SOLVED problems - Convolution Sum - part 1

Discrete Time Systems (DTS)

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Example - Problem 1

Given:

$$x[0] = 2, \quad x[1] = -3, \quad x[n] = 0 \text{ otherwise}$$

$$h[-2] = 1, h[-1] = 5, h[0] = 0, h[1] = -3, \quad h[n] = 0 \text{ otherwise}$$

Obtain the output of the LTI system y[n], with impulse response h[n]

Namely, y[n]=?

Example - Problem 1

We know that:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

Example - Problem 1

Moreover, in this example x[n] and h[n] have <u>finite lengths</u> and, in this scenario, there are (*at least*) two ways/methods of computing the convolution sum.

We will do both (and also with two sub-versions).

Example - Problem 1

In both methods, it is useful to recall that with signals with finite length, we have the following "a-priori" information:

(1) y[n] starts at = start of x[n] + start of h[n]
(2) y[n] finishes at = end of x[n] + end of h[n]
(3) the length of y[n] = length of x[n]+ length of h[n]-1

The rest of y[n] is for "sure" zero …

Example - Problem 1

Then, in this example:

- start of x[n] at 0, start of h[n] at -2
- end of x[n] at 1, end of h[n] at 1
- the length of x[n] is 2
- the length of h[n] is 4

(1) y[n] starts at = 0+2=-2

(2) y[n] finishes at = 1+1=2

(3) the length of y[n] = 2+4-1=5

Example - Problem 1

FIRST METHOD - first way: Note that we can write

$$x[n] = 2\delta[n] - 3\delta[n-1]$$

And for the property of convolution with deltas, we have:

$$y[n] = x[n] * h[n] = 2h[n] - 3h[n-1]$$

Example - Problem 1

FIRST METHOD - first way

$$y[n] = x[n] * h[n] = 2h[n] - 3h[n-1]$$

We build a table:

n	$y[n]$
-3	$2h[-3] - 3h[-4] = 0$
-2	$2h[-2] - 3h[-3] = 2$
-1	$2h[-1] - 3h[-2] = 7$
0	$2h[0] - 3h[-1] = -15$
1	$2h[1] - 3h[0] = -6$
2	$2h[2] - 3h[1] = 9$
3	$2h[3] - 3h[2] = 0$
4	$2h[4] - 3h[3] = 0$

Example - Problem 1

FIRST METHOD - first way

Solution:

$$y[-2] = 2, y[-1] = 7, y[0] = -15, y[1] = -6, y[2] = 9$$

$$y[n] = 0 \text{ otherwise}$$

This is already the solution… but we solve in other ways…

Example - Problem 1

FIRST METHOD - second way: Note that we can write

$$h[n] = \delta[n+2] + 5\delta[n+1] - 3\delta[n-1]$$

And for the property of convolution with deltas, we have:

$$y[n] = x[n] * h[n] = x[n+2] + 5x[n+1] - 3x[n-1]$$

(IMP. REMARK: note that in this case, we have y[n] expressed as a finite linear combination of translated version of x[n]… we have already the *linear difference equation* - with constant coefficients - associated to our LTI system.)

Example - Problem 1

FIRST METHOD - second way

$$y[n] = x[n] * h[n] = x[n+2] + 5x[n+1] - 3x[n-1]$$

We build a table:

n	$y[n]$
-3	$x[-1] + 5x[-2] - 3x[-4] = 0$
-2	$x[0] + 5x[-1] - 3x[-3] = 2$
-1	$x[1] + 5x[0] - 3x[-2] = 7$
0	$x[2] + 5x[1] - 3x[-1] = -15$
1	$x[3] + 5x[2] - 3x[0] = -6$
2	$x[4] + 5x[3] - 3x[1] = 9$
3	$x[5] + 5x[4] - 3x[2] = 0$
4	$x[6] + 5x[5] - 3x[3] = 0$

…obtaining the same solution !!!

Example - Problem 1

SECOND METHOD - first way: Note that

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

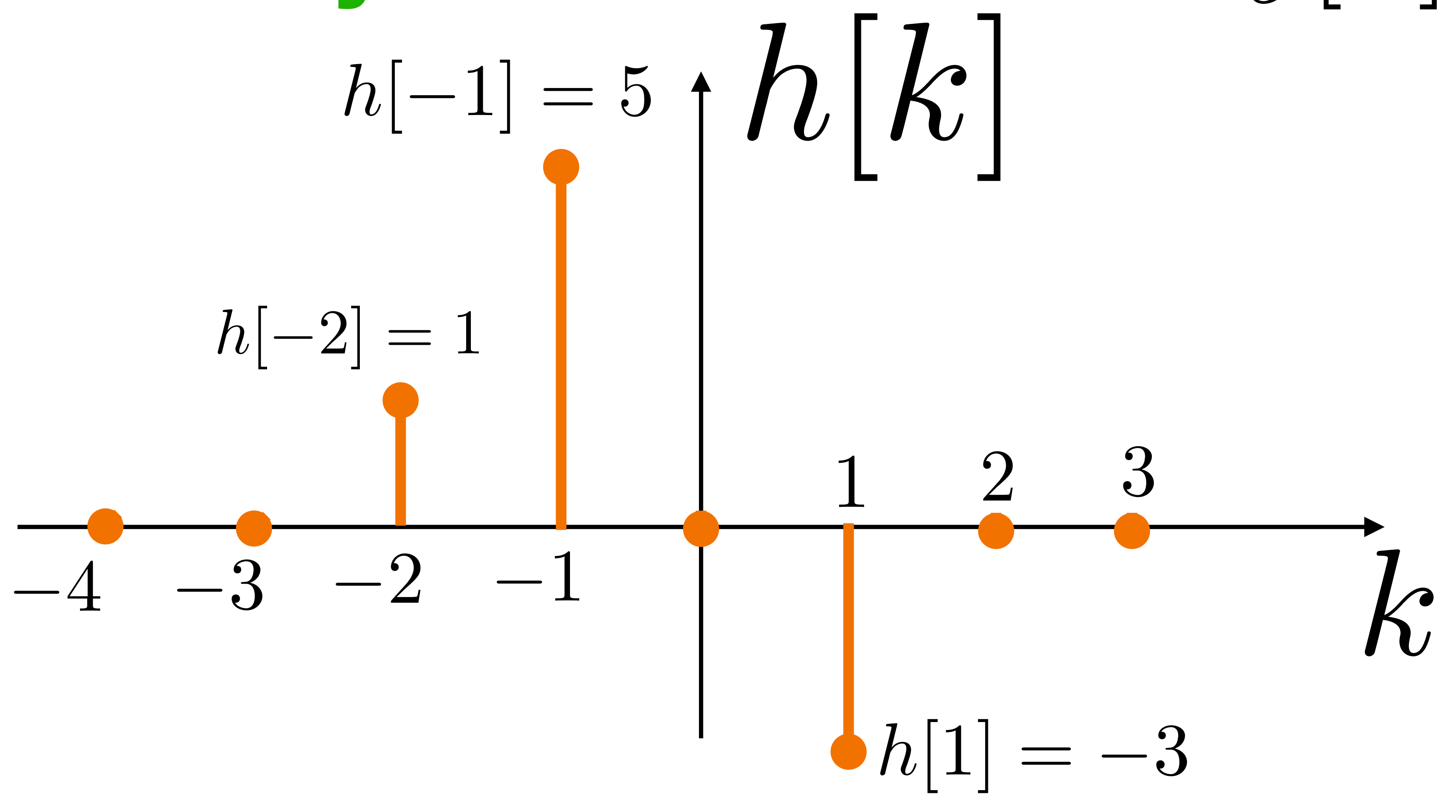
$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]z[k-n]$$

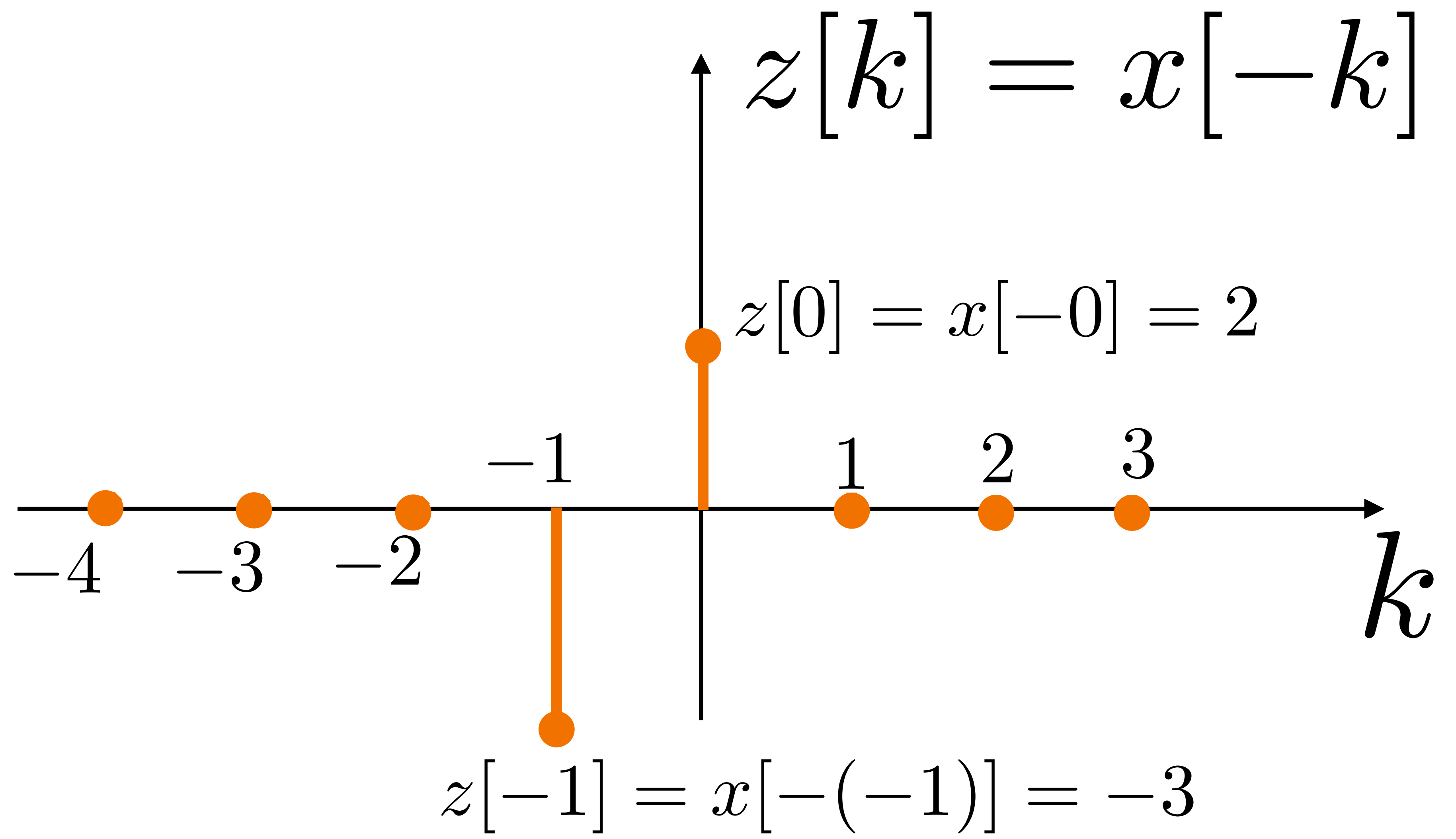
$$z[k] = x[-k]$$

Example - Problem 1

SECOND METHOD - first way: Note that

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]z[k-n]$$





Example - Problem 1

SECOND METHOD - first way

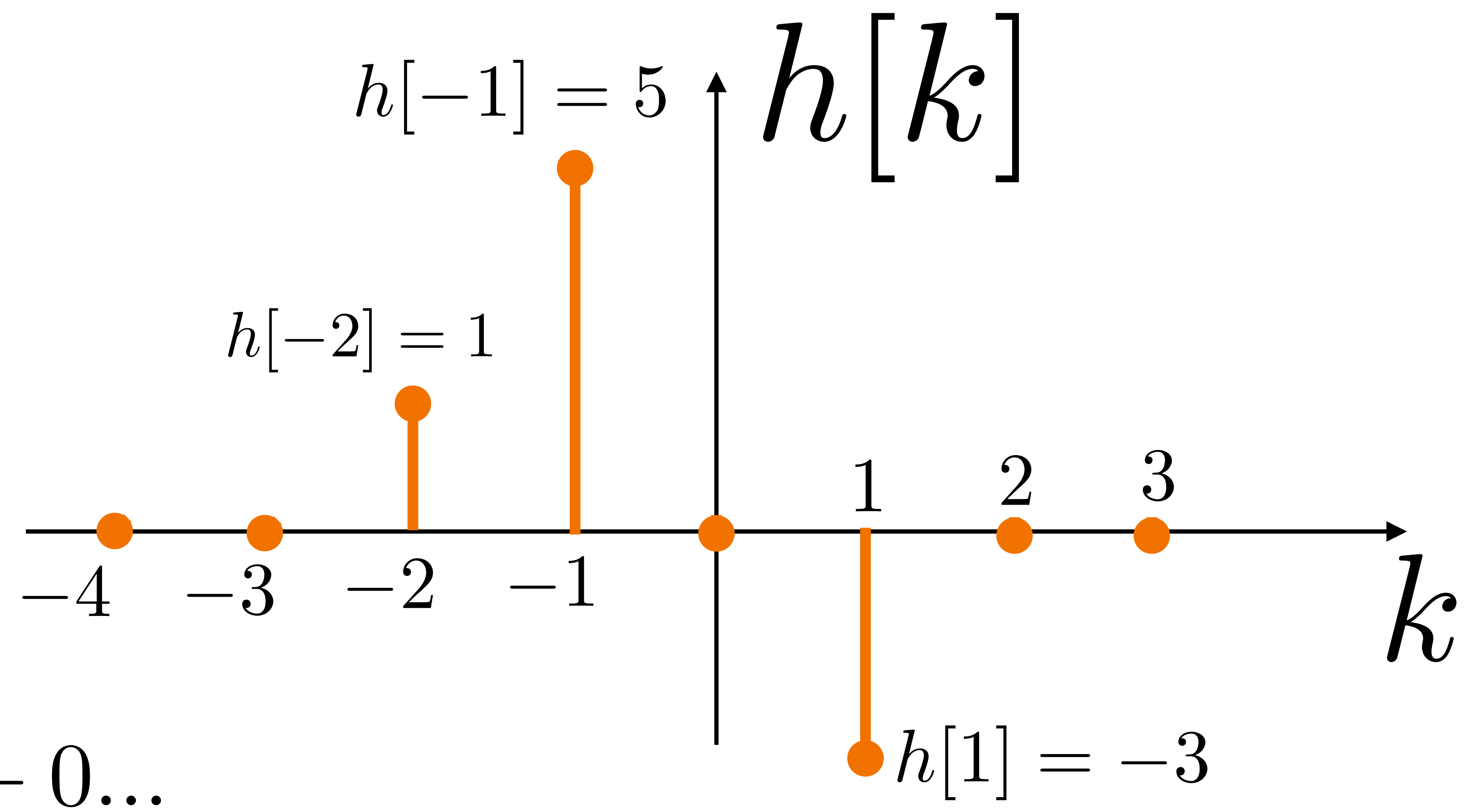
$$y[0] = \sum_{k=-\infty}^{+\infty} h[k]z[k-0]$$

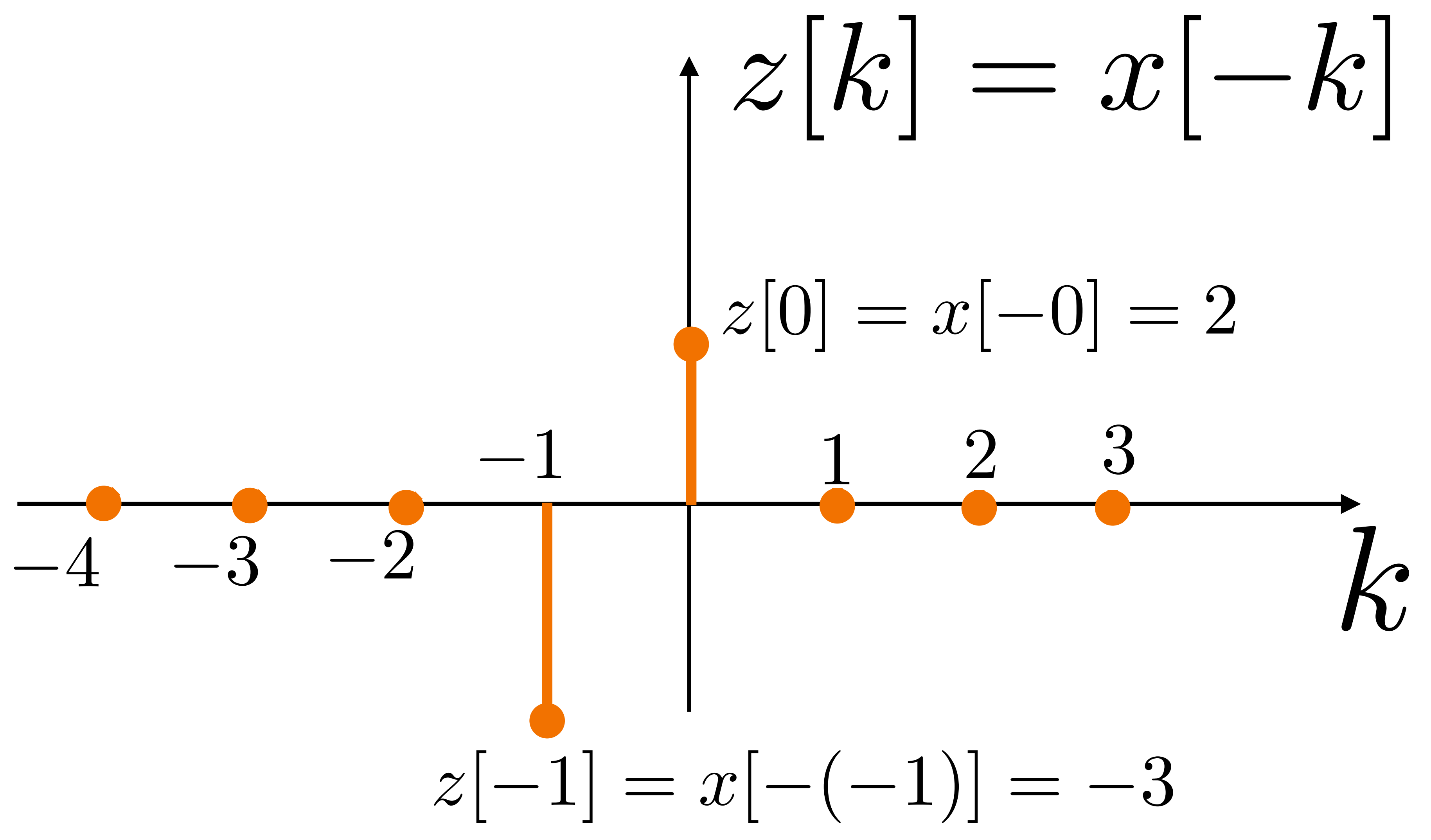
$$= \sum_{k=-\infty}^{+\infty} h[k]z[k]$$

$$= ...0 + 0 + (1)0 + 5(-3) + 2(0) - 3(0) + 0 + 0...$$

$$= ...0 + 0 + 0 - 15 + 0 + 0 + 0 + 0...$$

$$y[0] = -15$$





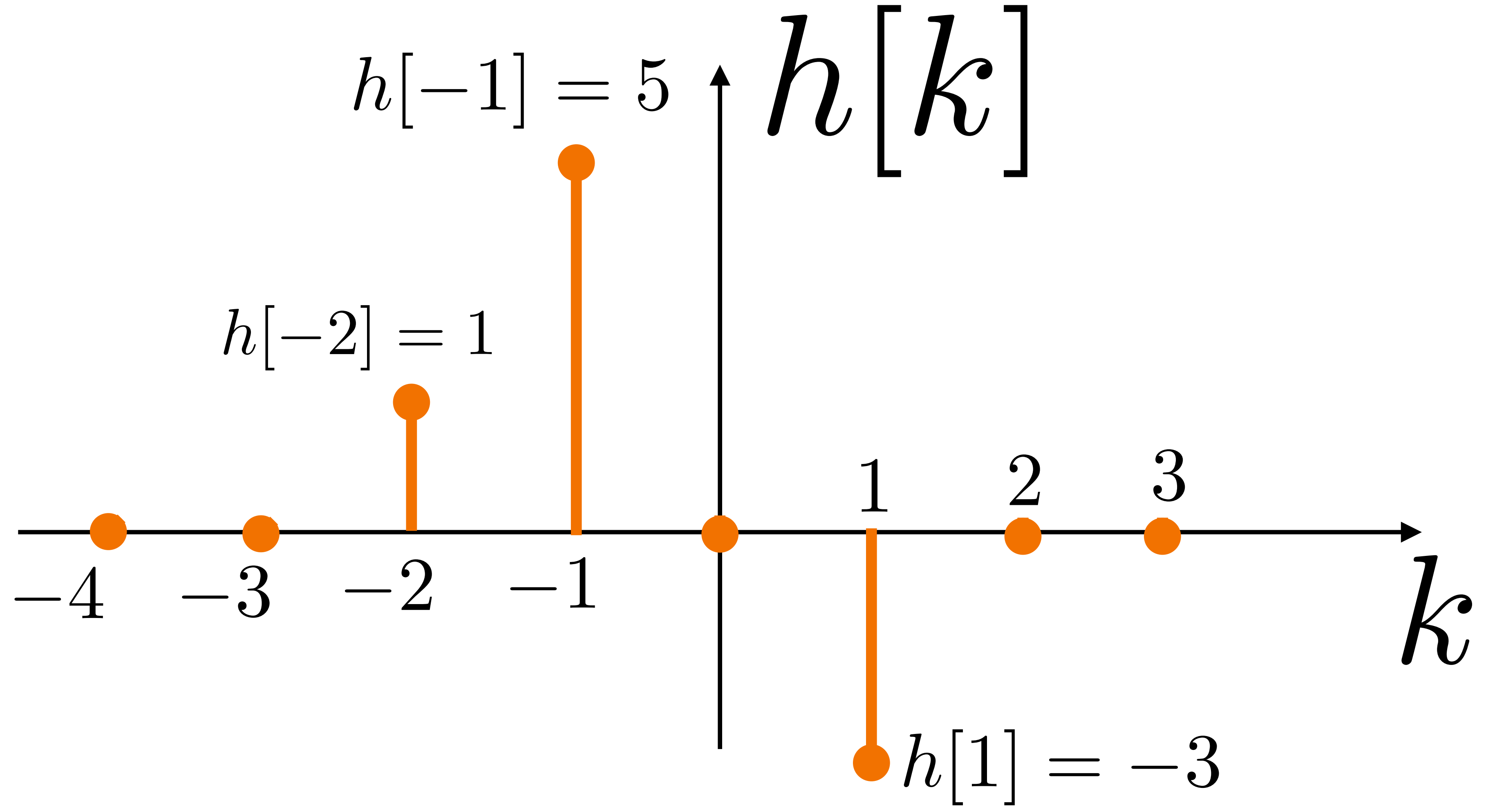
Example - Problem 1

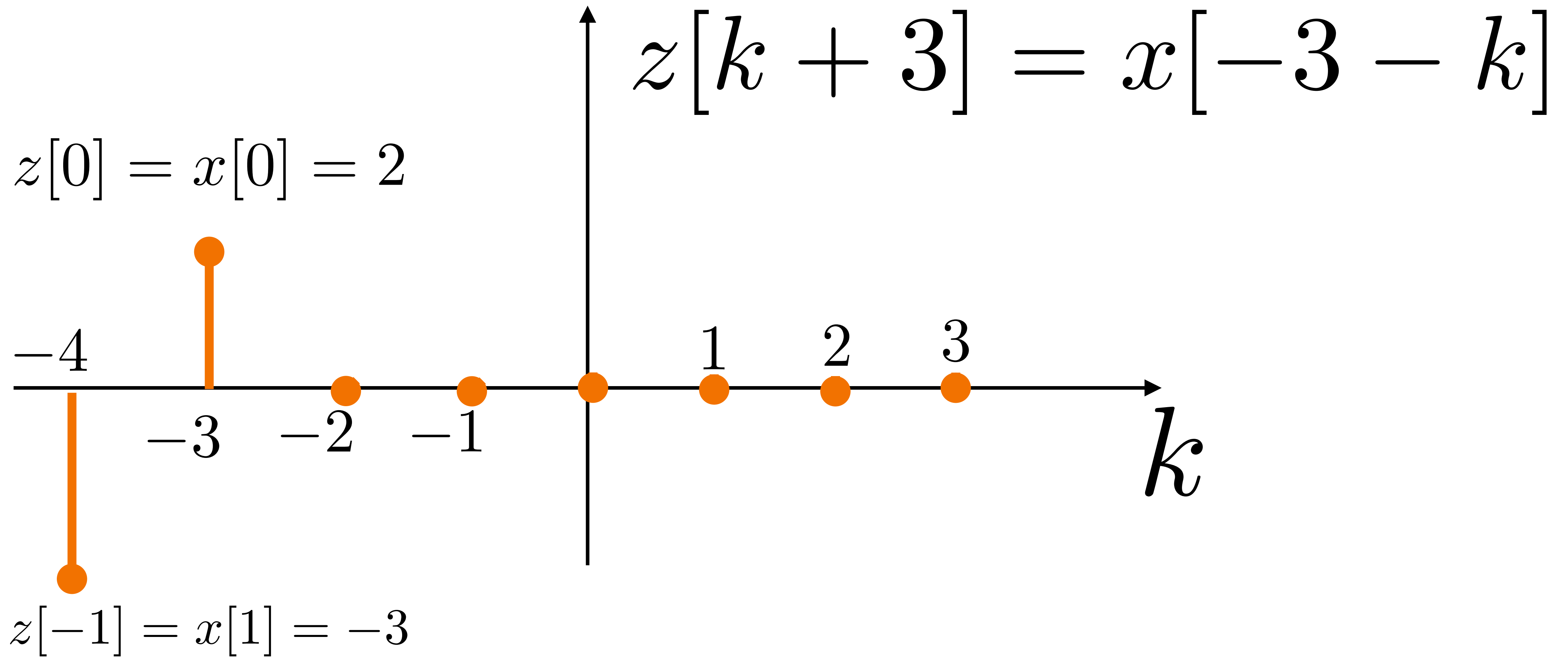
SECOND METHOD - first way

$$y[-3] = \sum_{k=-\infty}^{+\infty} h[k]z[k+3]$$

$$= ...0+0+0+0+0+0+0+0....$$

$$y[-3] = 0$$





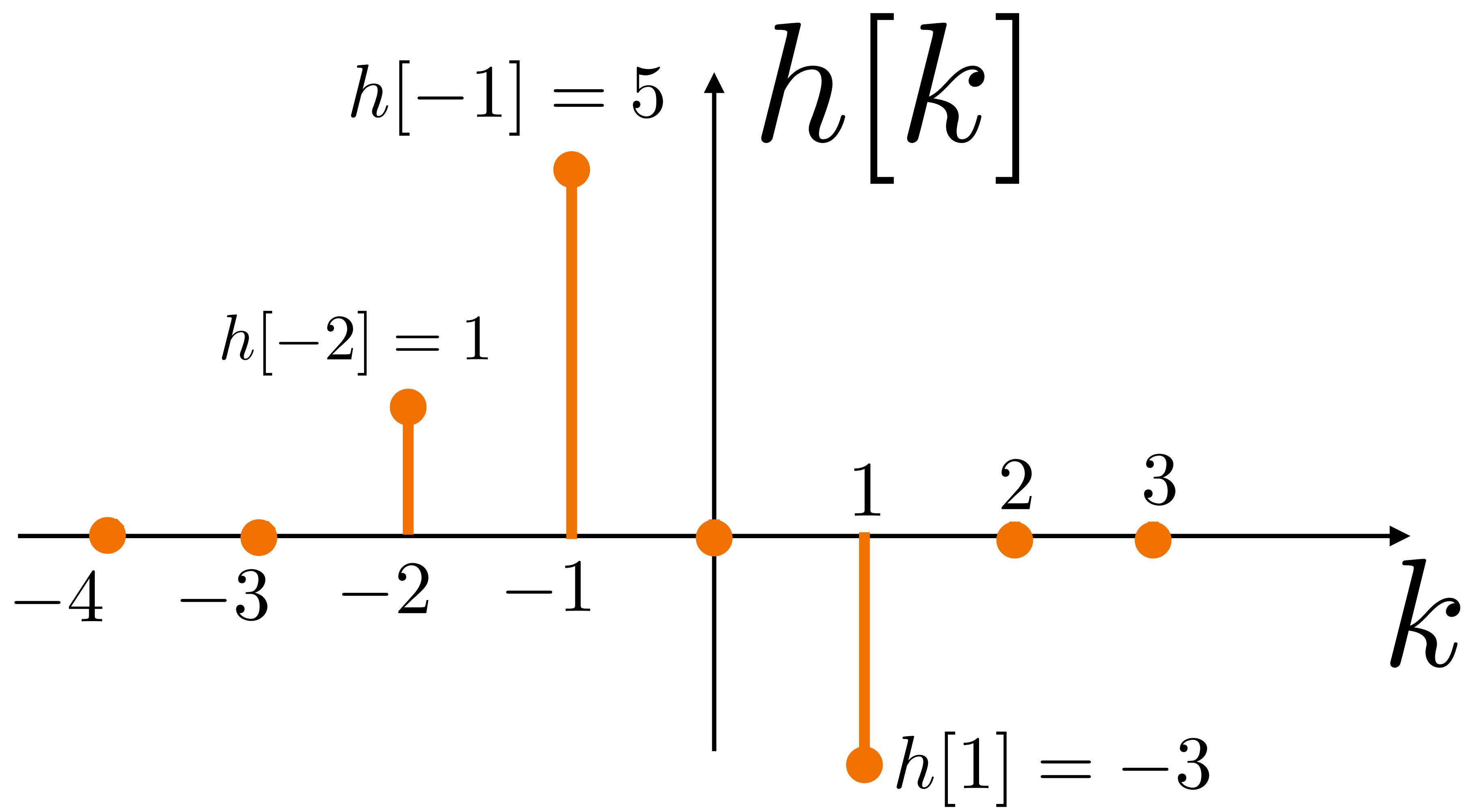
Example - Problem 1

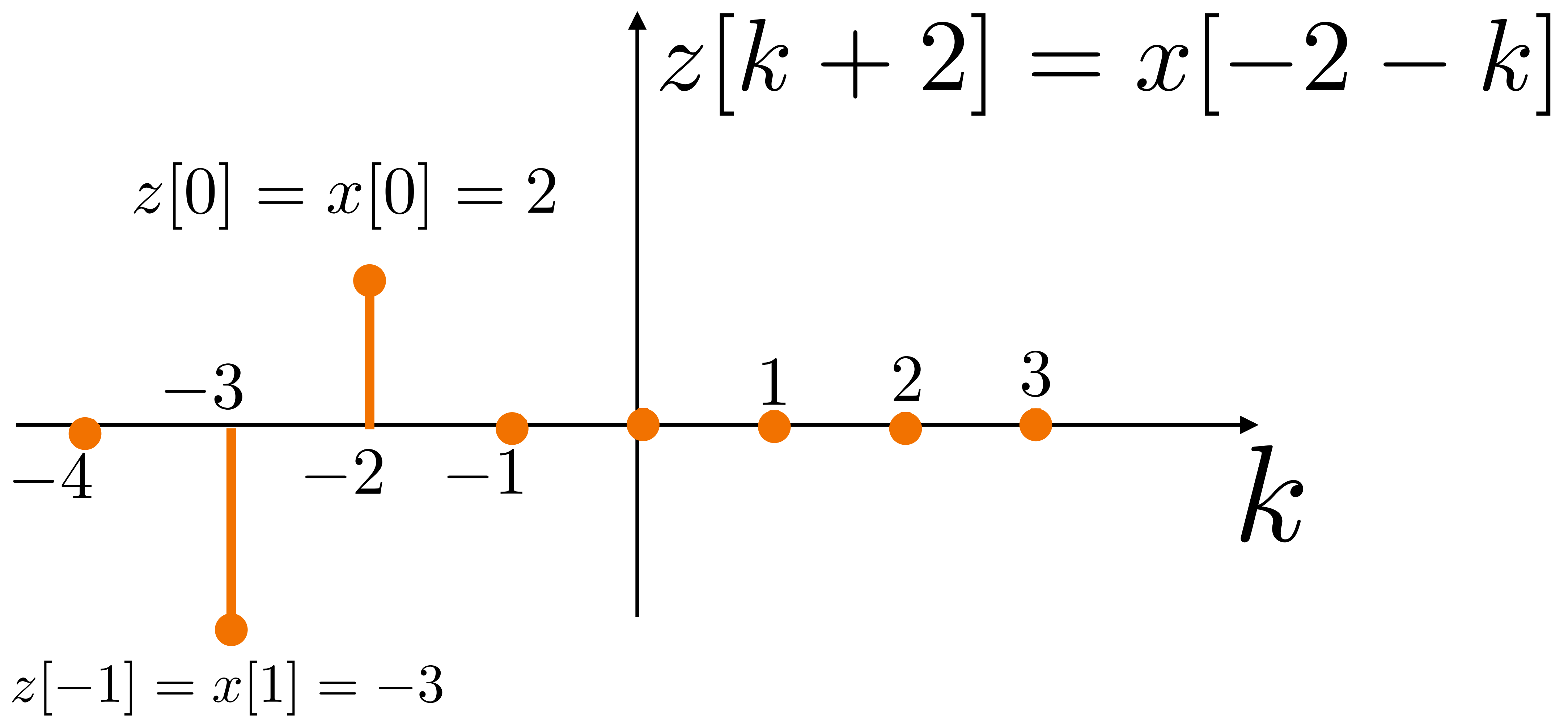
SECOND METHOD - first way

$$y[-2] = \sum_{k=-\infty}^{+\infty} h[k]z[k+2]$$

$$= ...0 + 0 + 2(1) + 0 + 0 + 0 + 0 + 0....$$

$$y[-2] = 2$$





Example - Problem 1

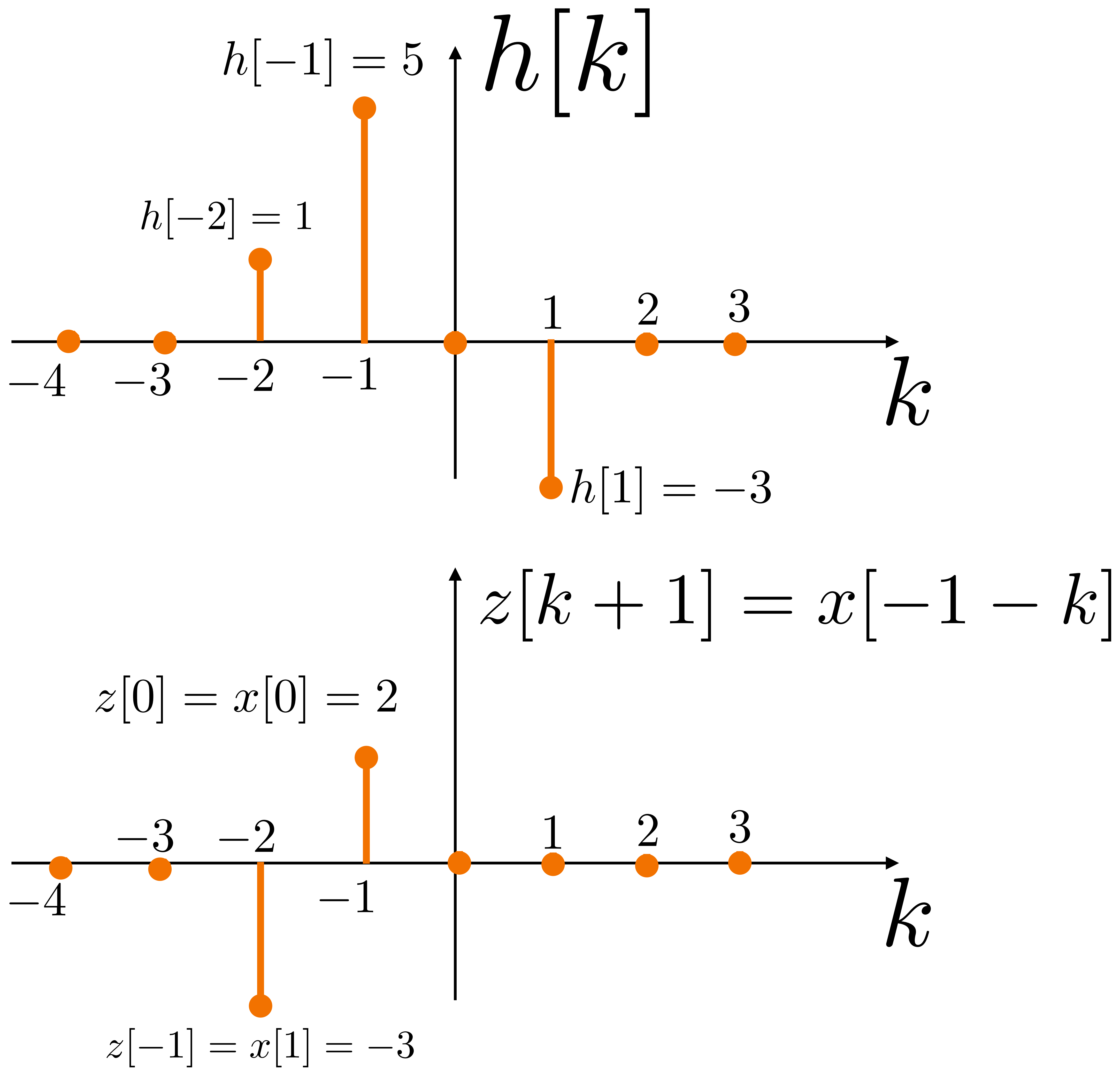
SECOND METHOD - first way

$$y[-1] = \sum_{k=-\infty}^{+\infty} h[k]z[k+1]$$

$$= ...0 + 1(-3) + 5(2) + 0 + 0 + 0 + 0....$$

$$= ...0 - 3 + 10 + 0 + 0 + 0 + 0....$$

$$y[-1] = 7$$



Example - Problem 1

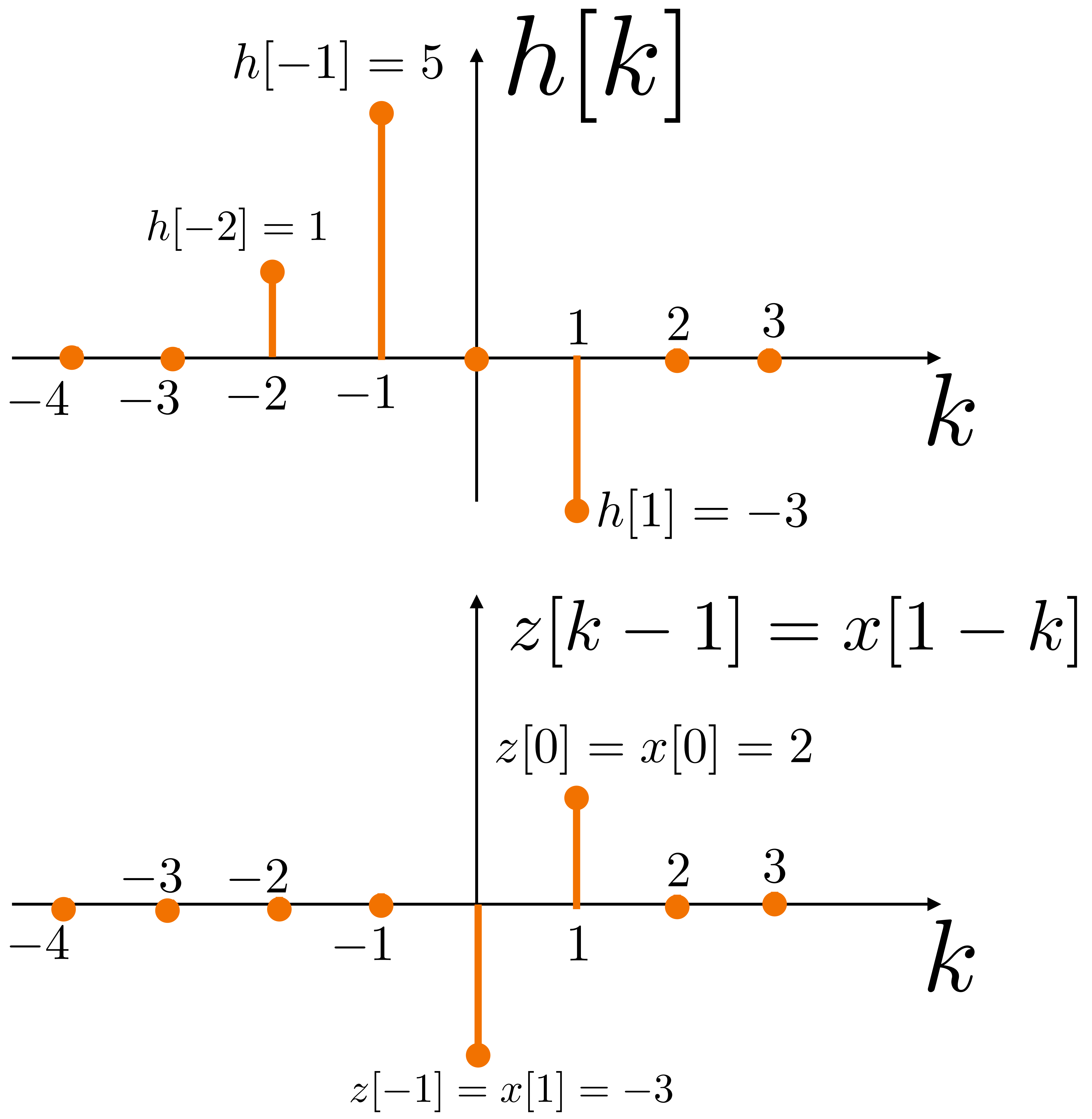
SECOND METHOD - first way

$$y[1] = \sum_{k=-\infty}^{+\infty} h[k]z[k-1]$$

$$= ...0+0+0+0+0-3(2)+0+0....$$

$$= ...0+0+0+0+0-6+0+0....$$

$$y[1] = -6$$



Example - Problem 1

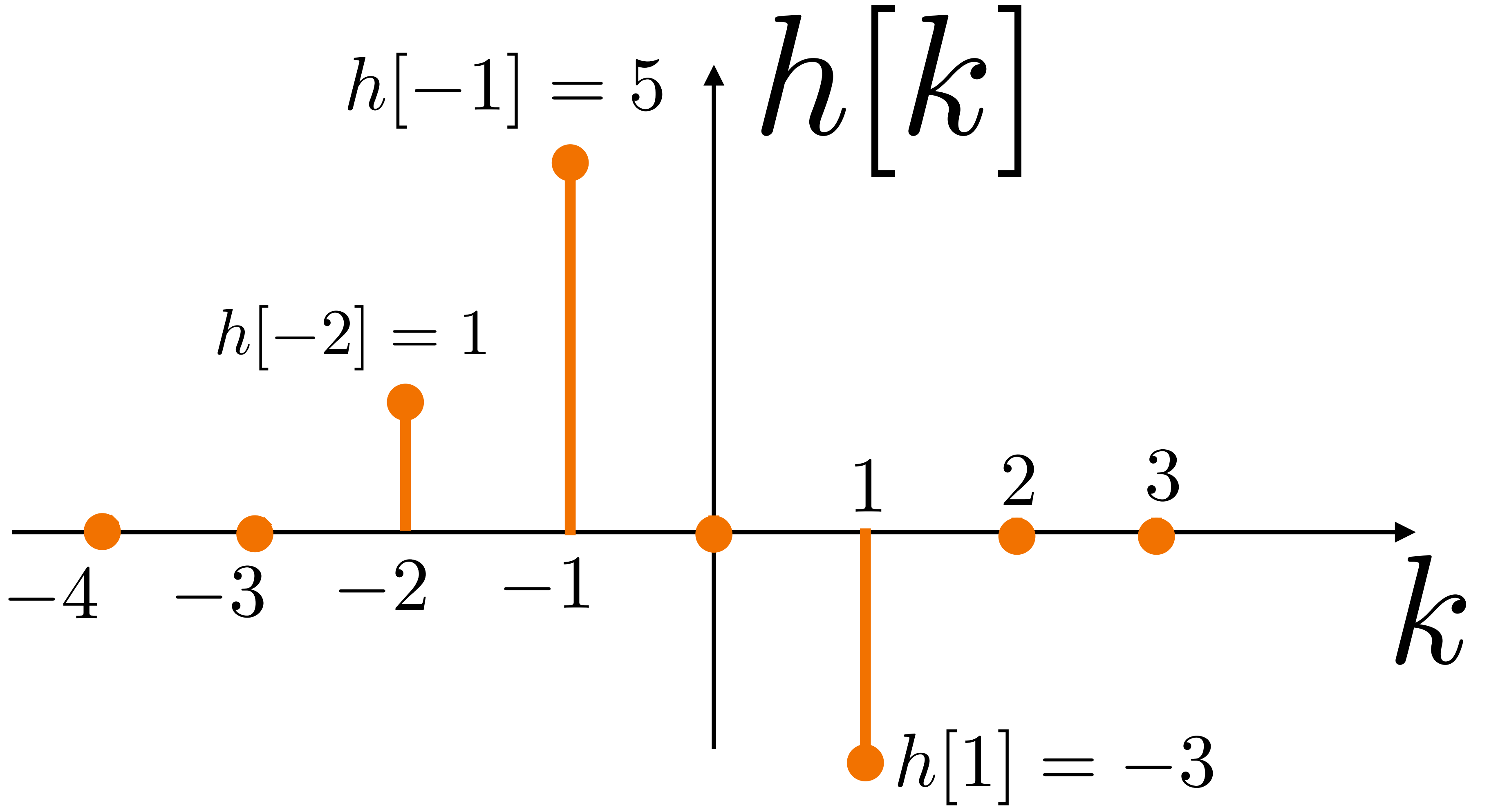
SECOND METHOD - first way

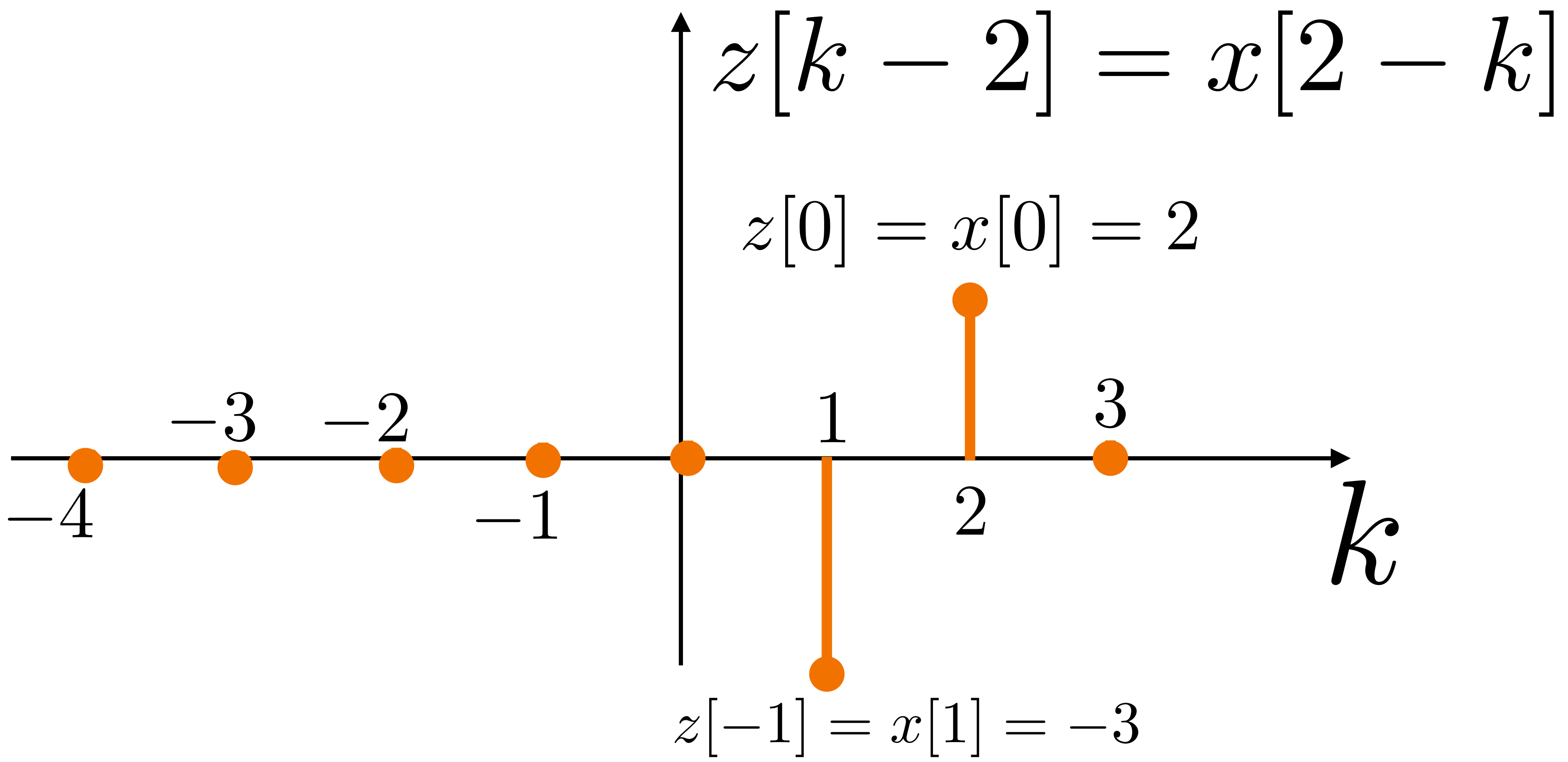
$$y[2] = \sum_{k=-\infty}^{+\infty} h[k]z[k-2]$$

$$= ...0+0+0+0+0+0-3(-3)+0+0....$$

$$= ...0+0+0+0+0+9+0+0....$$

$$y[2] = 9$$



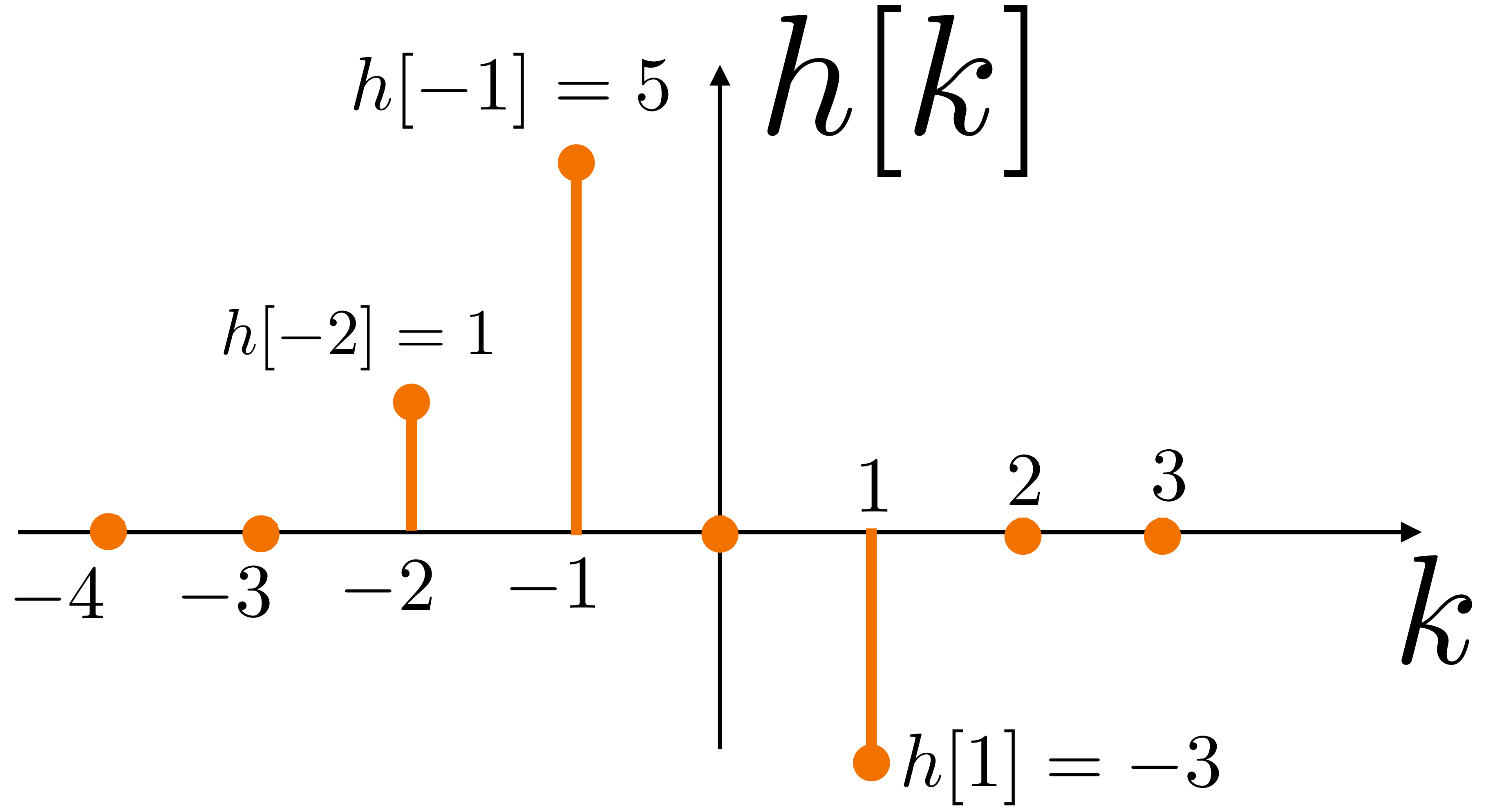


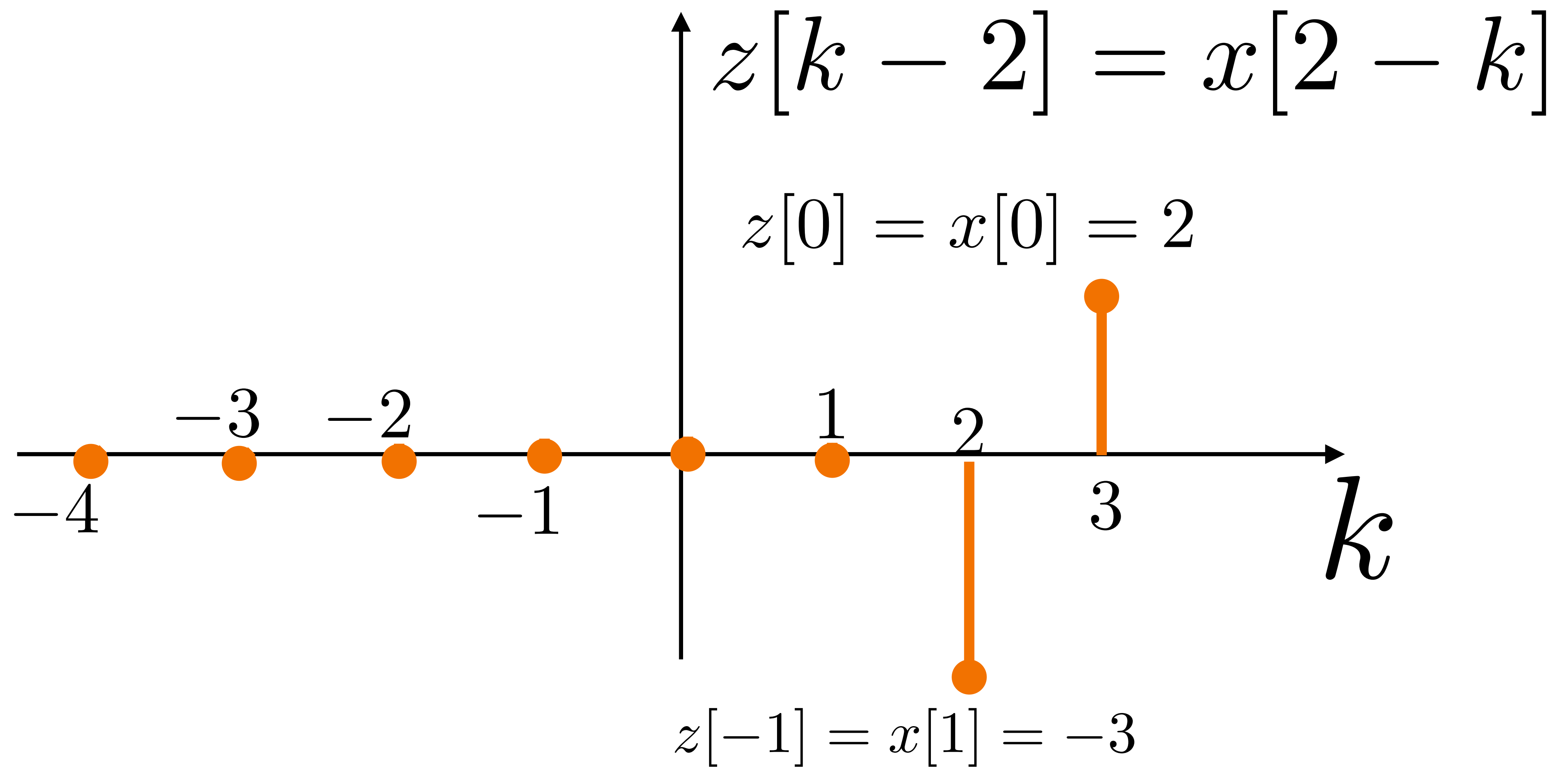
Example - Problem 1

SECOND METHOD - first way

$$y[3] = \sum_{k=-\infty}^{+\infty} h[k]z[k-3]$$

$$= ...0+0+0+0+0+0+0+0+0....$$





$$y[3] = 0$$

Example - Problem 1

SECOND METHOD - SECOND way: Note that

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]z[k+n]$$

$$z[k] = h[-k]$$

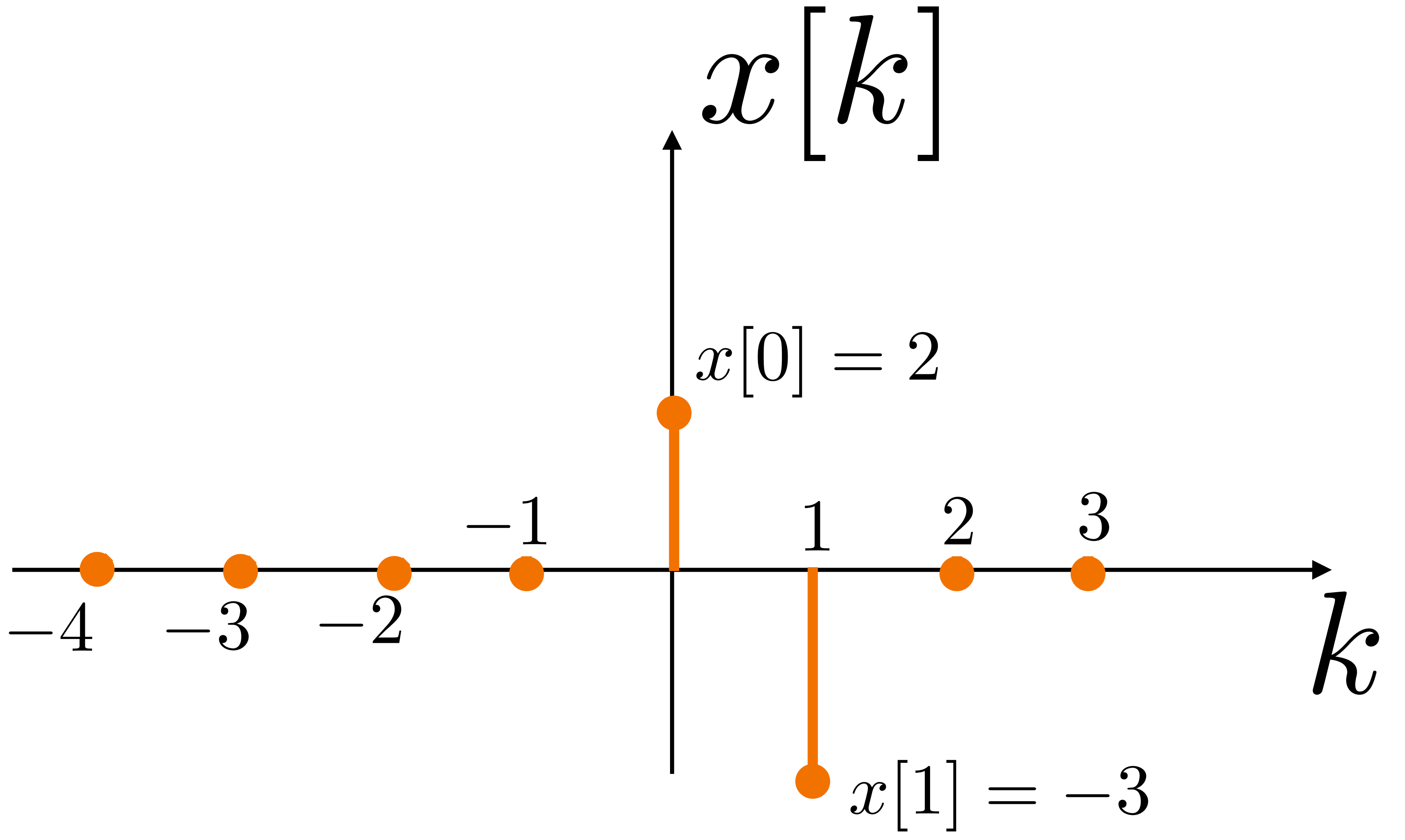
Example - Problem 1

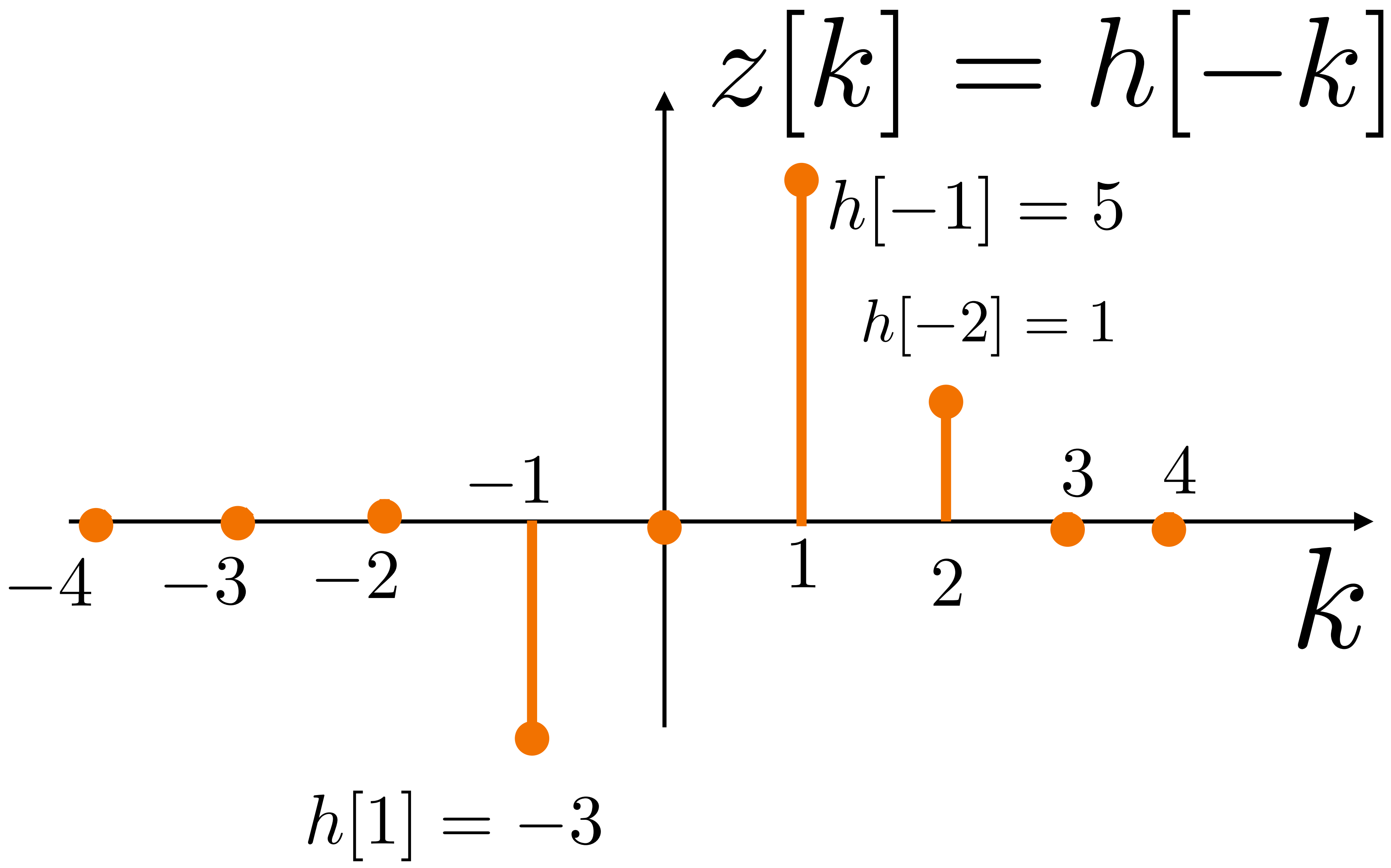
SECOND METHOD -
SECOND way

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]z[k]$$

$$= ...0 + 0 + 0 - 15 + 0 + 0 + 0 + 0...$$

$$y[0] = -15$$





Example - Problem 1

SECOND METHOD - SECOND way

FINISH the second method - second way at home !!!
…. and deliver to me your solution, please :)

Example 2

Given:

$$x[0] = 1, \quad x[1] = 1, \quad x[2] = 1, \quad x[n] = 0 \text{ otherwise}$$

$$h[0] = 0.5, h[1] = 2, \quad h[n] = 0 \text{ otherwise}$$

Obtain the output of the LTI system y[n], with impulse response h[n]

Namely, y[n]=?

Example 2

Then, in this example:

- start of x[n] at 0, start of h[n] at 0
- end of x[n] at 2, finish of h[n] at 1
- the length of x[n] is 3
- the length of h[n] is 2

(1) y[n] starts at = 0+0=0

(2) y[n] finishes at = 2+1=3

(3) the length of y[n] = 3+2-1=4

Example 2

Solution (try to do at home as the previous one):

$$y[n] = 0.5x[n] + 2x[n-1]$$

$$y[0] = 0.5, \quad y[1] = 2.5, \quad y[2] = 2.5, \quad y[3] = 2, \quad y[n] = 0 \text{ otherwise}$$

Example 3

Given:

$$x[-3] = -3, \quad x[1] = 1, \quad x[2] = 1, \quad x[n] = 0 \text{ otherwise}$$

$$h[0] = -0.5, \quad h[2] = 2, \quad h[n] = 0 \text{ otherwise}$$

Obtain the output of the LTI system y[n], with impulse response h[n]

Namely, y[n]=?

Example 3

Then, in this example:

- start of $x[n]$ at -3, start of $h[n]$ at 0
- end of $x[n]$ at 2, finish of $h[n]$ at 2
- the length of $x[n]$ is 6
- the length of $h[n]$ is 3

(1) $y[n]$ starts at $= -3+0=-3$
(2) $y[n]$ finishes at $= 2+2=4$
(3) the length of $y[n] = 6+3-1=8$

Example 3

Solution (try to do at home as the previous one):

$$y[n] = -0.5x[n] + 2x[n-2]$$

$$y[-3] = 1.5, \quad y[-2] = 0, \quad y[-1] = -6, \quad y[0] = 0 \quad y[1] = -0.5,$$
$$y[2] = -0.5 \quad y[3] = 2, \quad y[4] = 2, \quad y[n] = 0 \text{ otherwise}$$

Example 4

Given:

$$x[0] = 2, \quad x[1] = -3, \quad x[n] = 0 \text{ otherwise}$$

$$h[-2] = 1, \quad h[-1] = 5, \quad h[0] = 0, \quad h[1] = -3, \quad h[n] = 0 \text{ otherwise}$$

Obtain the output of the LTI system y[n], with impulse response h[n]

Namely, y[n]=?

Example 5

Then, in this example:

- start of $x[n]$ at 0, start of $h[n]$ at -2
- end of $x[n]$ at 1, finish of $h[n]$ at 1
- the length of $x[n]$ is 2
- the length of $h[n]$ is 4

(1) $y[n]$ starts at = $0+(-2)=-2$
(2) $y[n]$ finishes at = $1+1=2$
(3) the length of $y[n]$ = $4+2-1=5$

Example 4

Solution (try to do at home as the previous one):

$$y[n] = 2h[n] - 3h[n-1]$$

$$y[-2] = 2, \quad y[-1] = 7, \quad y[0] = -15, \quad y[1] = -6 \quad y[2] = 9, \quad y[n] = 0 \text{ otherwise}$$

Questions?