OVERVIEW: Fourier series and standard Fourier Transform in CT

Linear systems and circuit applications

Discrete Time Systems

Luca Martino — luca.martino@urjc.es — http://www.lucamartino.altervista.org

Transformations for signal in continuous time

For Periodic signals

For non-periodic signals

Fourier Series (FS)

Stand. Fourier Transform (FT)

Laplace Transform (LT)

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

TABLE SUMMARIZING ALL THE FOURIER SERIES AND ALL THE STANDARD FOURIER TRANSFORMS

	Periódica en el tiempo	No periódica en el tiempo	
Continua en el tiempo $\omega_0 = \frac{2\pi}{T}$	**CTFS** $a_k = X[k] = \frac{1}{T}\int_{<T>} x(t) e^{-jk\omega_0 t} dt$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	**CTFT** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	No periódica en frecuencia
Discreta en el tiempo $\Omega_0 = \frac{2\pi}{N}$	**DTFS** $x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$ $a_k = X[k] = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\Omega_0 n}$	**DTFT** $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$ $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$	Periódica en frecuencia
	Discreta en frecuencia	Continua en frecuencia	

WE WILL SEE:

(in this course)

FOURIER SERIES

STANDARD FOURIER TRANSFORM

Fourier series for signal in CT

FOURIER SERIES =

PROPER MATHEMATICAL TOOL FOR PERIODIC SIGNALS

$$x(t) = x(t + T_0)$$

FUNDAMENTAL FREQUENCY:

$$\omega_0 = \frac{2\pi}{T_0}$$

Periodic signals

- Then the signals also contains **ONLY** the frequencies which are multiple of the fundamental frequency:

$$\omega^{(k)} = k\frac{2\pi}{T_0} = k\omega_0$$

Fourier Series:
first from time, x(t), to frequency (a_k) …

Fourier Series: frequency information?

- Frequency information is **contained in the a_k's**

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

$$a_k = X_k$$

- **INTEGRAL IN A PERIOD - from 0 to To or -To/2 to To/2, for instance.**
- a_k: complex coefficients
- k integer variable

Fourier Series: frequency information?

- Definition of the a_k (it is possible to prove it):

- Analysis equation $$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{T_0}\int_{0}^{T_0} x(t)e^{-jk\omega_0 t}dt \qquad a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt$$

- the definition is a definite integral in a period (**analysis equation**)

Fourier Series:
first from frequency (a_k) to time, x(t)…

Fourier Series (decomposition of the signal)

- Definition of the Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- It is a series then we have to consider the convergence.
- it is a decomposition of the signal x(t) with respect to *the periodic bases*.

Fourier Series

- Definition of the Fourier series (**synthesis equation**):

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- **a_k: complex coefficients**
- **k integer variable**

Fourier Series

- Definition of the Fourier series :

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- **Synthesis equation:** from a_k to x(t)

$$a_k \Longrightarrow x(t)$$

- we need to know a_k

Fourier Series: frequency information?

- Relationship with the frequencies contained in the signal:

$$a_k \Longrightarrow \omega^{(k)} = k\omega_0$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

Which frequencies are into a periodic signal?

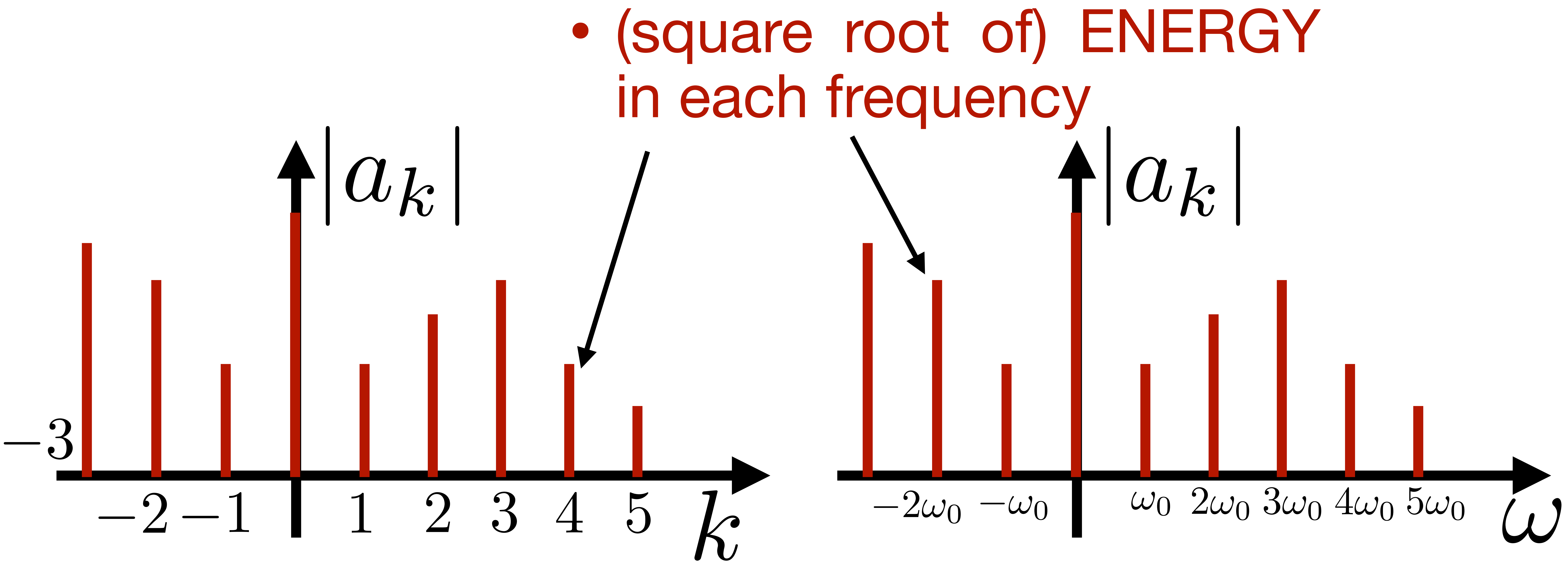
- Does a periodic signal contain all the frequencies?
- **NO!! only the multiple of the fundamental frequency.**

$$a_k \Longrightarrow \omega^{(k)} = k\omega_0$$

$$|a_k|^2\text{: energy at frequency } k\omega_0$$

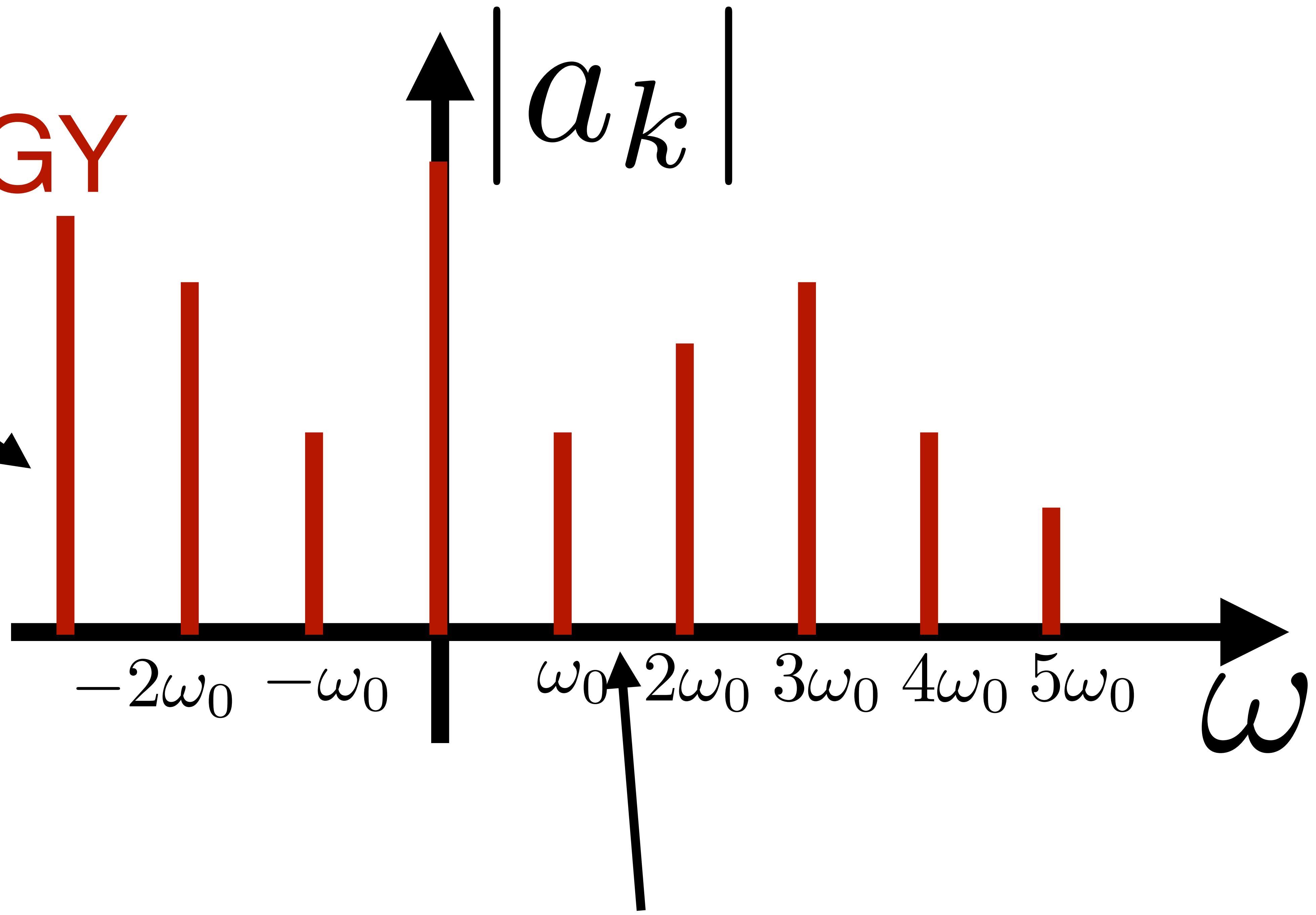
Which frequencies are into a periodic signal?

- The coefficients a_k's are complex numbers



Which frequencies are into a periodic signal?

- Important observation:
- (square root of) ENERGY in each frequency



- In the middle **is not defined**... these frequencies are not contained

Now a brief summary

- Summary for the Fourier series representation for continuous-time periodic signals:

  - **Synthesis equation:** $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
  - **Analysis equation:** $$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

- The integral can be done in each interval of length To (the period)

Now a brief summary

- Synthesis equation: from frequency to time
- Analysis equation: from time to frequency

Other forms of seeing the FS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k [\cos(k\omega_0 t) + j\sin(k\omega_0 t)]$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cos(k\omega_0 t) + j \sum_{k=-\infty}^{+\infty} a_k \sin(k\omega_0 t)$$

Other forms of seeing the FS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cos(k\omega_0 t) + \sum_{k=-\infty}^{+\infty} b_k \sin(k\omega_0 t)$$

with $b_k = +j a_k$

Recall: a_k are complex numbers, in general

Other forms of seeing the FS

- **There are many other mathematical forms: depending if the signal is real, if the signal is odd or even, or odd and real, or even and real etc…** (you will see some examples later)
- **But it is the same "mathematical tool"**
- (se puede escribir de diferentes formas, basta ser coherentes con la ecuación de análisis y síntesis, etc.)

Convergence

- Consider a **truncated Fourier Series:**

$$x(t) \approx \sum_{k=-N}^{N} a_k e^{+jk\omega_0 t}$$

when N goes to infinity, we recover the Fourier Series.

Convergence

when N goes to infinity, we recover the Fourier Series:

$$x(t) \approx a_{-N}e^{-jN\omega_0 t} + \ldots a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots + a_N e^{jN\omega_0 t}$$

2N+1 is the number of total components in the sum!

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{+jk\omega_0 t}$$

Convergence

- We can define the "error signal" - error in approximation as:

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- y definimos el error (en un periodo) como:

$$E_N = \int_{T_0} |e_N(t)|^2 dt$$

Convergence

- **For a fixed N:** It is possible to prove that the best choice (i.e., that minimizes) of the coefficient a_k is

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

- i.e., the coefficients of the Fourier Series (that we have already defined and studied).

- **Increasing N, with the a_k above:** the error E_N decreases as N grows, i.e.,

$$E_N \to 0 \quad \text{as} \quad N \to +\infty$$

Convergence: example

think to a periodic "rectangular" signal ....

Convergence example

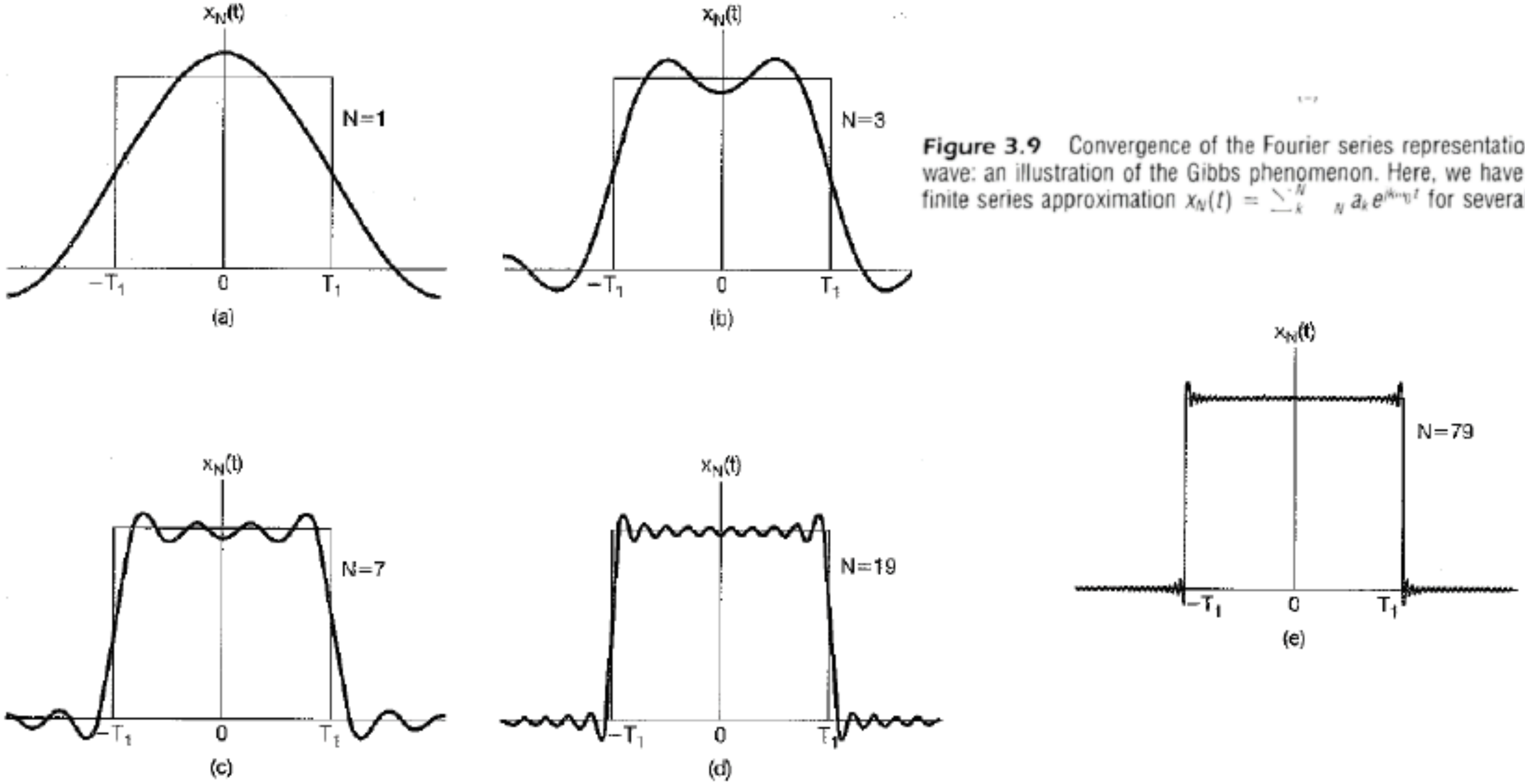


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

Convergence: Gibbs phenomenon

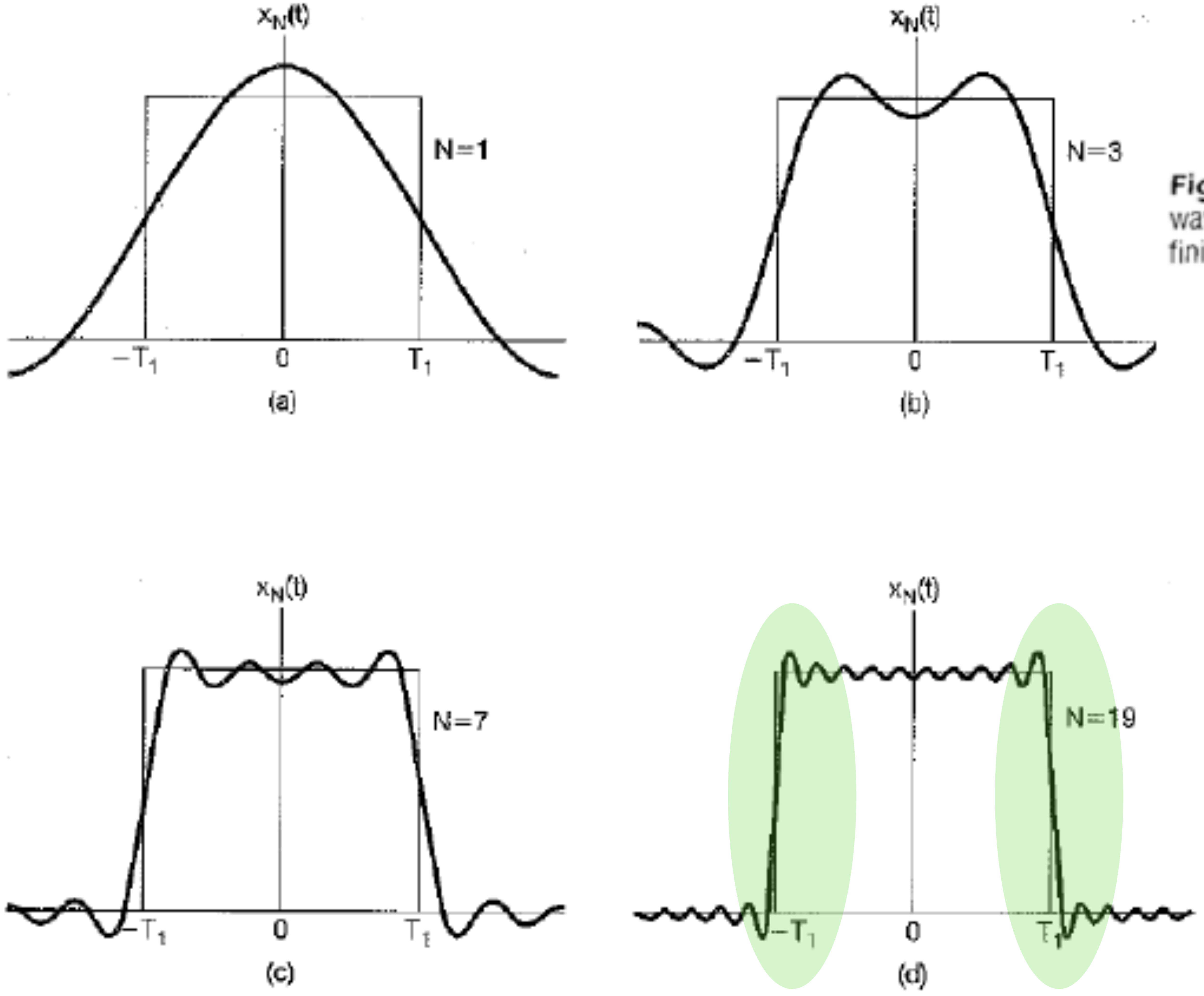
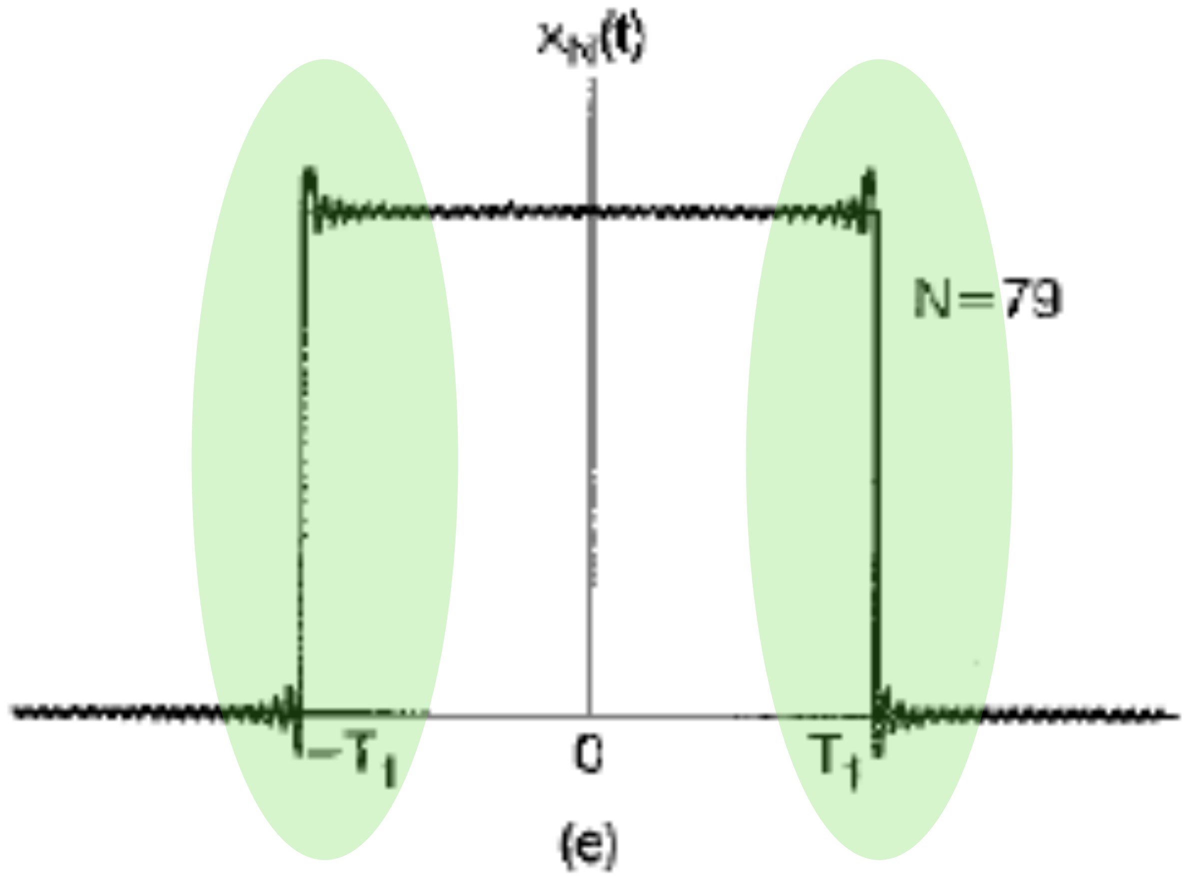


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.



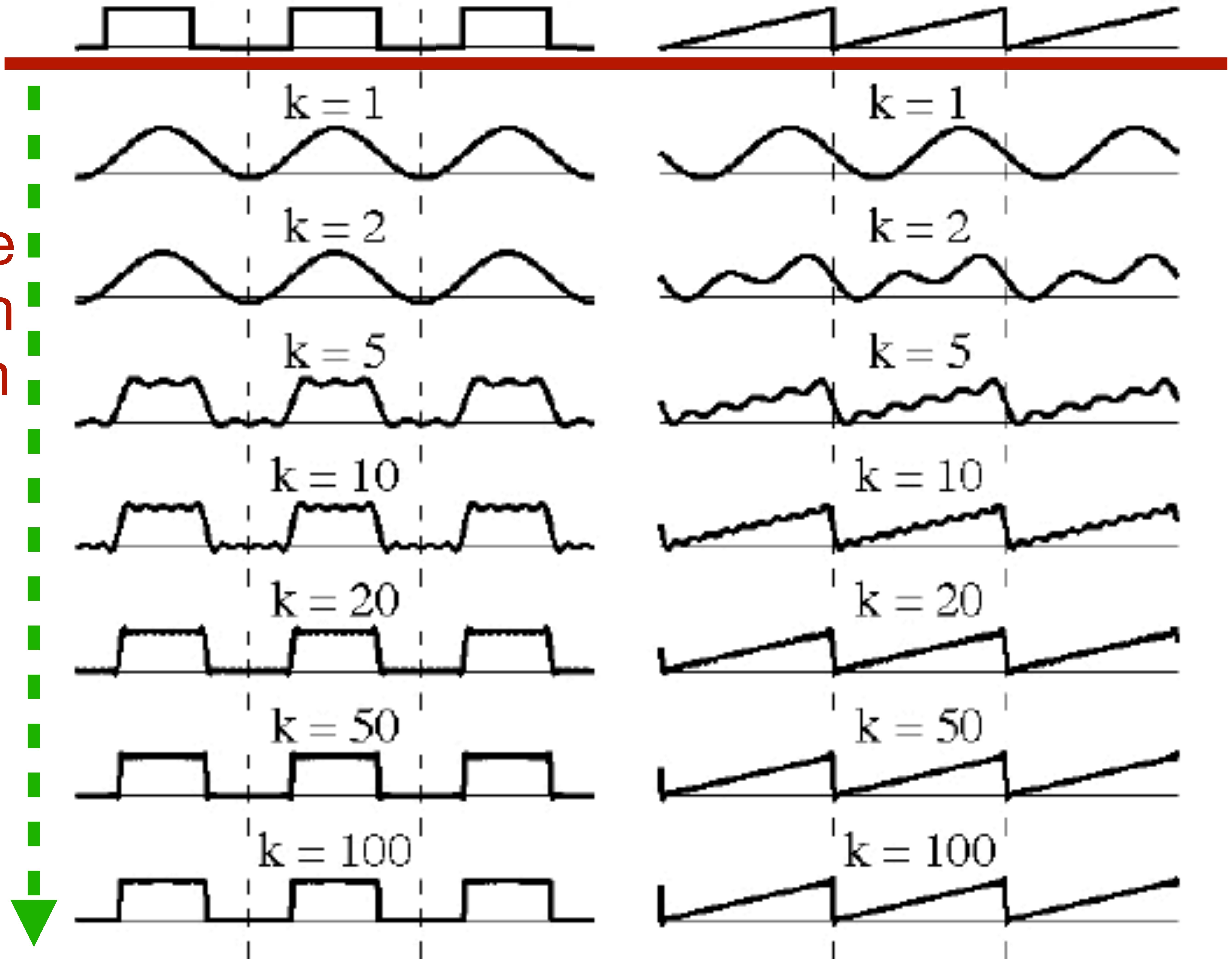
- some problem in the discontinuities

Convergence: examples

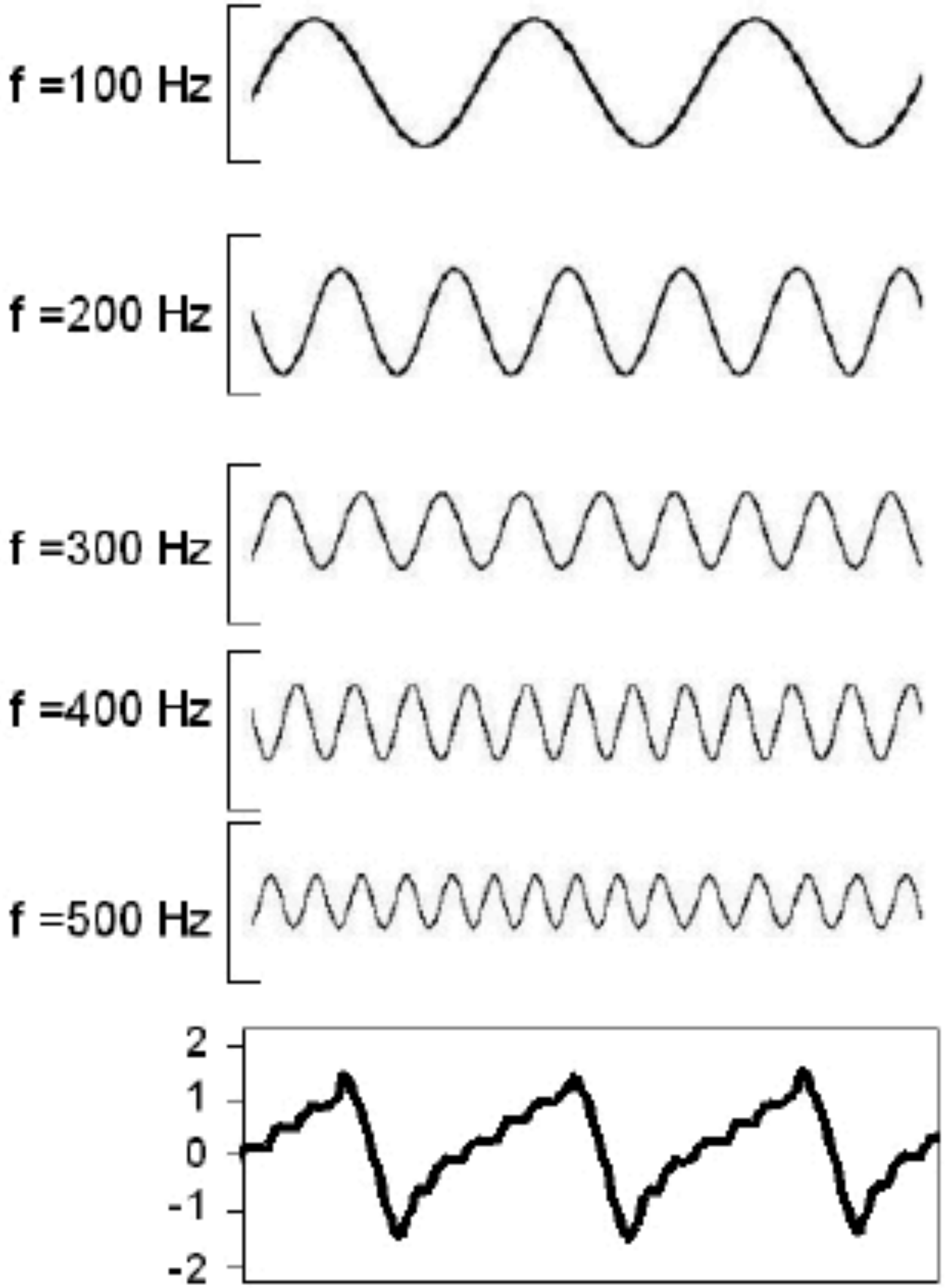
- Cualquier función periódica puede ser representada por la suma de senos y cosenos de diferentes amplitudes y frecuencias

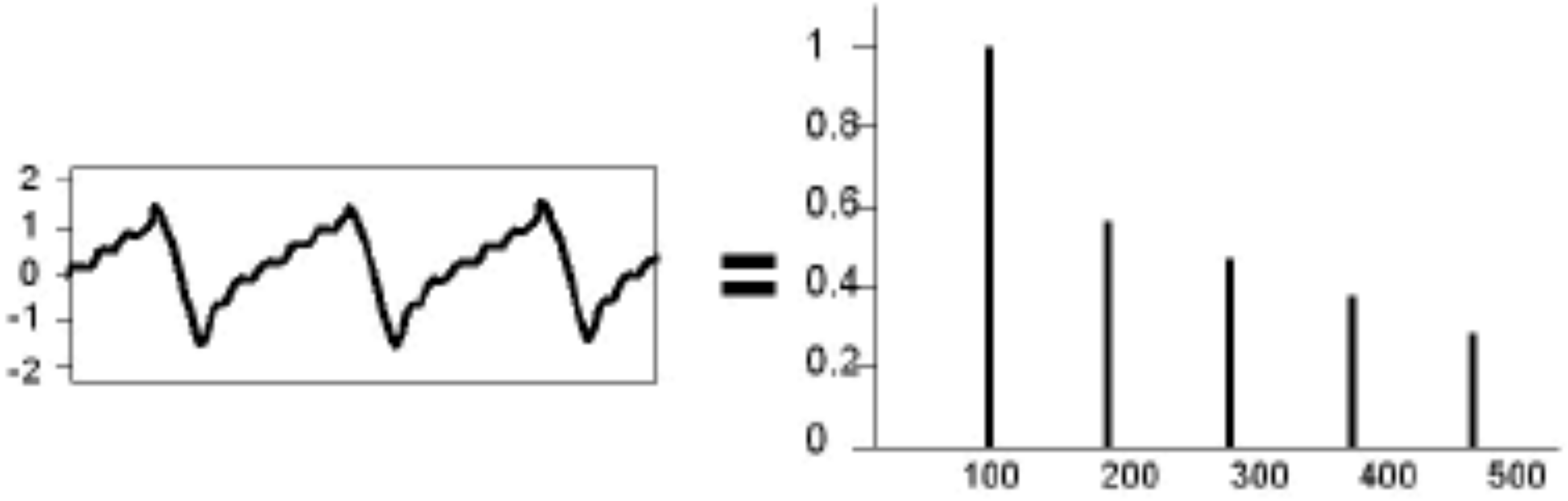
• Signal:

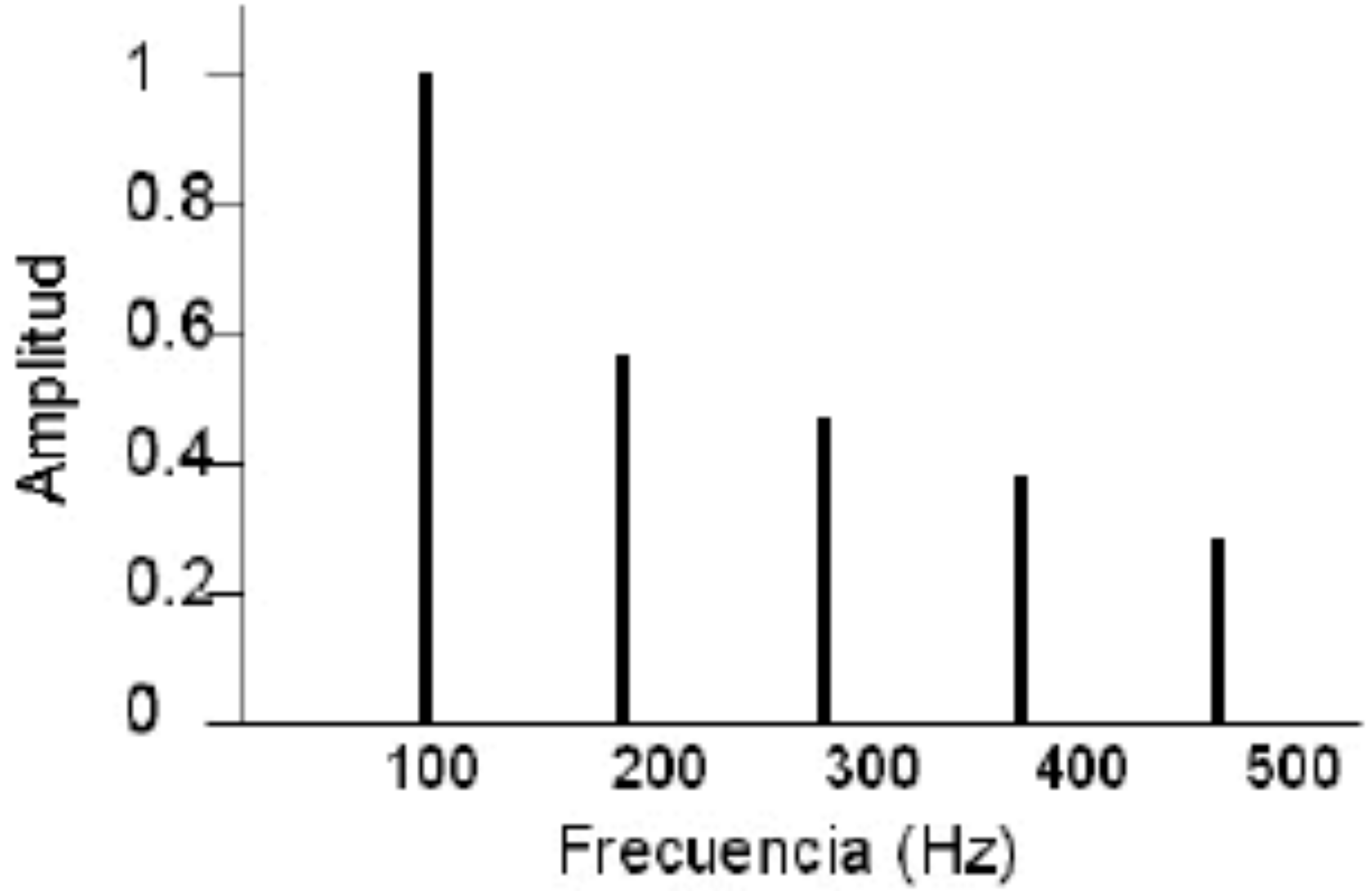
Increasing the components in the truncated sum



Convergence: examples







Example: Fourier series of the cosine

$$x(t) = \cos(\omega_0 t)$$

- For Euler…

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

- Sum of complex exponentials… this is already the Fourier Series !!!

Example: Fourier series of the cosine

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

- Sum of complex exponentials… this is already the Fourier Series !!!

$$a_1 = \frac{1}{2}, \qquad a_{-1} = \frac{1}{2},$$

$$a_k = 0 \text{ for all } k \neq -1, 1$$

Example: Fourier series of the sine

$$x(t) = \sin(\omega_0 t)$$

- For Euler…

$$\sin(\omega_0 t) = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

- Sum of complex exponentials… this is already the Fourier Series !!!

Example: Fourier series of the cosine

$$\sin(\omega_0 t) = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

- Sum of complex exponentials... this is already the Fourier Series !!!

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0 \text{ for all } k \neq -1, 1$$

Does The Fourier series exist always?

- The Fourier series decomposition exists ***almost for all*** the periodic signals.
- The integral of synthesis equation must exist:

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

since it is computed in a finite region - such as [0,To] - there are "a lot of chances" that this integral exists and is a finite value, for all k.

Next?

- non-periodic signals in frequency domain
- some non-periodic signals…
- All the non-periodic signals, *with finite energy*, have *Standard Fourier Transform*…

Transformations for signal in continuous time

For Periodic signals | **For non-periodic signals**

Fourier Series (FS) | **Stand. Fourier Transform (FT)** | **Laplace Transform (FT)**

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- **Analysis equations:** from x(t) to the transformed domain

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$\omega \in \mathbb{R}$$

- **Special case of the Laplace Transform when:**

$$\sigma = 0, \qquad s = 0 + j\omega$$

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\omega \in \mathbb{R}$$

- **But it takes complex values, i.e.,**

$$X(\omega) \in \mathbb{C}$$

Stand. Fourier Transform (analysis equation)

- Definition (analysis equation):

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)dt$$

$$\omega \in \mathbb{R}$$

- **But it takes complex values, i.e.,**

$$X(\omega) \in \mathbb{C}$$

Stand. Fourier Transform

- **But it takes complex values, i.e.,**

$$X(\omega) \in \mathbb{C}$$

- **then, in general, we can plot module/phase, real part and imaginary part…**

Synthesis equation: Inverse Fourier Transform

- Definition (synthesis equation):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

- **Synthesis equations:** from the transformed domain to x(t)

Brief summary

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Inverse Fourier Transform
  Ec. de síntesis

- Stand. Fourier Transform
  Ec. de análisis,

Existence of the Fourier Transform

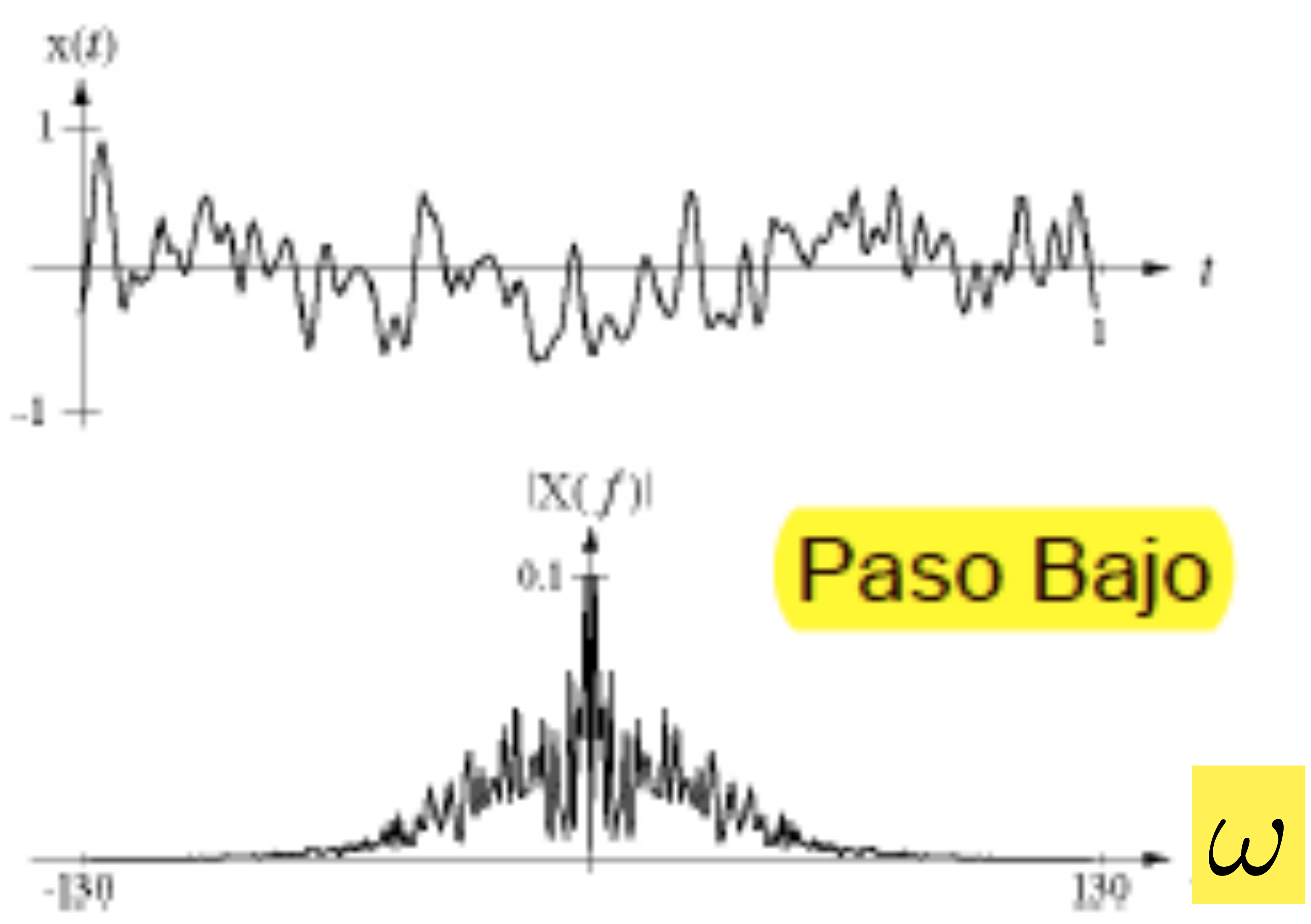
- The integral below must exist:

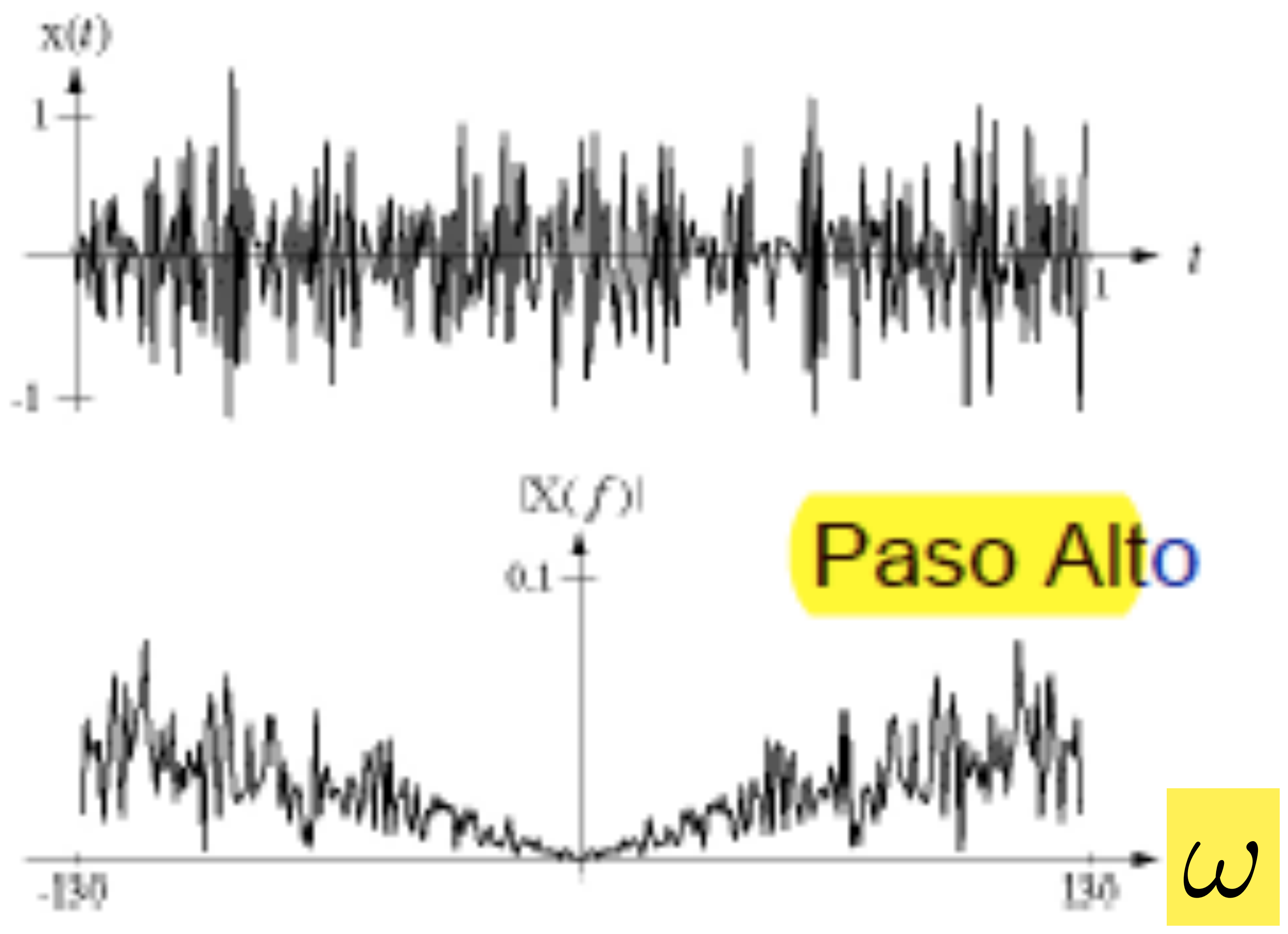
$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

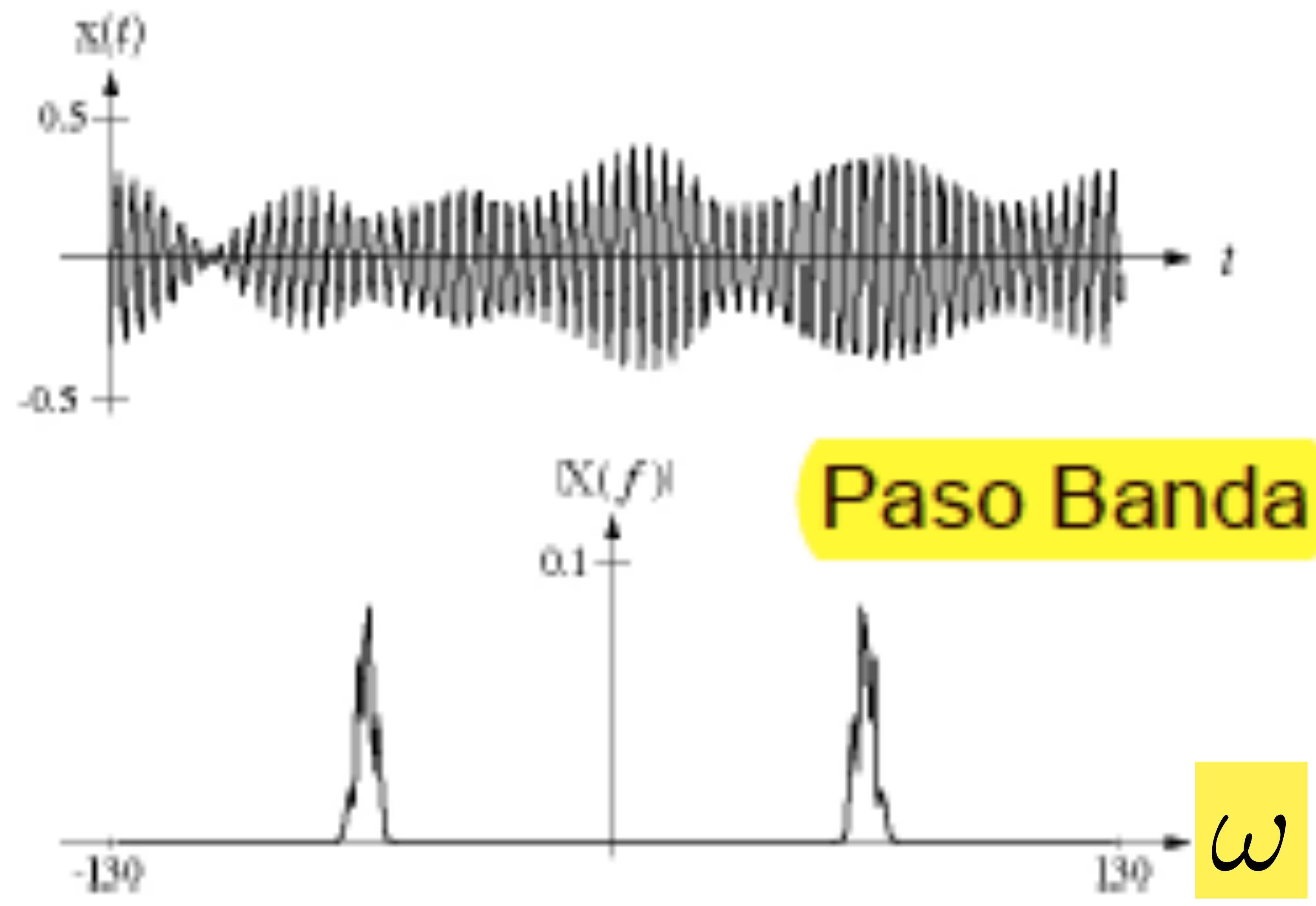
- **Sufficient condition (signal with finite energy):**

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

Low-pass, High-pass, Band-pass filters







Questions?