About the periodic brother of the truncated cosine in CT

Discrete Time Systems

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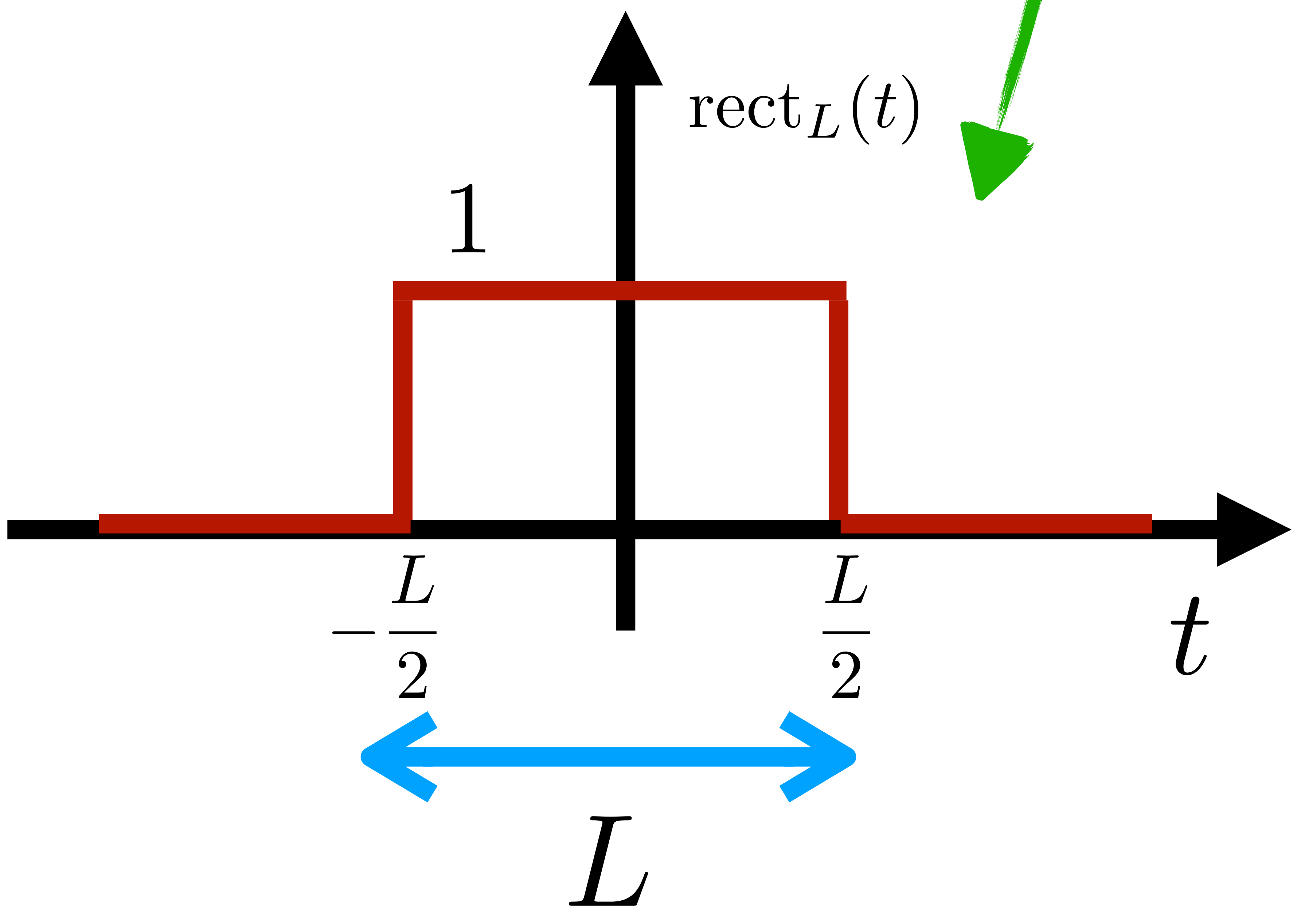
Truncated cosine signal

$$x(t) = \cos(\omega_0 t)\mathrm{rect}_L(t)$$



Truncated cosine signal in frequency

This is already the solution:

$$X(\omega) = \frac{\sin((\omega - \omega_0)\frac{L}{2})}{\omega - \omega_0} + \frac{\sin((\omega + \omega_0)\frac{L}{2})}{\omega + \omega_0}$$

That can be written as sum of two "octupus"/pulpos: (i.e., two sinc functions)

$$X(\omega) = \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega - \omega_0) + \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega + \omega_0)$$

Truncated cosine signal in frequency

$$X(\omega) = \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega - \omega_0) + \frac{1}{2}\text{sinc}_{\frac{L}{2}}(\omega + \omega_0)$$

where:

$$\text{sinc}_{\frac{L}{2}}(\omega) = \frac{2\sin(\omega\frac{L}{2})}{\omega}$$

We consider now a *special case*
of truncated cosine and “periodic brother”

SPECIAL CASE of truncated cosine signal

$$x(t) = \cos(\omega_0 t)\mathrm{rect}_L(t)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

We choose L such that: $L = T_0$

Special case

i.e., we consider just one period of the cosine

$$x(t) = \cos\left(\omega_0 t\right)\mathrm{rect}_{T_0}(t)$$

Special case of truncated cosine

In this case, we saw that we have:

$$X(\omega) = \frac{\sin\left((\omega - \omega_0)\frac{T_0}{2}\right)}{\omega - \omega_0} + \frac{\sin\left((\omega + \omega_0)\frac{T_0}{2}\right)}{\omega + \omega_0}$$

Now, we look for the w such that X(w)=0

Now, we look for the w such that X(w)=0, for this special case.

Zeros of X(w) of a "special" truncated cosine

First of all, we focus on the first numerator:

$$\sin\left((\omega-\omega_0)\frac{T_0}{2}\right)=0 \quad \xrightarrow{\omega_0=\frac{2\pi}{T_0}} \quad \sin\left(\left(\omega-\frac{2\pi}{T_0}\right)\frac{T_0}{2}\right)=0$$

$$\sin\left(\omega\frac{T_0}{2}-\pi\right)=0$$

$$\omega\frac{T_0}{2}-\pi=k\pi, \quad \text{with } k \text{ any positive or negative integer, } k=...-3,-2,-1,0,1,2,3,..$$

$$\omega=(k+1)\pi\frac{2}{T_0}=(k+1)\frac{2\pi}{T_0}=(k+1)\omega_0$$

Zeros of X(w) of a "special" truncated cosine

Then, we obtain:

$$\omega = (k+1)\omega_0$$

$$\omega = k'\omega_0$$

with k' any positive or negative integer, $k' = ... -3, -2, -1, 0, 1, 2, 3, ..$

So that, we have obtained:

$$\sin\left((\omega - \omega_0)\frac{T_0}{2}\right) = 0 \longrightarrow \omega = k'\omega_0$$

Zeros of X(w) of a "special" truncated cosine

Secondly, we focus on the second numerator:

$$\sin\left((\omega+\omega_0)\frac{T_0}{2}\right)=0 \quad \xrightarrow{\omega_0=\frac{2\pi}{T_0}} \quad \sin\left(\left(\omega+\frac{2\pi}{T_0}\right)\frac{T_0}{2}\right)=0$$

$$\sin\left(\omega\frac{T_0}{2}+\pi\right)=0$$

$$\omega\frac{T_0}{2}+\pi=k\pi,$$ with k any positive or negative integer, $k=\ldots-3,-2,-1,0,1,2,3,..$

$$\omega=(k-1)\pi\frac{2}{T_0}=(k-1)\frac{2\pi}{T_0}=(k-1)\omega_0$$

Zeros of X(w) of a “special” truncated cosine

Then, we obtain:

$$\omega = (k-1)\omega_0$$

$$\omega = k''\omega_0$$

with k'' any positive or negative integer, $k'' = ... -3, -2, -1, 0, 1, 2, 3, ..$

So that, we have obtained:

$$\sin\left((\omega + \omega_0)\frac{T_0}{2}\right) = 0 \longrightarrow \omega = k''\omega_0$$

Zeros of X(w) of a "special" truncated cosine

$$\sin\left((\omega - \omega_0)\frac{T_0}{2}\right) = 0 \quad \longleftrightarrow \quad \omega = k'\omega_0$$

$$\sin\left((\omega + \omega_0)\frac{T_0}{2}\right) = 0 \quad \longleftrightarrow \quad \omega = k''\omega_0$$

Where k' and k" are positive or negative integers.

Therefore, finally we get:

$$X(\omega) = \frac{\sin\left((\omega - \omega_0)\frac{T_0}{2}\right)}{\omega - \omega_0} + \frac{\sin\left((\omega + \omega_0)\frac{T_0}{2}\right)}{\omega + \omega_0}$$

$$X(\omega) = 0 \quad \longleftrightarrow \quad \omega = k\omega_0$$

IMP: for $k \neq 1, -1$

Zeros of X(w) of a “special” truncated cosine

$$X(\omega) = \frac{\sin\left((\omega - \omega_0)\frac{T_0}{2}\right)}{\omega - \omega_0} + \frac{\sin\left((\omega + \omega_0)\frac{T_0}{2}\right)}{\omega + \omega_0}$$

$$X(\omega) = 0 \longleftrightarrow \omega = k\omega_0$$

IMP:

for $k \neq 1, -1$

The reason of this is the denominators:

$$k = 1 \Longrightarrow X(\omega_0) = \frac{0}{0} + 0$$

$$k = -1 \Longrightarrow X(-\omega_0) = 0 + \frac{0}{0}$$

Zeros of X(w) of a “special” truncated cosine

$$k = 1 \implies X(\omega_0) = \frac{0}{0} + 0$$

$$k = -1 \implies X(-\omega_0) = 0 + \frac{0}{0}$$

…and recalling the sinc function, we can prove:

$$k = 1 \implies X(\omega_0) = \frac{T_0}{2}$$

$$k = -1 \implies X(-\omega_0) = \frac{T_0}{2}$$

NOW, we consider the “PERIODIC BROTHER” (for this special case) REPEATING THE SIGNAL EXACTLY EACH T_o ==> *NAMELY, WE OBTAIN THE STANDARD COSINE !!!!*

PERIODIC BROTHER…with period To

$$x(t) = \cos\left(\omega_0 t\right) \mathrm{rect}_{T_0}(t)$$

If we considering a periodic PERIODIC BROTHER with period To, WE OBTAIN THE STANDARD COSINE !!!

$$\widetilde{x}(t) = \cos(\omega_0 t)$$

PERIODIC BROTHER…with period To

Using the formula for the "periodic brother":

$$a_k = \frac{1}{T_0} X(k\omega_0)$$

where:

$$X(\omega) = \frac{\sin\left((\omega - \omega_0)\frac{T_0}{2}\right)}{\omega - \omega_0} + \frac{\sin\left((\omega + \omega_0)\frac{T_0}{2}\right)}{\omega + \omega_0}$$

PERIODIC BROTHER…with period To

We obtain:

$$a_k = \frac{1}{T_0} X(k\omega_0) = \frac{1}{T_0} \frac{\sin\left(\left(k\omega_0 - \omega_0\right)\frac{T_0}{2}\right)}{k\omega_0 - \omega_0} + \frac{1}{T_0} \frac{\sin\left(\left(k\omega_0 + \omega_0\right)\frac{T_0}{2}\right)}{k\omega_0 + \omega_0}$$

and recalling:

$$X(\omega) = 0 \longleftrightarrow \omega = k\omega_0$$

IMP:

for $k \neq 1, -1$

$$k = 1 \Longrightarrow X(\omega_0) = \frac{T_0}{2}$$

$$k = -1 \Longrightarrow X(-\omega_0) = \frac{T_0}{2}$$

PERIODIC BROTHER…with period To

$$a_k = \frac{1}{T_0} X(k\omega_0) = \frac{1}{T_0} \frac{\sin\left(\left(k\omega_0 - \omega_0\right)\frac{T_0}{2}\right)}{k\omega_0 - \omega_0} + \frac{1}{T_0} \frac{\sin\left(\left(k\omega_0 + \omega_0\right)\frac{T_0}{2}\right)}{k\omega_0 + \omega_0}$$

We finally get:

$$a_k = 0 \text{ for } k \neq -1, 1$$

$$a_1 = \frac{1}{T_0}\frac{T_0}{2} + 0 = \frac{1}{2}$$

as we expect !!!!!

$$a_{-1} = 0 + \frac{1}{T_0}\frac{T_0}{2} = \frac{1}{2}$$