Spectral Analysis of x(t)=1 and x[n]=1 (and more considerations)

Linear systems and circuit applications

Discrete Time Systems

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Example 1

Consider the signal x(t)=1:

- **Say which mathematical tools can you apply in a transformed domain (and why) in in order to perform a Spectral analysis of the signal.**
- **Obtain/compute the mathematical tools that you can apply.**
- **Say which frequency is contained in x(t)=1.**

Example 1

- **The signal x(t)=1 has (clearly) infinite energy. For this reason the Standard Fourier Transform cannot be applied (i.e., does not exist).**

- **Since x(t)=1 does not converge to zero in both sides, i.e., to -Inf and +Inf, and the Laplace transform can help *"only in one side"*, *the Laplace transform does not exist neither.* In fact, it we try to compute the Laplace transform, we have:**

$$X(s) \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-st} dt =$$

$$= \left[-\frac{e^{-st}}{s} \right]_{-\infty}^{\infty} = \left[-\frac{e^{-(\sigma + j\omega)t}}{(\sigma + j\omega)} \right]_{-\infty}^{\infty} = +\infty \text{ or } -\infty (\text{depending on the sign of } \sigma)$$

$X(s)$ does not exist

Example 1

- In the next slides, we will see if we can express this signal as a Fourier Series…
- First of all, we will consider a generic period T_0 (and, as a consequence, a generic fundamental frequency w_0)
- Later we will check if we need to define specific values of T_0 or w_0.

Example 1

- For a periodic signal, we can compute the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- for any values of the fund. frequency w_0, we can directly see that:

$$x(t) = ... + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{2j\omega_0 t} + ...$$

- and setting:

$$a_0 = 1, \quad \text{and } a_k = 0 \quad \forall k \neq 1$$

- we get exactly: $x(t) = 1$ (for any possible w_0 !!!)

Example 1

- Note that the previous result is valid for any possible values of the fundamental frequency w_0 (and/or the period T_0).
- The signal x(t)=1 can be considered a periodic signal, but we do not need to specify the period T_0 or the fundamental frequency w_0.

Example 1

- The corresponding GENERALIZED Fourier Transform (GFT) is with a_0=1, the rest of coefficients zero, we have:

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\omega_0\right)$$

$$X_G(\omega) = 2\pi\delta(\omega)$$

- LOOKING THE FS and the GFT (which just contains the information in the FS), we can see that, among all the frequency that a periodic signal can contain:

$$\omega^{(k)} = k\omega_0$$

- the only contained frequency is $\omega^{(0)} = 0$ (see the delta above or think in the unique non-null coefficient a_0=1)

Example 1

- **Hence, the only frequency contained is the null frequency:**

$$\omega^{(0)} = 0$$

- **This should be clear and obvious (from the beginning of the example) since x(t)=1 is a *constant signal*, and a constant signal "means" no variations, zero variations, NO OSCILLATIONS that is exactly the meaning of the null frequency !**

ADDITIONAL OBSERVATIONS TO Example 1

- Let consider that we have a signal x(t) with “zero mean”. In this case, this signal does not contain the null frequency.
- If we add a constant *A* to the signal x(t) above with “zero mean”, we get

$$z(t) = A + x(t)$$

- This signal z(t) has mean “A” and contains the null frequency with energy “A^2” (“A” al cuadrado).

Example 1.5

- Let consider the signals

$$x_1(t) = \cos(\omega_0 t), \quad x_2(t) = A + \cos(\omega_0 t)$$

- write the two Generalized Fourier Transforms,
- and say the frequencies contained in each one.

Example 1.5

- The two Generalized Fourier Transforms are

$$X_{G,1}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

$$X_{G,2}(\omega) = 2\pi A\delta(\omega) + \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

- and the frequencies in x_1(t) **are just two frequencies,**

$$-\omega_0, \qquad \text{and } \omega_0$$

- and the frequencies in x_2(t) **are just three frequencies,**

$$0, \quad -\omega_0, \qquad \text{and } \omega_0$$

Example 2

Consider the signal $x[n]=1$:

- **Say which mathematical tools can you apply in a transformed domain (and why) in in order to perform a Spectral analysis of the signal.**
- **Obtain/compute the mathematical tools that you can apply.**
- **Say which frequency is contained in $x[n]=1$.**

Example 2

- **For the same reasons of x(t)=1, the signal x[n]=1 does not admit Stand. Fourier Transform and Zeta Transform.**

- **We can just compute Fourier Series and GFT, as for x(t)=1.**

Example 2

- The signal x[n]=1 is a periodic signal with period

$$N = 1$$

- As for x(t)=1, we have

$$a_0 = 1$$

- but for the peridicity of the a_k's (in discrete time), we have

$$a_k = a_{k+N} = a_{k+1}$$

- so that all the a_k's are equal to 1,

$$\ldots = a_{-1} = a_0 = a_1 = a_2 = \ldots = 1$$

Example 2

- Moreover, since N=1, the fundamental frequency is

$$\Omega_0 = \frac{2\pi}{N} = 2\pi$$

- and since, all the a_k's are equal to 1, the signal x[n]=1 contains all these frequencies *(but are different frequencies?)*

$$\Omega^{(k)} = k\Omega_0 = 2\pi k$$

- ***but are different frequencies? no!*** **IT IS JUST THE NULL FREQUENCY !** (in discrete time, the Omega's are "angles"...)

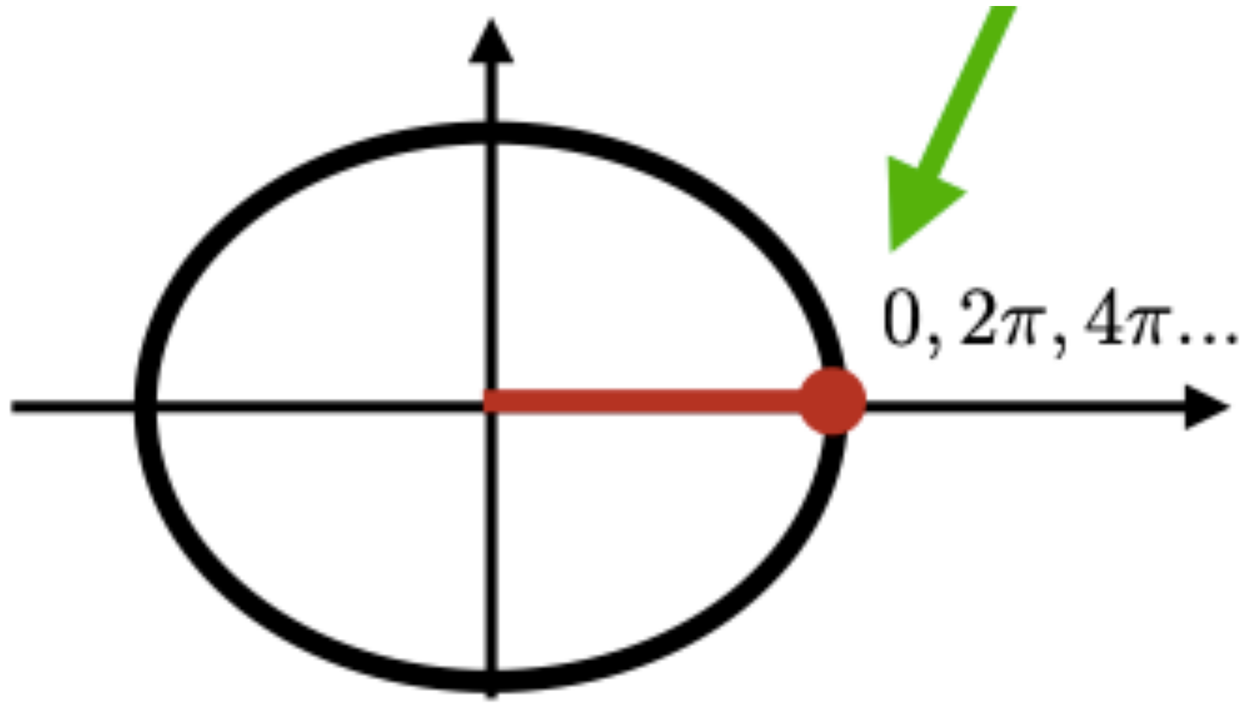
Example 2

- ...*but are different frequencies? no!* IT IS JUST THE NULL FREQUENCY ! (in discrete time, the Omega's are "angles"...)

$$\Omega^{(k)} = k\Omega_0 = 2\pi k$$

- It represents the angle:

$$0, 2\pi, 4\pi, -2\pi, \text{etc...}$$



- that is always the same angle (and then the same frequency): ZERO

Example 2

- **Again, this should be clear and obvious (from the beginning of the example) since x[n]=1 is a *constant signal in discrete time*, and a constant signal "means" no variations, zero variations, NO OSCILLATIONS that is exactly the meaning of the null frequency !**

Example 2

- **THE GENERALIZED FOURIER TRANSFORM IN THIS CASE IS:**
- **RECALL THAT** $\ldots = a_{-1} = a_0 = a_1 = a_2 = \ldots = 1$
- **and**

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right)$$

- **and**

$$\Omega_0 = \frac{2\pi}{N} = 2\pi$$

- **we obtain:**

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right)$$

Example 2

- we have obtained:

$$X_G(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right)$$

- Note that it is periodic with period 2\pi as any other FTs and GFTs for signals defined in the discrete time domain:

$$X_G(\Omega) = X_G(\Omega + 2\pi)$$

Questions?