Solved Problems - DFT - part 2

Linear systems and circuit applications

Discrete Time Systems

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Example 8

Let consider the following signal:

$$x[0] = 3, x[1] = -5, x[2] = 1.5, \quad x[n] = 0 \quad \text{for the rest of n}$$

Let us consider that is obtained sampling a continuos signal x(t) with sampling period T=0.1 sec.

(a) Say what is the maximum frequency of the signal x(t) that you can detect.

(b) Interpret the output of an DFT for a generic N and with N=3,4,5, as X(Omega) - FT of x[n] - and as X(omega) - FT of x(t).

Example 8

(a) If the signal x(t) has been well-sampled, we can "see" until the frequency:

$$\frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T} = \frac{\pi}{0.1} = 31.4159 \quad \text{rad/sec}$$

Example 8

(b) Each output of DFT from k=0,…,N-1 (fft in matlab) can be interpreted as associate to the frequencies

$$\Omega = 0, \Omega_0, 2\Omega_0, ..., (N-1)\Omega_0 \qquad \Omega_0 = \frac{2\pi}{N}$$

for x[n] (discrete time), and

$$\omega = 0, \omega_0, 2\omega_0, ..., (N-1)\omega_0$$

for x(t) (continuos time), where

$$\omega_0 = \frac{\Omega_0}{T} = \frac{2\pi}{NT}$$

Example 8

However, recall that in continuous time we can "see" only until

$$\omega_{\texttt{max}} = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T} = \frac{\pi}{0.1} = 31.4159 \quad \text{rad/sec}$$

therefore some outputs, in this case, have not "sense"….

$$\omega = 0, \omega_0, 2\omega_0, ..., (N-1)\omega_0$$

exactly only an half or half+1…. (solo mitad! o mitad+1) since fft gives you exactly N values…

$$\omega_0 = \frac{\Omega_0}{T} = \frac{2\pi}{NT} = \frac{2}{N}\omega_{\texttt{max}}$$

$$\omega_{\texttt{max}} = \frac{N}{2}\omega_0$$

Example 8

(b) N=3

$$X(\Omega) = 3 - 5e^{-j\Omega} + 1.5e^{-j2\Omega}$$

$$X_N[k] = 3 - 5e^{-jk\frac{2\pi}{N}} + 1.5e^{-j2k\frac{2\pi}{N}}$$

In an exam, you have to write all the "complete" formulas with exponentials etc. here I will give you the numerical results in order to check your calculus... (see Example 2)

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$$

Example 8

$$X(\Omega) = 3 - 5e^{-j\Omega} + 1.5e^{-j2\Omega}$$

$$X_N[k] = 3 - 5e^{-jk\frac{2\pi}{N}} + 1.5e^{-j2k\frac{2\pi}{N}}$$

(b) N=3

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$$

```
here the FFT:

ans =

  -0.5000 + 0.0000i   4.7500 + 5.6292i   4.7500 - 5.6292i

Omega_0:

Omega_0 =

    2.0944

Frequencies (discrete):

Omega =

         0    2.0944    4.1888
```

Example 8

(b) N=3

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3} \qquad \omega_0 = \frac{2\pi}{NT} = 20.9440 \qquad \omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
sampling period:

T =

    0.1000


omega_0 =

   20.9440

maximum frequency that you can see:

omega_max =

   31.4159
```

```
Frequencies (continuos):

omega =

         0   20.9440   41.8879

Just frequencies that you can see:

omega =

         0   20.9440
```

31.4159

Example 8

(b) N=4

```
here the FFT:

ans =

  -0.5000 + 0.0000i   1.5000 + 5.0000i   9.5000 + 0.0000i   1.5000 - 5.0000i

Omega_0:

Omega_0 =

    1.5708

Frequencies (discrete):

Omega =

         0    1.5708    3.1416    4.7124
```

Example 8

(b) N=4

```
sampling period:

T =

    0.1000


omega_0 =

   15.7080

maximum frequency that you can see:

omega_max =

   31.4159
```

```
Frequencies (continuous):

omega =

         0   15.7080   31.4159   47.1239

Just frequencies that you can see:

omega =

         0   15.7080   31.4159
```

Example 8

(b) N=5

```
here the FFT:

ans =

  Columns 1 through 4

  -0.5000 + 0.0000i   0.2414 + 3.8736i   7.5086 + 4.3655i   7.5086 - 4.3655i

  Column 5

   0.2414 - 3.8736i

Omega_0:

Omega_0 =

    1.2566

Frequencies (discrete):

Omega =

         0    1.2566    2.5133    3.7699    5.0265
```

Example 8

(b) N=5

```
sampling period:

T =

    0.1000


omega_0 =

   12.5664

maximum frequency that you can see:

omega_max =

   31.4159
```

```
Frequencies (continuous):

omega =

         0   12.5664   25.1327   37.6991   50.2655

Just frequencies that you can see:

omega =

         0   12.5664   25.1327
```

Example 8

Please, compare Example 8 and Example 7, which consider different signals/data, and see what is changed and what remains the same (basically only the fft changes…).

Example 9

Let consider the following signal:

$$x[0] = 1, x[1] = -5, x[2] = 2, x[3] = 0.1 \quad x[n] = 0 \text{ for the rest of n}$$

(a) Show the output of fft(x) (without specify "N") and provide the correspondence with the values of X_N[k] and X(Omega).

(b) Considering a signal obtained repeating x[n] periodically with period N=4. Obtain the coefficients a_k of the Fourier Series.

Example 9

$$X(\Omega) = 1 - 5e^{-j\Omega} + 2e^{-j2\Omega} + 0.1e^{-j3\Omega}$$

$$\Omega = k\Omega_0 = k\frac{2\pi}{N}$$

$$X_N[k] = 1 - 5e^{-jk\frac{2\pi}{N}} + 2e^{-j2k\frac{2\pi}{N}} + 0.1e^{-j3k\frac{2\pi}{N}}$$

Example 9

$$X(\Omega) = 1 - 5e^{-j\Omega} + 2e^{-j2\Omega} + 0.1e^{-j3\Omega}$$

$$X_N[k] = 1 - 5e^{-jk\frac{2\pi}{N}} + 2e^{-j2k\frac{2\pi}{N}} + 0.1e^{-j3k\frac{2\pi}{N}}$$

(a) The fft(x) returns the same result of fft(x,L), i.e., N=L. In this case, L=4. Then, fft(x) returns X_4[k] for k=0,1,2,3:

$$X_4[k] = 1 - 5e^{-jk\frac{2\pi}{4}} + 2e^{-j2k\frac{2\pi}{4}} + 0.1e^{-j3k\frac{2\pi}{4}}$$

$$X_4[k] = 1 - 5e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + 0.1e^{-jk\frac{3\pi}{2}}$$

Example 9

(a) The fft(x) returns the same result of fft(x,L), i.e., N=L. In this case, L=4. Then, fft(x) returns X_4[k] for k=0,1,2,3:

$$X_4[k] = 1 - 5e^{-jk\frac{2\pi}{4}} + 2e^{-j2k\frac{2\pi}{4}} + 0.1e^{-j3k\frac{2\pi}{4}}$$

$$X_4[k] = 1 - 5e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + 0.1e^{-jk\frac{3\pi}{2}}$$

$$X_4[0] = 1 - 5 + 2 + 0.1 = -1.9$$

$$X_4[1] = -1 + 5.1j$$

```
>> 1-5*exp(-j*1*pi/2)+2*exp(-j*1*pi)+0.1*exp(-j*1*3/2*pi)

ans =

  -1.0000 + 5.1000i
```

Example 9

$$X_4[0] = 1 - 5 + 2 + 0.1 = -1.9$$

$$X_4[1] = -1 + 5.1j$$

```
>> 1-5*exp(-j*1*pi/2)+2*exp(-j*1*pi)+0.1*exp(-j*1*3/2*pi)

ans =

  -1.0000 + 5.1000i
```

$$X_4[2] = 7.9$$

```
>> 1-5*exp(-j*2*pi/2)+2*exp(-j*2*pi)+0.1*exp(-j*2*3/2*pi)

ans =

   7.9000 + 0.0000i
```

$$X_4[3] = -1 - 5.1j$$

```
>> 1-5*exp(-j*3*pi/2)+2*exp(-j*3*pi)+0.1*exp(-j*3*3/2*pi)

ans =

  -1.0000 - 5.1000i
```

Example 9

CHECK IT:

$X_4[k]$ →

```
here the FFT:

ans =

  -1.9000 + 0.0000i  -1.0000 + 5.1000i   7.9000 + 0.0000i  -1.0000 - 5.1000i

Omega_0:

Omega_0 =

    1.5708

Frequencies (discrete):

Omega =

         0    1.5708    3.1416    4.7124
```

Example 9

(b) Since the period of the signal is exactly N=4, we do need more computations since, by the theory, we have:

$$a_k = \frac{1}{4} X_4[k]$$

and can use the X_4[0], X_4[1], X_4[2], X_4[3] computed before.

Generally:

$$a_k = \frac{1}{N} X_N[k]$$

Example 10

Let consider the following signal:

$$x[0] = 1, x[1] = -5, x[2] = 2, x[3] = 0.1 \quad x[n] = 0 \text{ for the rest of n}$$

Considering N=4, check that X_4[0]=X_4[4], X_4[1]=X_4[5], etc…, i.e., the periodicity of X_N[k].

Example 10

From the previous example:

$$X_4[k] = 1 - 5e^{-jk\frac{2\pi}{4}} + 2e^{-j2k\frac{2\pi}{4}} + 0.1e^{-j3k\frac{2\pi}{4}}$$

$$X_4[k] = 1 - 5e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + 0.1e^{-jk\frac{3\pi}{2}}$$

$$X_4[0] = 1 - 5 + 2 + 0.1 = -1.9$$

$$X_4[1] = -1 + 5.1j$$

$$X_4[2] = 7.9$$

$$X_4[3] = -1 - 5.1j$$

Example 10

$$X_4[4] = -1.9$$

```
>> 1-5*exp(-j*4*pi/2)+2*exp(-j*4*pi)+0.1*exp(-j*4*3/2*pi)

ans =

  -1.9000 - 0.0000i
```

$$X_4[5] = -1 + 5.1j$$

```
>> 1-5*exp(-j*5*pi/2)+2*exp(-j*5*pi)+0.1*exp(-j*5*3/2*pi)

ans =

  -1.0000 + 5.1000i
```

$$X_4[6] = 7.9$$

```
>> 1-5*exp(-j*6*pi/2)+2*exp(-j*6*pi)+0.1*exp(-j*6*3/2*pi)

ans =

   7.9000 + 0.0000i
```

Example 10

$$X_4[7] = -1 - 5.1j$$

```
>> 1-5*exp(-j*7*pi/2)+2*exp(-j*7*pi)+0.1*exp(-j*7*3/2*pi)

ans =

  -1.0000 - 5.1000i
```

$$X_4[8] = -1.9$$

```
>> 1-5*exp(-j*8*pi/2)+2*exp(-j*8*pi)+0.1*exp(-j*8*3/2*pi)

ans =

  -1.9000 - 0.0000i
```

etc… we proved that by numerical check.

Example 11

Considering N=4, write the Vandermonde matrix and the vectorial form for the computation of the DFT.

Example 11

Example: evaluating DFT with N=4 at 0, 1,2,3

$$X_4[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3]$$

$$\begin{aligned} X_4[1] &= \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}1n} = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\frac{\pi}{2}})^n \sum_{n=0}^{3} x[n] \cdot (-j)^n \\ &= x[0](-j)^0 + x[1](-j)^1 + x[2](-j)^2 + x[3](-j)^3 = x[0] - x[1]j - x[2] + x[3]j \end{aligned}$$

$$X_4[2] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}2n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\pi})^n = x[0] + x[1](-1) + x[2](-1)^2 + x[3](-1)^3$$

$$X_4[3] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}3n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\frac{3\pi}{2}})^n = x[0] + x[1]j + x[2]j^2 + x[3]j^3$$

The same as with the Fourier Series (different only for a factor 1/N)

Example 11

We arrive to:

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$
$$X_4[1] = x[0] - jx[1] - x[2] + x[3]$$
$$X_4[2] = x[0] - x[1] + x[2] - x[3]$$
$$X_4[3] = x[0] + jx[1] - x[2] - jx[3]$$

We can write it as a linear system !

Example 11

With the previous case (N=4):

$$\begin{bmatrix} X_4[0] \\ X_4[1] \\ X_4[2] \\ X_4[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$\mathbf{X}_4$ $\quad$ $\mathbf{F}$ $\quad$ $\mathbf{x}$

F is a Vandermonde matrix ! Each row is "geometric progression" (see next slide)

$$\mathbf{X}_4 = \mathbf{F}\mathbf{x}$$

Example 11

Generic N:

$$\mathbf{F} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{2\pi}{N}2} & \cdots & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}_{N\times N}$$

$$\mathbf{X}_N = \mathbf{F}\mathbf{x}$$

Example 11

Considering N=2, write the Vandermonde matrix and the vectorial form for the computation of the DFT.

Example 12

$$X_2[0] = \sum_{n=0}^{1} x[n] = x[0] + x[1]$$

$$X_2[1] = \sum_{n=0}^{1} x[n] \cdot e^{-j\frac{2\pi}{2}1n} = \sum_{n=0}^{1} x[n] \cdot e^{-j\pi n} = \sum_{n=0}^{1} x[n] \cdot \left(e^{-j\pi}\right)^n = \sum_{n=0}^{1} x[n] \cdot (-1)^n$$

$$= x[0](-1)^0 + x[1](-1)^1 = x[0] - x[1]$$

Example 12

$$X_2[0] = x[0] + x[1]$$
$$X_2[1] = x[0] - x[1]$$

Example 12

$$\begin{bmatrix} X_2[0] \\ X_2[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

Example 13

Considering N=3, write the Vandermonde matrix and the vectorial form for the computation of the DFT.

Example 13

$$X_3[0] = \sum_{n=0}^{2} x[n] = x[0] + x[1] + x[2]$$

$$X_3[1] = \sum_{n=0}^{2} x[n] \cdot e^{-j\frac{2\pi}{3}1n} = \sum_{n=0}^{2} x[n] \cdot e^{-j\frac{2\pi}{3}n} = \sum_{n=0}^{2} x[n] \cdot \left(e^{-j\frac{2\pi}{3}}\right)^n$$

$$= x[0](e^{-j\frac{2\pi}{3}})^0 + x[1](-e^{-j\frac{2\pi}{3}})^1 + x[2](e^{-j\frac{2\pi}{3}})^2 = x[0] + e^{-j\frac{2\pi}{3}}x[1]j + e^{-j\frac{4\pi}{3}}x[2]$$

$$X_3[2] = \sum_{n=0}^{2} x[n] \cdot e^{-j\frac{2\pi}{3}2n} = \sum_{n=0}^{2} x[n] \cdot \left(e^{-j\frac{4\pi}{3}}\right)^n = x[0] + x[1](e^{-j\frac{4\pi}{3}}) + x[2](e^{-j\frac{8\pi}{3}})$$

Example 13

$$\begin{bmatrix} X_3[0] \\ X_3[1] \\ X_3[2] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2}{3}\pi} & e^{-j\frac{4}{3}\pi} \\ 1 & e^{-j\frac{4}{3}\pi} & e^{-j\frac{8}{3}\pi} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}$$

Example 14

Let consider some data (discrete signal) x[n] which is obtained sampling a continuos signal x(t) with sampling period T=0.1 sec.

(a) Say what is the maximum frequency of the signal x(t) that you can detect.

(b) Interpret the output of an DFT for a generic N and with N=3,4,5, 6 and 7, as X(Omega) - FT of x[n] - and as X(omega) - FT of x(t).

Example 14

(a) If the signal x(t) has been well-sampled, we can "see" until the frequency:

$$\frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T} = \frac{\pi}{0.1} = 31.4159 \quad \text{rad/sec}$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

Note that this does not depend on N…

Example 14

(b) Each output of DFT from k=0,...,N-1 (fft in matlab) can be interpreted as associate to the frequencies

$$\Omega = 0, \Omega_0, 2\Omega_0, ..., (N-1)\Omega_0 \qquad \Omega_0 = \frac{2\pi}{N}$$

for x[n] (discrete time), and

$$\omega = 0, \omega_0, 2\omega_0, ..., (N-1)\omega_0$$

for x(t) (continuos time), where

$$\omega_0 = \frac{\Omega_0}{T} = \frac{2\pi}{NT}$$

THIS DEPENDS on N...

Example 14

However, recall that in continuous time we can “see” only until

$$\omega_{\mathtt{max}} = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T} = \frac{\pi}{0.1} = 31.4159 \quad \text{rad/sec}$$

therefore some outputs, in this case, have not “sense”….

$$\omega = 0, \omega_0, 2\omega_0, ..., (N-1)\omega_0$$

exactly only an half or half+1…. (solo mitad! o mitad+1) since fft gives you exactly N values…

$$\omega_0 = \frac{\Omega_0}{T} = \frac{2\pi}{NT} = \frac{2}{N}\omega_{\mathtt{max}} \qquad \omega_{\mathtt{max}} = \frac{N}{2}\omega_0$$

Example 14

(b) N=3

```
Omega_0 =

    2.0944
```

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$$

```
Frequencies (discrete):

Omega =

         0    2.0944    4.1888
```

Example 14

(b) N=3

```
omega_0 =

   20.9440
```

$$\omega_0 = \frac{2\pi}{NT} = 20.9440$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

$$\omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
Frequencies (continuous):

omega =

         0   20.9440   41.8879

Just frequencies that you can see:

omega =

         0   20.9440
```

Example 14

(b) N=4

```
Omega_0 =

    1.5708
```

$$\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2} = 1.5708$$

```
Frequencies (discrete):

Omega =

         0    1.5708    3.1416    4.7124
```

Example 14

(b) N=4

```
omega_0 =

   15.7080
```

$$\omega_0 = \frac{2\pi}{NT} = 15.7080$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

$$\omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
Frequencies (continuous):

omega =

         0   15.7080   31.4159   47.1239

Just frequencies that you can see:

omega =

         0   15.7080   31.4159
```

Example 14

(b) N=5

```
Omega_0 =

    1.2566
```

$$\Omega_0 = \frac{2\pi}{5} = 1.2566$$

```
Frequencies (discrete):

Omega =

         0    1.2566    2.5133    3.7699    5.0265
```

Example 14

(b) N=5

```
omega_0 =

   12.5664
```

$$\omega_0 = \frac{2\pi}{NT} = 12.5664$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

$$\omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
Frequencies (continuous):

omega =

         0   12.5664   25.1327   37.6991   50.2655

Just frequencies that you can see:

omega =

         0   12.5664   25.1327
```

Example 14

(b) N=6

```
Omega_0 =

    1.0472
```

$$\Omega_0 = \frac{2\pi}{N}$$

```
Frequencies (discrete):

Omega =

         0    1.0472    2.0944    3.1416    4.1888    5.2360
```

Example 14

(b) N=6

```
omega_0 =

   10.4720
```

$$\omega_0 = \frac{2\pi}{NT}$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

$$\omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
Frequencies (continuous):

omega =

         0   10.4720   20.9440   31.4159   41.8879   52.3599

Just frequencies that you can see:

omega =

         0   10.4720   20.9440   31.4159
```

Example 14

(b) N=7

```
Omega_0 =

    0.8976
```

$$\Omega_0 = \frac{2\pi}{N}$$

```
Frequencies (discrete):

Omega =

         0    0.8976    1.7952    2.6928    3.5904    4.4880    5.3856
```

Example 14

(b) N=7

```
omega_0 =

    8.9760
```

$$\omega_0 = \frac{2\pi}{NT}$$

```
maximum frequency that you can see:

omega_max =

   31.4159
```

$$\omega_{\mathtt{max}} = \frac{\pi}{T} = 31.4159$$

```
Frequencies (continuous):

omega =

         0    8.9760   17.9520   26.9279   35.9039   44.8799   53.8559

Just frequencies that you can see:

omega =

         0    8.9760   17.9520   26.9279
```

Questions?