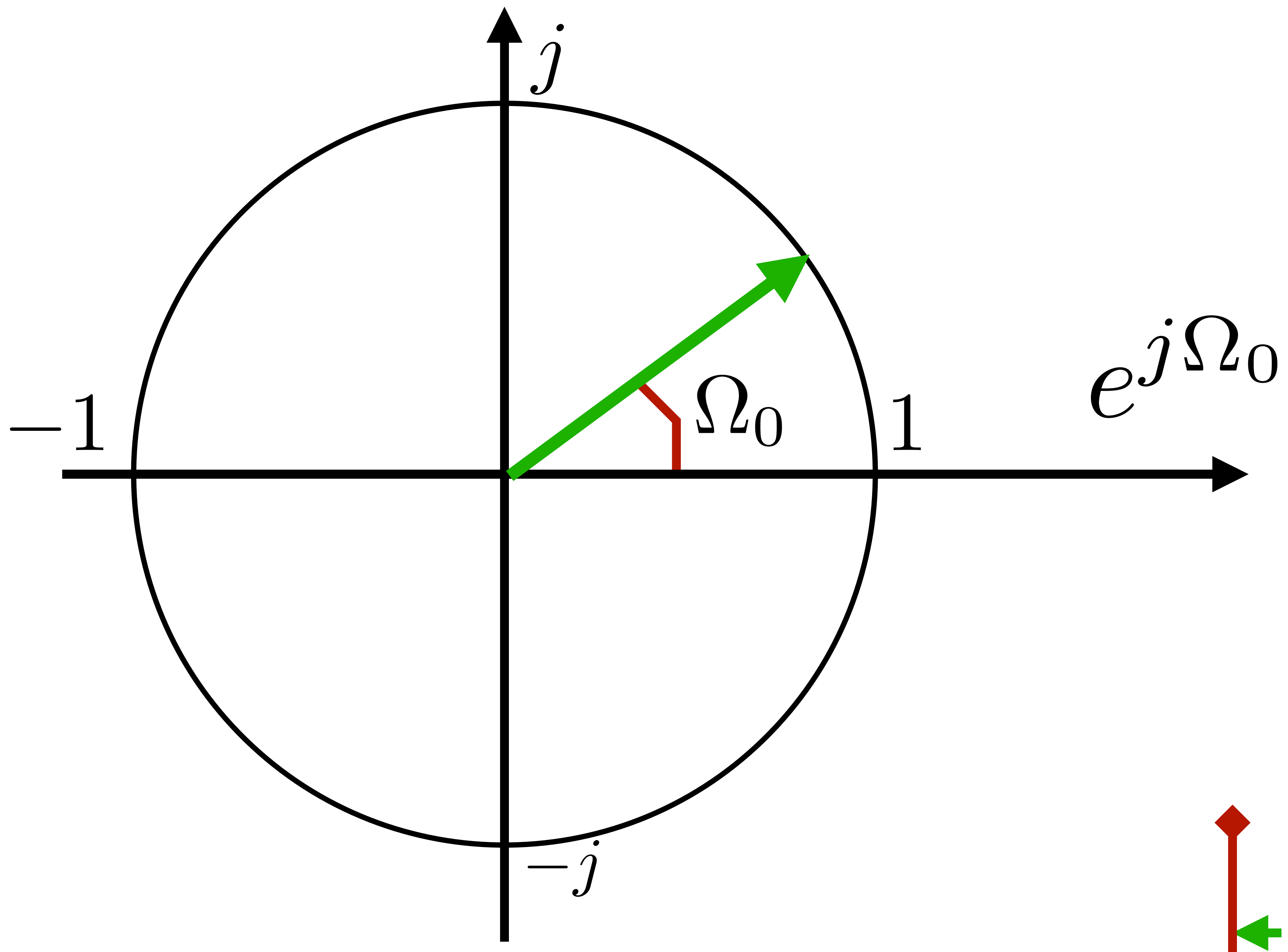
Recall that in discrete time, a complex exponential can be also non-periodic (with respect to the discrete time variable n)

Example

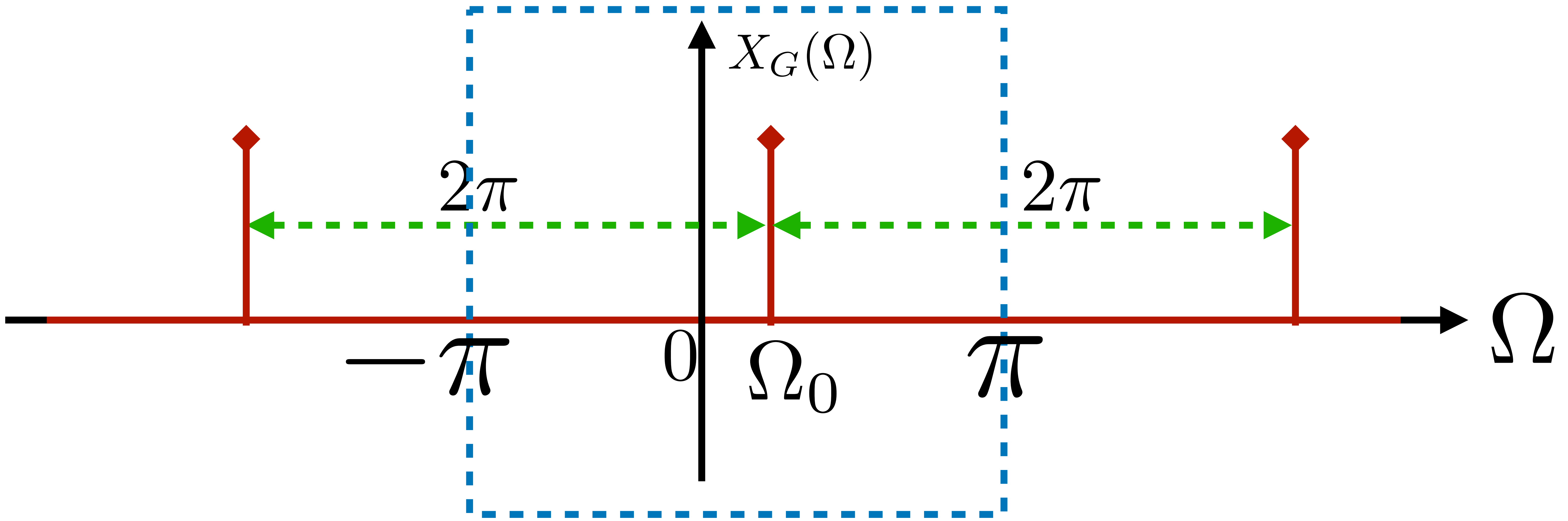
It is periodic if $\Omega_0 = 2\pi \frac{m}{N}$

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi\delta(\Omega - \Omega_0)$$

periodic with period 2π



$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$



Example

It is periodic if $\Omega_0 = 2\pi \frac{m}{N}$

- For instance:

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi\delta(\Omega - \Omega_0)$$

periodic with period 2π

- Note that alreadyis a “train” in delta (due to the 2\pi-periodicity):

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$

Generalized Fourier Transform for periodic signals

Discrete Time Systems

Generalized Fourier Transform of periodic signals

- **Periodic signals have infinite energy**
- **For periodic signals, the Fourier Series is well-defined with our calculus rules.**

$$x[n] \longleftrightarrow a_k$$

a_k periodic with period N

$$a_k = a_{k+N}$$

Generalized Fourier Transform of periodic signals

- **The direct transformation of standard does not exist (in our word - for a periodic signal)**

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

Generalized Fourier Transform of periodic signals

- **We could use the "inverse" definition idea…**

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X_G(\Omega) e^{j\Omega n} d\Omega$$

Generalized Fourier Transform of periodic signals

- GTF of a periodic signal of period N:

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

$$\Omega_0 = \frac{2\pi}{N}$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

- We will check that **is periodic with period 2\pi**

GFT of periodic signals

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

$$a_{k+N}\delta\left(\Omega - \frac{2\pi(k+N)}{N}\right) = a_{k+N}\delta\left(\Omega - \frac{2\pi k}{N} - \frac{2\pi N}{N}\right)$$

$$= a_{k+N}\delta\left(\Omega - \frac{2\pi k}{N} - 2\pi\right)$$

Moreover, since:

a_k periodic with period N

$$a_k = a_{k+N}$$

GFT of periodic signals

Since: a_k periodic with period N

$$a_k = a_{k+N}$$

$$a_{k+N}\delta\left(\Omega - \frac{2\pi(k+N)}{N}\right) = a_k\delta\left(\Omega - \frac{2\pi k}{N} - 2\pi\right) \longrightarrow$$

So each 2\pi in Omega,
we have deltas with the same "values"/"areas"
(see next slide)

GFT of periodic signals

$k+N$ → $a_{k+N}\delta\left(\Omega-\frac{2\pi(k+N)}{N}\right)=a_k\delta\left(\Omega-\frac{2\pi k}{N}-2\pi\right)$

k → $a_k\delta\left(\Omega-\frac{2\pi k}{N}\right)$

→ $$X_G(\Omega)=2\pi\sum_{k=-\infty}^{\infty}a_k\delta\left(\Omega-\frac{2\pi k}{N}\right)$$

$$X_G(\Omega)=X_G(\Omega+2\pi)=X_G(\Omega-2\pi)$$

periodic with period 2π

Examples

Example

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

$$x[n] = 1$$

==> It has infinite energy

It can be considered as a signal of periodic of period N=1;
Indeed, it has a Fourier Series with coefficients:

$$a_k = 1 \quad \text{for all } k \qquad \text{note that } a_k = a_{k+N} = a_{k+1}$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} 1\delta(\Omega - 2\pi k)$$

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

Other Example

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

==> It has infinite energy

It can be considered as a signal of periodic of period N;

with N=1 we come back to the previous example;

It has a Fourier Series with coefficients:

$$a_k = \frac{1}{N} \text{ for all } k$$

note that $a_k = a_{k+N}$

$$X_G(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi}{N} k\right)$$

Example: again the complex exponential

$$x[n] = e^{j\Omega_0 n}$$

Such that $\Omega_0 = 2\pi \frac{m}{N}$

Considering $k = 0, \ldots, N-1$

$$a_1 = 1 \xrightarrow{a_k = a_{k+N}} a_{N+1} = 1 \longrightarrow a_{mN+1} = 1$$

$$a_k = 0 \text{ for } k = 0 \text{ and } k = 2, \ldots, N-1$$

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi\delta(\Omega - \Omega_0)$$

periodic with period 2π

Example: again the complex exponential

$$x[n] = e^{j\Omega_0 n}$$ → Such that $\Omega_0 = 2\pi \frac{m}{N}$

Considering $k = 0, ..., N-1$

$$a_1 = 1 \xrightarrow{a_k = a_{k+N}} a_{N+1} = 1 \rightarrow a_{mN+1} = 1$$

$$a_k = 0 \text{ for } k = 0 \text{ and } k = 2, ..., N-1$$

If we replace this a_k's in the definition:

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \longrightarrow$$

Example: again the complex exponential

If we replace this a_k's in the definition:

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

$$a_1 = 1 \xrightarrow{a_k = a_{k+N}} a_{N+1} = 1 \longrightarrow a_{mN+1} = 1$$

$$a_k = 0 \text{ for } k = 0 \text{ and } k = 2, \ldots, N-1$$

We can rewrite as:

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$

Let see if the “inverse” definition works…

GFT of periodic signals

- Let prove that the GTF of a periodic signal is true by the the “inverse” definition:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X_G(\Omega)e^{j\Omega n}d\Omega$$

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right)\right] e^{j\Omega n} d\Omega$$

GFT of periodic signals

- Let prove that the GTF of a periodic signal is true by the the “inverse” definition:

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right) \right] e^{j\Omega n} d\Omega$$

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right) \right] e^{j\Omega n} d\Omega$$

GFT of periodic signals

- Let prove that the GTF of a periodic signal is true by the the “inverse” definition:

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi}\left[2\pi\sum_{k=0}^{N-1}a_k\delta\left(\Omega - k\Omega_0\right)\right]e^{j\Omega n}d\Omega$$

$$x[n] = \sum_{k=0}^{N-1}a_k\int_0^{2\pi}\delta\left(\Omega - k\Omega_0\right)e^{j\Omega n}d\Omega$$

GFT of periodic signals

- Let prove that the GTF of a periodic signal is true by the the “inverse” definition:

$$x[n] = \sum_{k=0}^{N-1} a_k \int_0^{2\pi} \delta\left(\Omega - k\Omega_0\right) e^{j\Omega n} d\Omega$$

$$\boxed{x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}}$$

- which is the FS of x[n], that is x[n] itself (we proved that).

Other Generalized Fourier Transforms (for signals with infinite energy)

Discrete Time Systems

GFT of non-periodic signals with infinite energy

- In different books/texts, we can find other GTFs.
- **(1)** For instance, a sum of periodic signals (that is not always periodic) has a GTF.
- **(2)** In discrete time, a complex exponential can be also non-periodic (with respect to the discrete time variable n) but always has a GTF.
- **(3)** Also cosine and sine are not always periodic in discrete time, and they can have GTF.

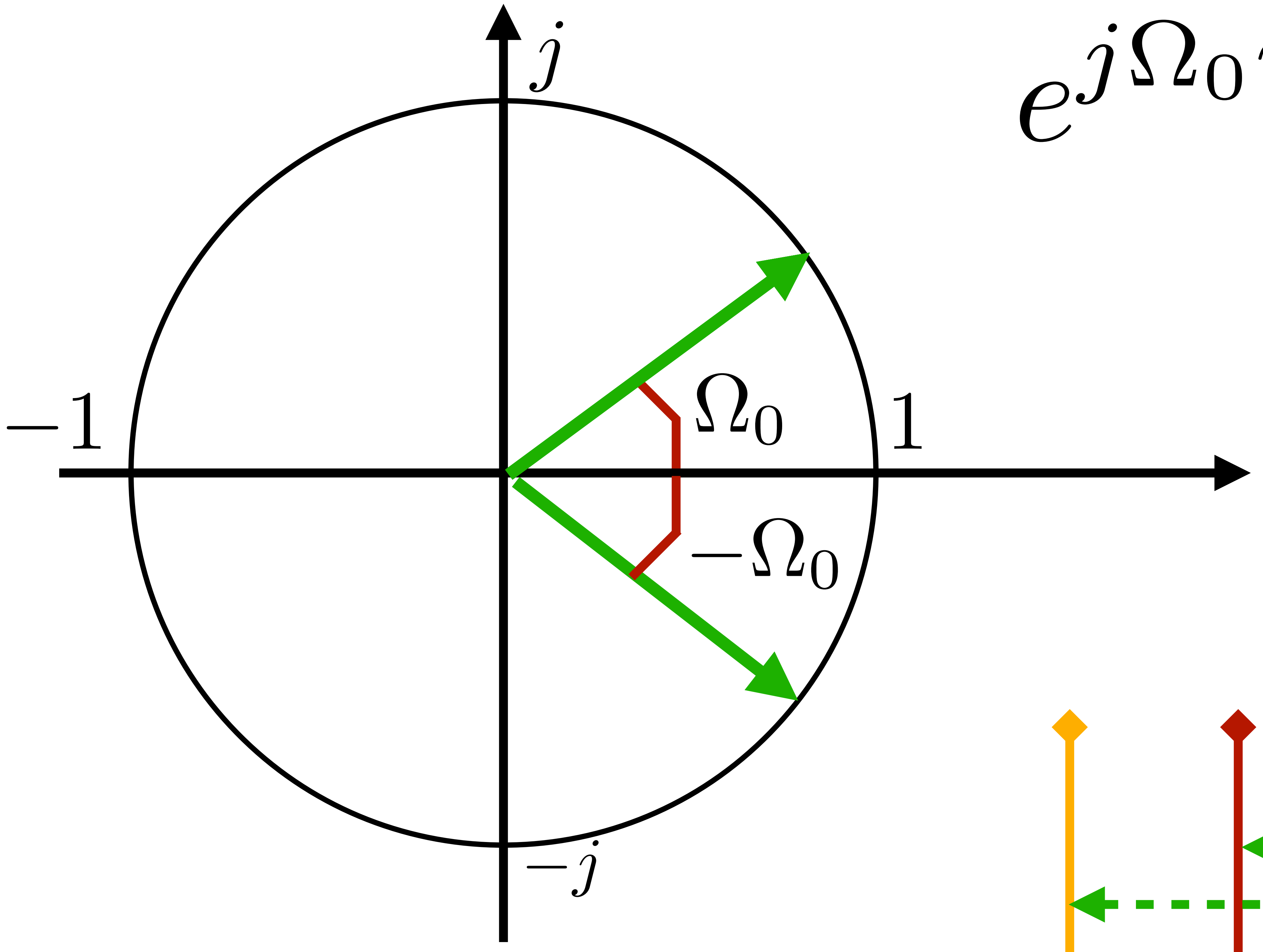
Example

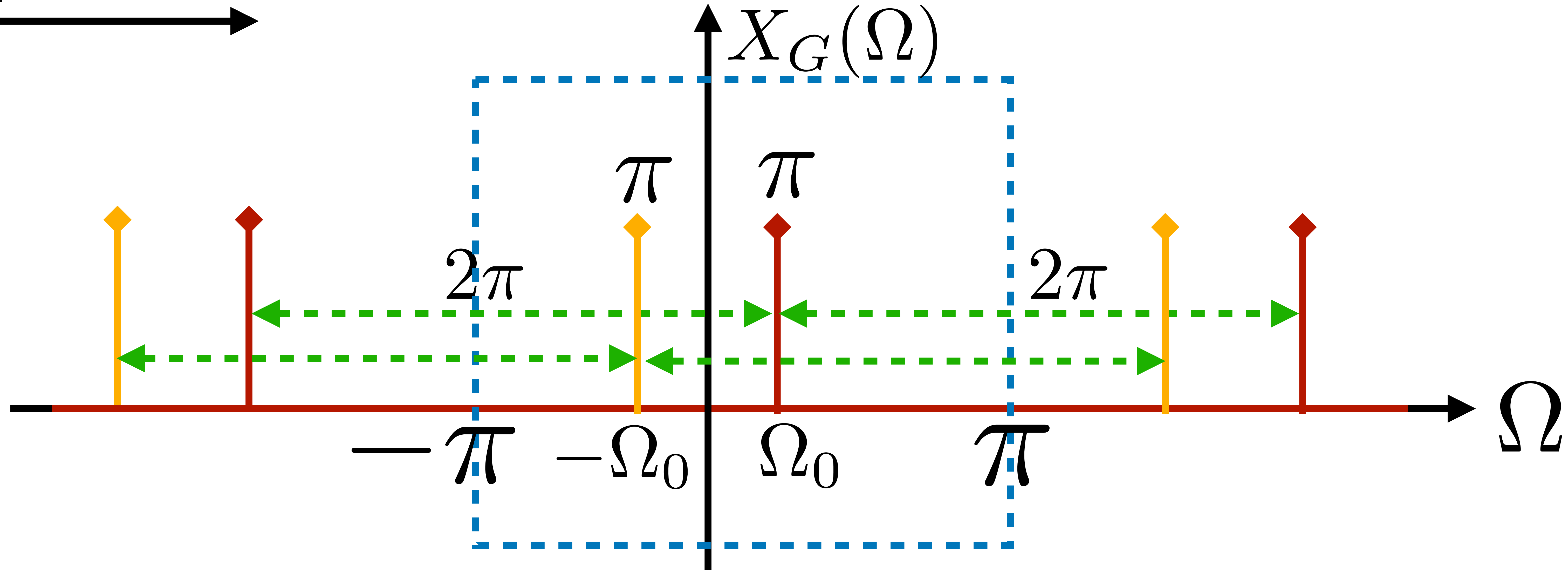
It is periodic $\Omega_0 = 2\pi \frac{m}{N}$

$$x[n] = \cos(\Omega_0 n) \implies \cos(\Omega_0 n) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$$

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi\delta(\Omega - \Omega_0)$$

periodic with 2π





Example

$$x[n] = \cos(\Omega_0 n)$$

It is periodic if $\Omega_0 = 2\pi \frac{m}{N}$

$$\cos(\Omega_0 n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n}$$

if it is periodic, we can write the FS:

$$a_1 = a_{-1} = \frac{1}{2}$$

Considering (N consecutive coefficients) from

$$k = -1, 0, 1..., N-2$$

$$a_k = 0, \text{ for all } k \neq -1, 1 \text{ and}$$

$$a_k = a_{k+N} \Rightarrow a_{mN+1} = a_{mN-1} = 1$$

We could replace all those values in the GTF of periodic signals !

Example

$$x[n] = \cos(\Omega_0 n)$$

Then, in any case, periodic or non-periodic:

$$\cos\Omega_0 n \longrightarrow X_G(\Omega) = \pi \sum_{l=-\infty}^{\infty} \{\delta(\Omega - \Omega_0 - 2\pi l) + \delta(\Omega + \Omega_0 - 2\pi l)\}$$

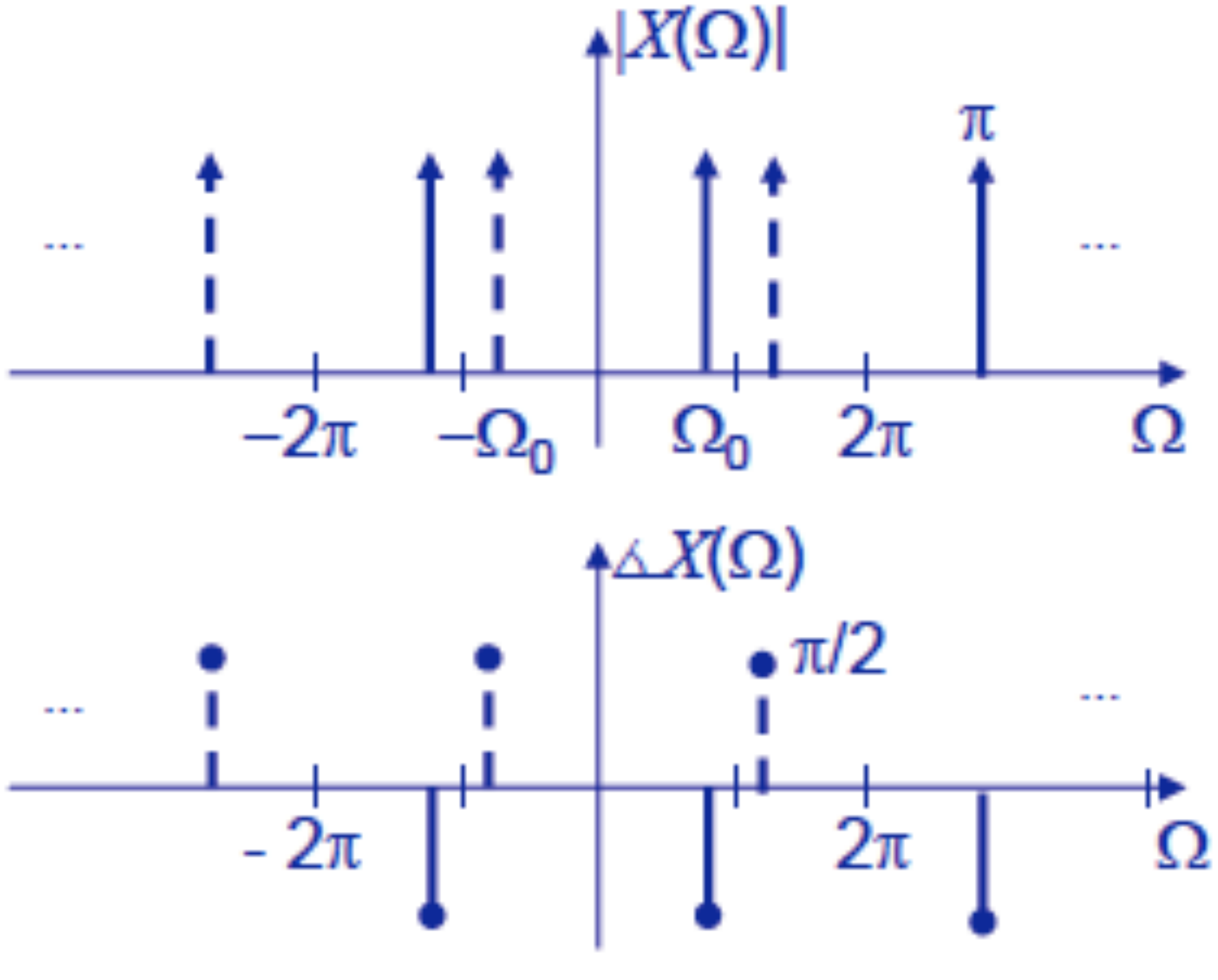
Other Example

$$e^{j\Omega_0 n} \xrightarrow{\mathcal{GTF}} 2\pi\delta(\Omega - \Omega_0)$$

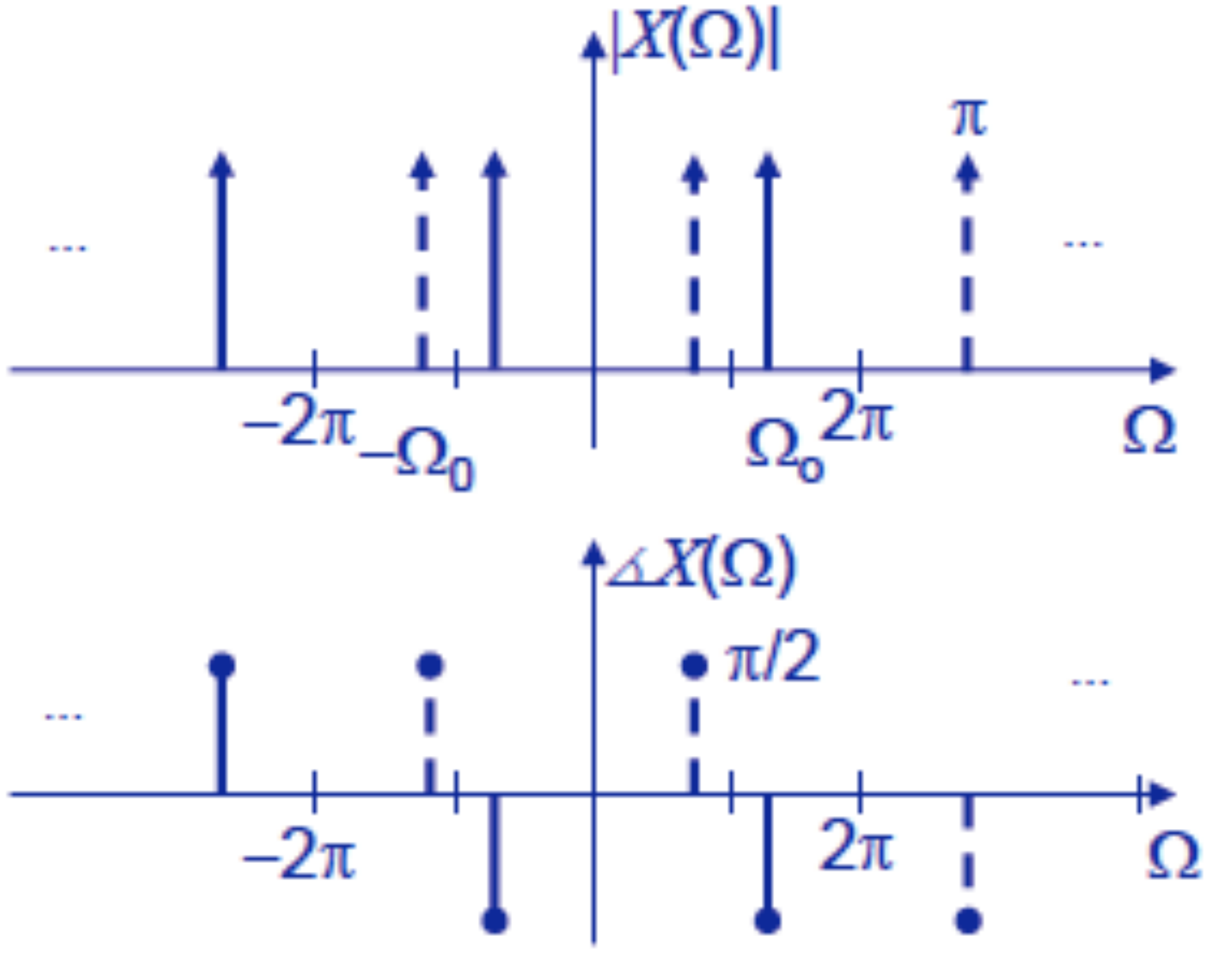
periodic with period 2π

$$x[n] = \text{sen}(\Omega_0 n) = \frac{e^{j\Omega_0 n}}{2j} - \frac{e^{-j\Omega_0 n}}{2j}$$

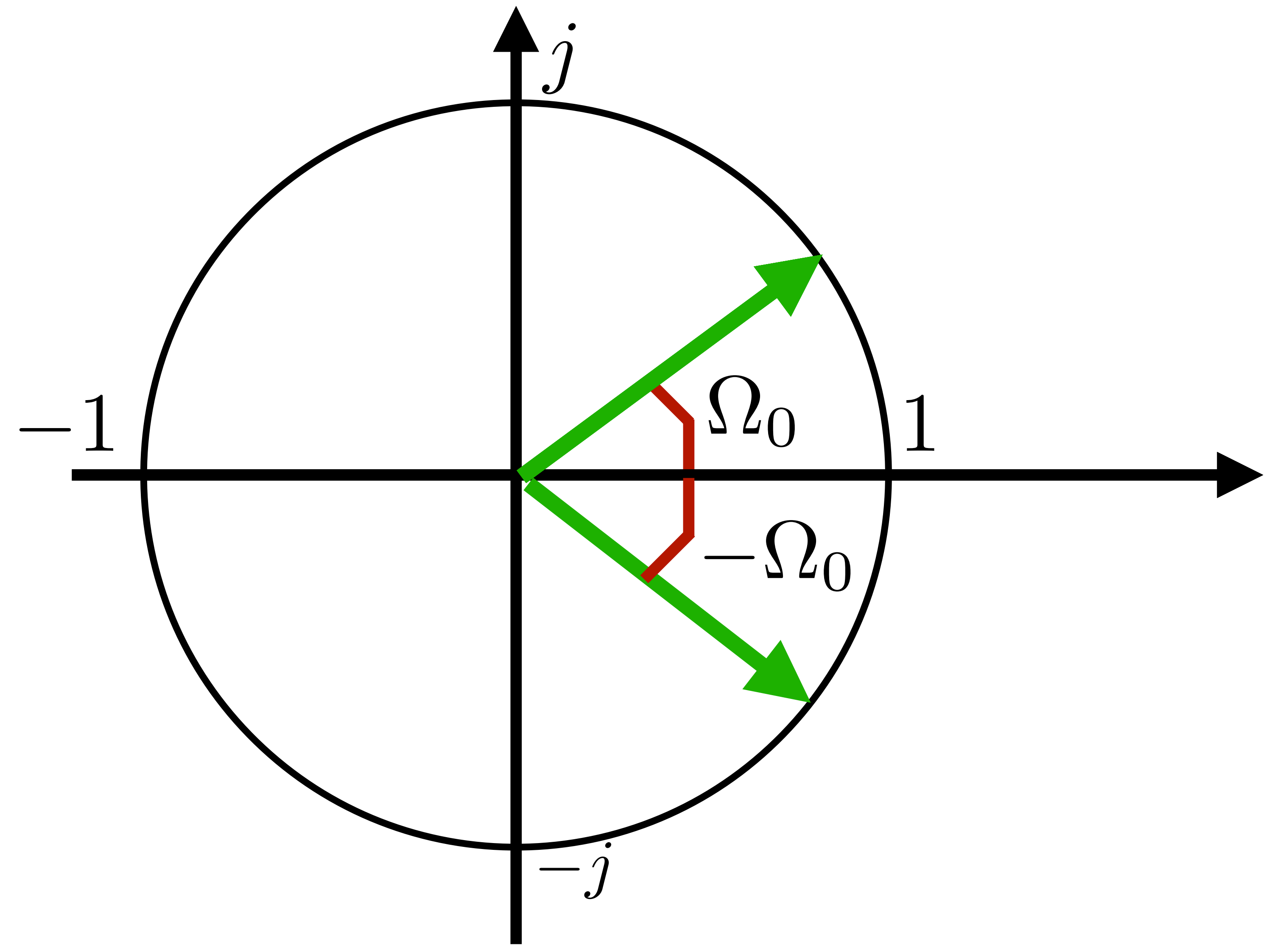
$$X_G(\Omega) = \frac{\pi}{j}\sum_{k=-\infty}^{\infty}\left[\delta(\Omega-\Omega_o-2k\pi)-\delta(\Omega+\Omega_o-2k\pi)\right]$$



Si 0<Ω0<π



Si Ω0>π



GFT of non-periodic signals with infinite energy

- In different books/texts, we could find other GTFs.
- **(4)** more generally, any signals with infinity energy where we can define GTF with the “inverse” definition, have a GTF.

Questions?