Solved Problems - FT (discrete time)

Linear systems and circuit applications

Discrete Time Systems

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Example 15

Consider the following FTs of an input x[n] and the impulse response h[n] of an LTI system:

$$X(\Omega) = \cos(\Omega) \qquad\qquad H(\Omega) = e^{j\frac{2\pi}{5}\Omega}$$

Compute the standard FT of the output of the system y[n], $Y(\Omega) = ?$

Example 15

We know that:

$$Y(\Omega) = X(\Omega)H(\Omega)$$

then:

$$\begin{aligned} Y(\Omega) &= \cos(\Omega)e^{j\frac{2\pi}{5}\Omega} \\ &= \frac{1}{2}\left(e^{j\Omega} + e^{-j\Omega}\right)e^{j\frac{2\pi}{5}\Omega} \\ &= \frac{1}{2}\left(e^{j\frac{2\pi+5}{5}\Omega} + e^{j\frac{2\pi-5}{5}\Omega}\right) \end{aligned}$$

This is the solution.

Example 16

Consider the following FT:

$$X(\Omega) = 1 \qquad \text{for } \frac{\pi}{4} \leq |\Omega| \leq \frac{3\pi}{4}$$

$$\text{and zero } (= 0) \text{ in the interval } [-\pi, \pi]$$

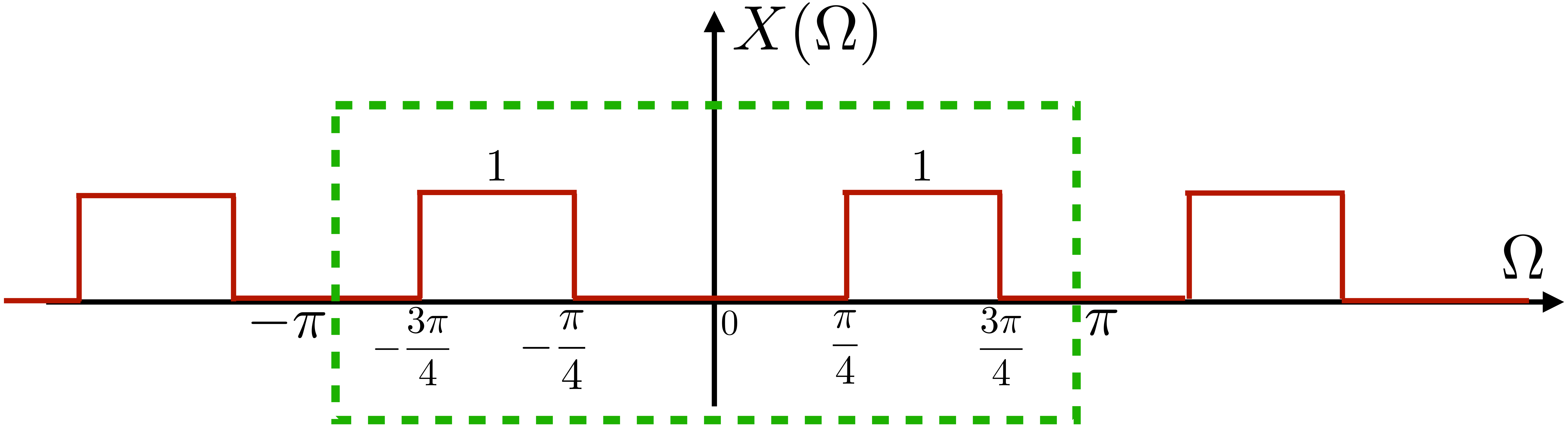
Compute the corresponding signal x[n].

Example 16

$$X(\Omega) = 1 \qquad \text{for } \frac{\pi}{4} \leq |\Omega| \leq \frac{3\pi}{4}$$

$$\text{and zero } (= 0) \text{ in the interval } [-\pi, \pi]$$

The FT is (recalling that is periodic with period 2\pi):



Example 16

The inverse transformation formula is:

Inverse
freq. ==> time

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Inverse Fourier Transform

One possible choice:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Example 16

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\Omega n}d\Omega$$

$$x[n] = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\Omega n}\right]_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\Omega n}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

Example 16

$$x[n] = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\Omega n}\right]_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\Omega n}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$x[n] = \frac{1}{2\pi}\frac{1}{jn}\left(e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n} - e^{j\frac{\pi}{4}n}\right)$$

This is already the solution; if required, we have to do more "arrangements".

Example 16

$$x[n] = \frac{1}{2\pi}\frac{1}{jn}\left(e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n} - e^{j\frac{\pi}{4}n}\right)$$

$$x[n] = \frac{1}{2\pi}\frac{1}{jn}\left(e^{-j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n}\right)$$

$$x[n] = \frac{1}{2\pi}\frac{1}{jn}\left(e^{-j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}n}\right) + \frac{1}{2\pi}\frac{1}{jn}\left(e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right)$$

$$x[n] = -\frac{1}{\pi n}\sin\left[\frac{\pi}{4}n\right] + \frac{1}{\pi n}\sin\left[\frac{3\pi}{4}n\right]$$

Example 17

Consider the following difference equation:

$$y[n] - 10y[n-1] = 3x[n] - \sqrt{2}x[n-10]$$

Compute the corresponding impulse response in frequency:

$$H(\Omega) = ?$$

Example 17

$$y[n] - 10y[n-1] = 3x[n] - \sqrt{2}x[n-10]$$

Recalling the property:

$$y[n-n_0] \Longleftrightarrow e^{-j\Omega n_0}Y(\Omega)$$

$$Y(\Omega) - 10e^{-j\Omega}Y(\Omega) = 3X(\Omega) - \sqrt{2}e^{-j10\Omega}X(\Omega)$$

Example 17

$$Y(\Omega) - 10e^{-j\Omega}Y(\Omega) = 3X(\Omega) - \sqrt{2}e^{-j10\Omega}X(\Omega)$$

$$Y(\Omega)\left(1 - 10e^{-j\Omega}\right) = X(\Omega)\left(3 - \sqrt{2}e^{-j10\Omega}\right)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{3 - \sqrt{2}e^{-j10\Omega}}{1 - 10e^{-j\Omega}}$$

This is the solution.

Example 17

Consider the following difference equation:

$$y[n] = 0.5y[n-1] + y[n-2] - 3x[n]$$

Compute the corresponding impulse response in frequency:

$$H(\Omega) = ?$$

Example 17

$$y[n] = 0.5y[n-1] + y[n-2] - 3x[n]$$

$$y[n] - 0.5y[n-1] - y[n-2] = -3x[n]$$

$$Y(\Omega) - 0.5e^{-j\Omega}Y(\Omega) - e^{-j2\Omega}Y(\Omega) = -3X(\Omega)$$

$$Y(\Omega)\left(1 - 0.5e^{-j\Omega} - e^{-j2\Omega}\right) = -3X(\Omega)$$

Example 17

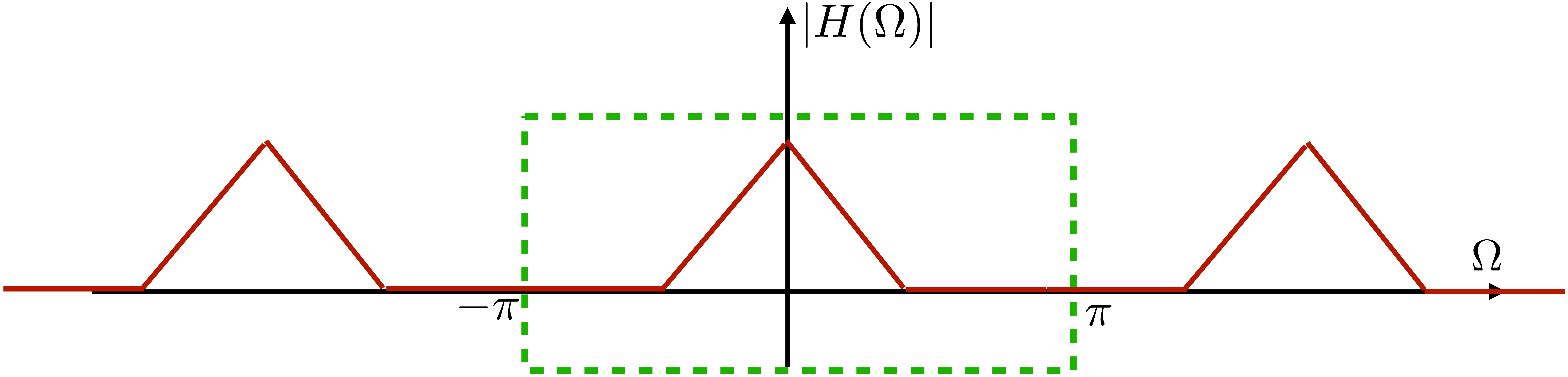
$$Y(\Omega)\left(1 - 0.5e^{-j\Omega} - e^{-j2\Omega}\right) = -3X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{-3}{1 - 0.5e^{-j\Omega} - e^{-j2\Omega}}$$

This is the solution.

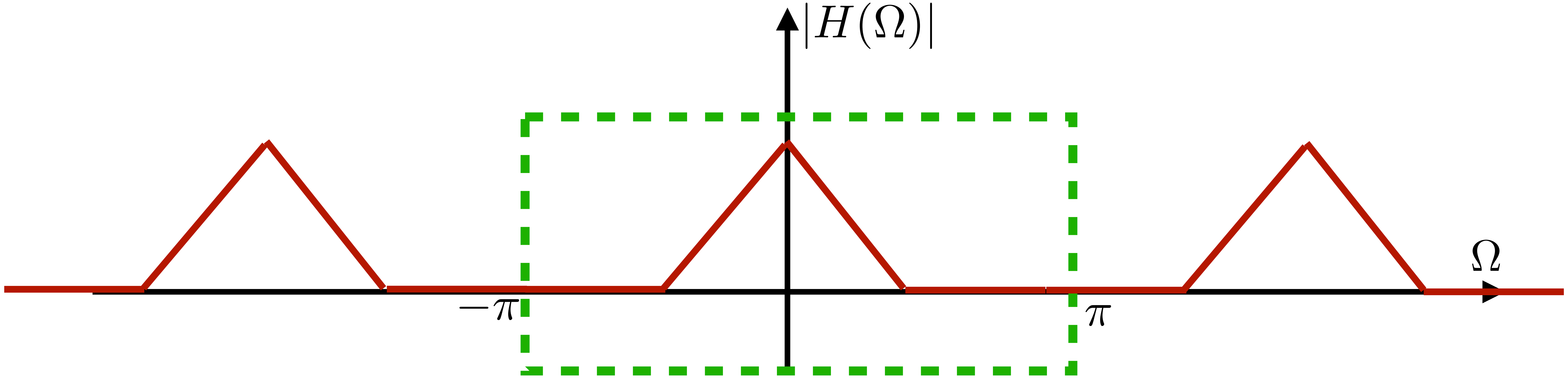
Example 18

Consider the following module of FT:



Say if it represents a low-pass filter, high-pass filter or a band-pass filter, and give a proper explanation.

Example 18

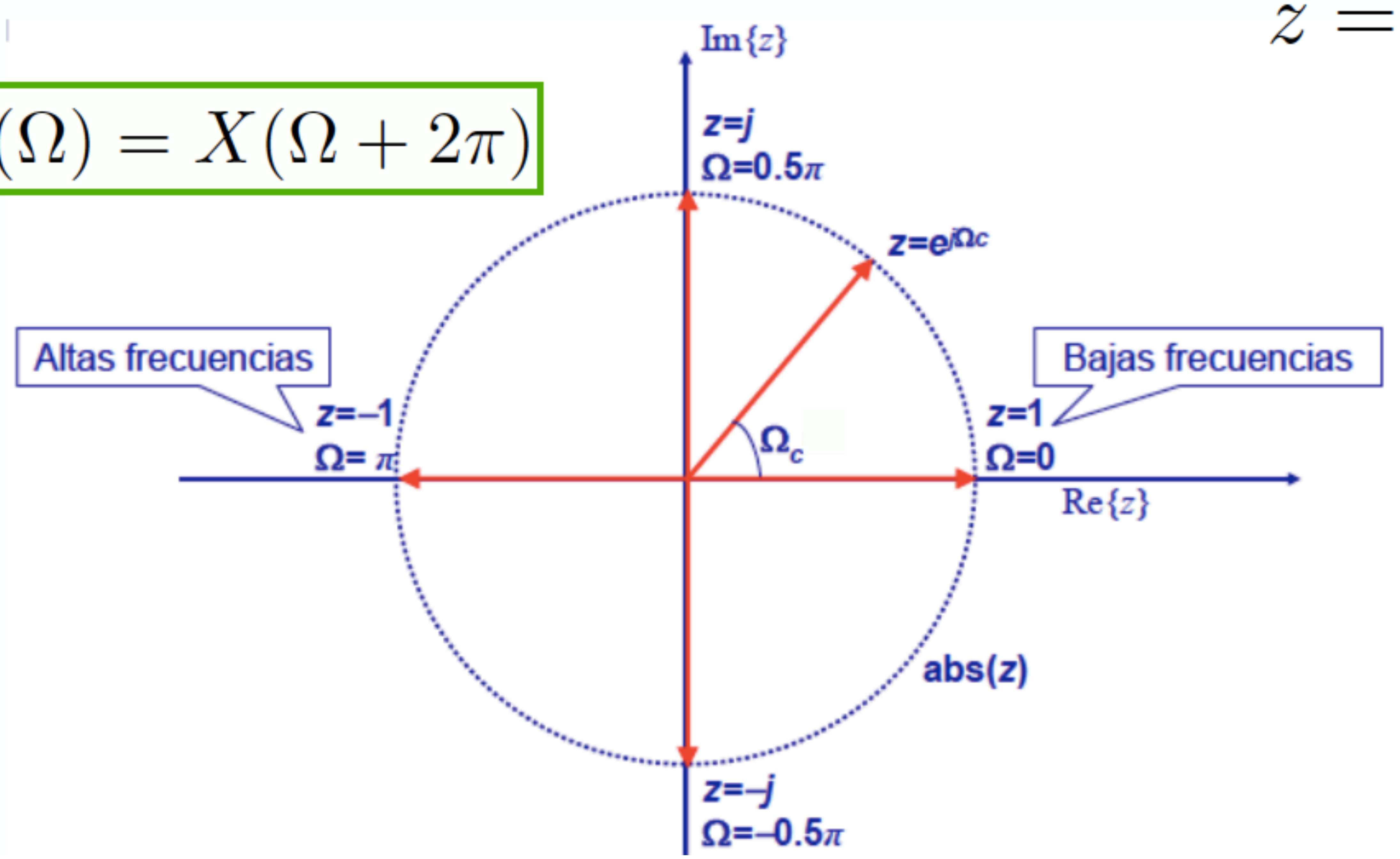


This is a low-pass filter. **(proper explanation leave to the student but recall figure below)**

Frequencies-“Omega” as an angle

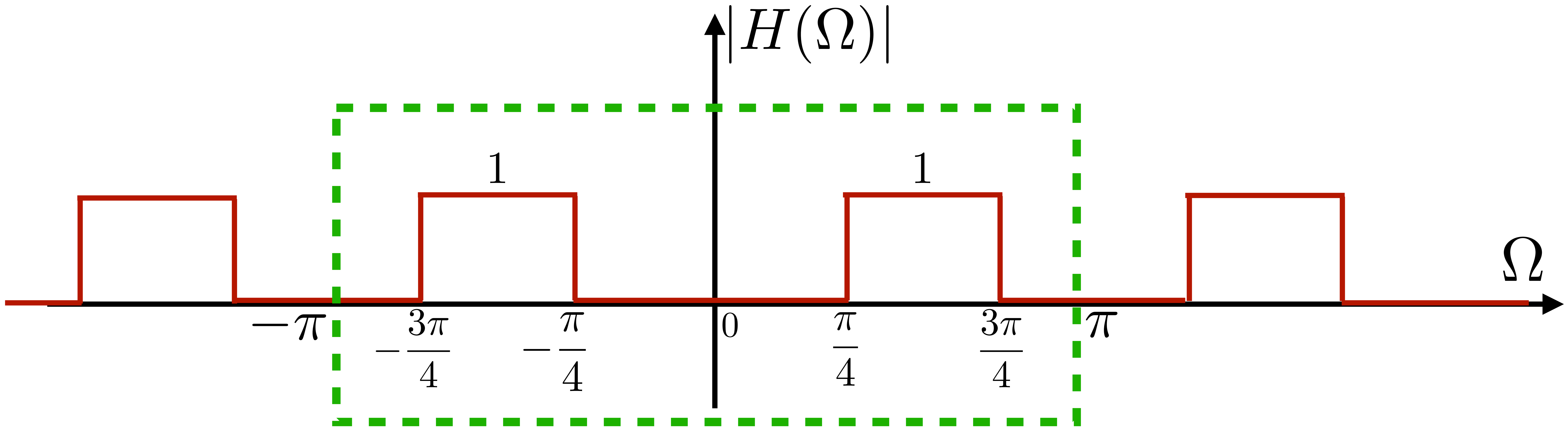
$$z = re^{j\Omega}$$

$$X(\Omega) = X(\Omega + 2\pi)$$



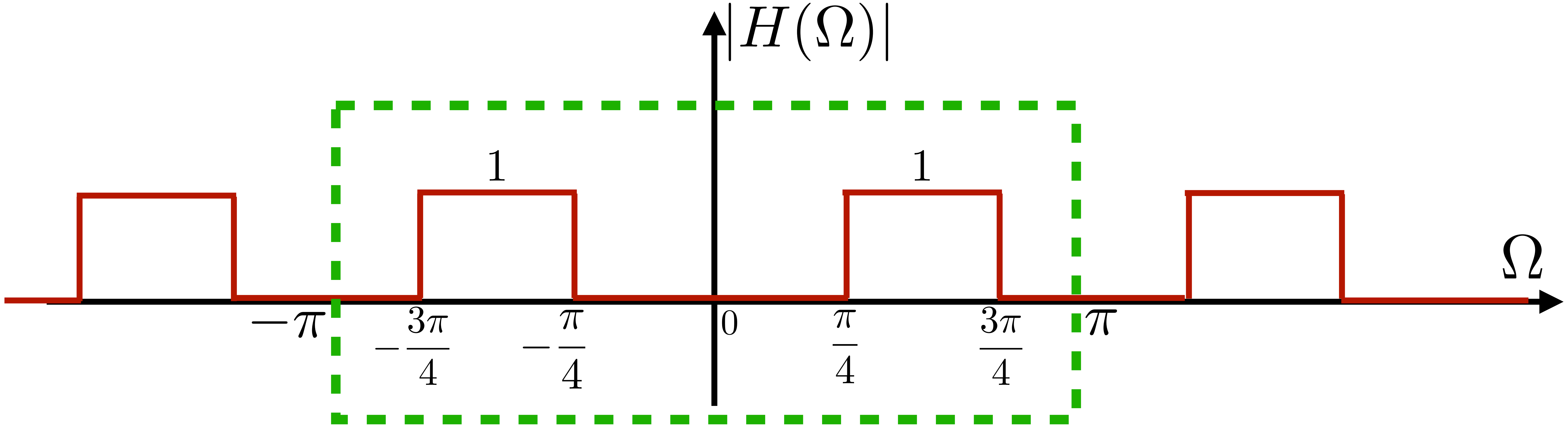
Example 19

Consider the following module of FT:

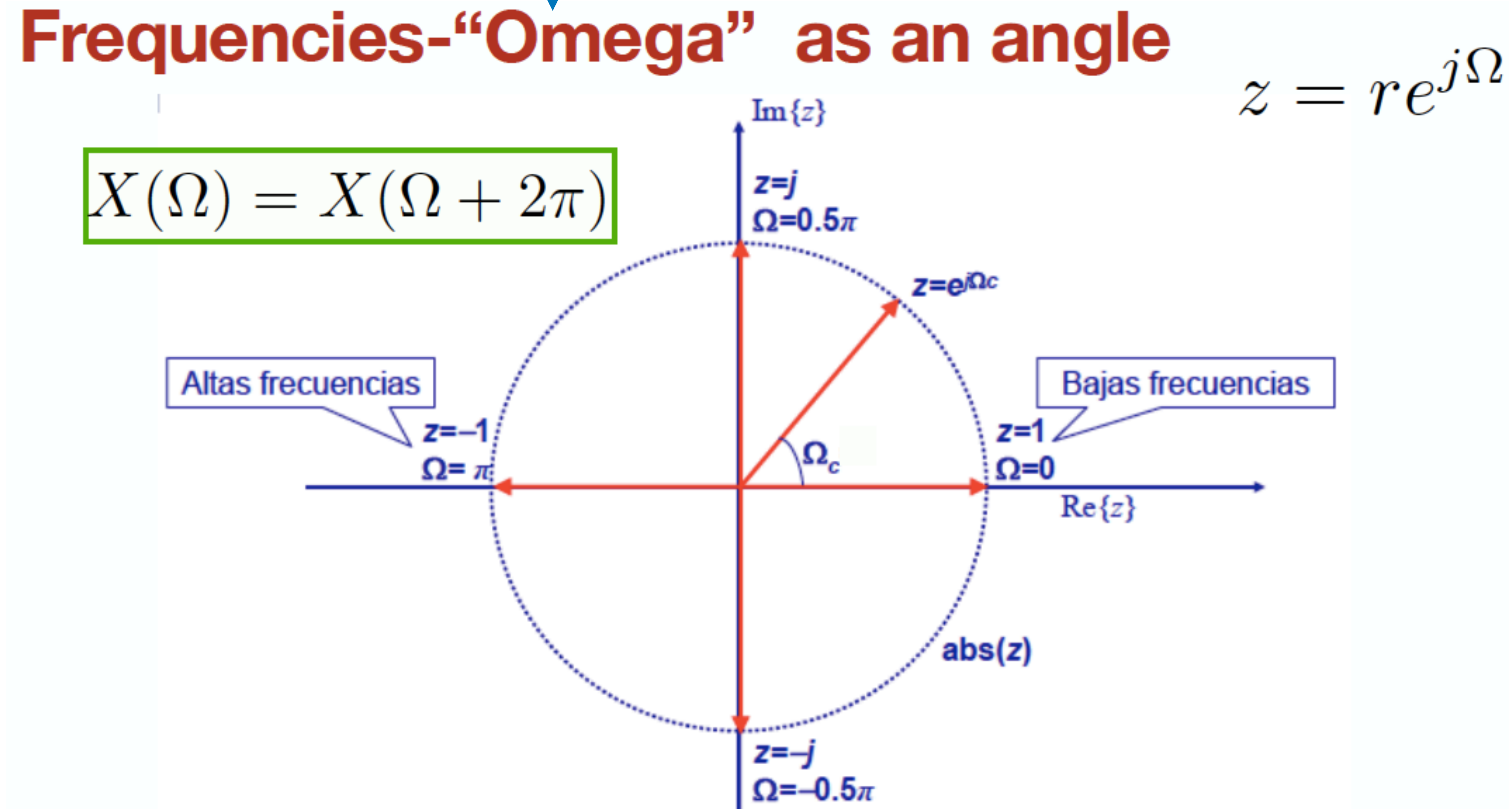


Say if it represents a low-pass filter, high-pass filter or a band-pass filter, and give a proper explanation.

Example 19

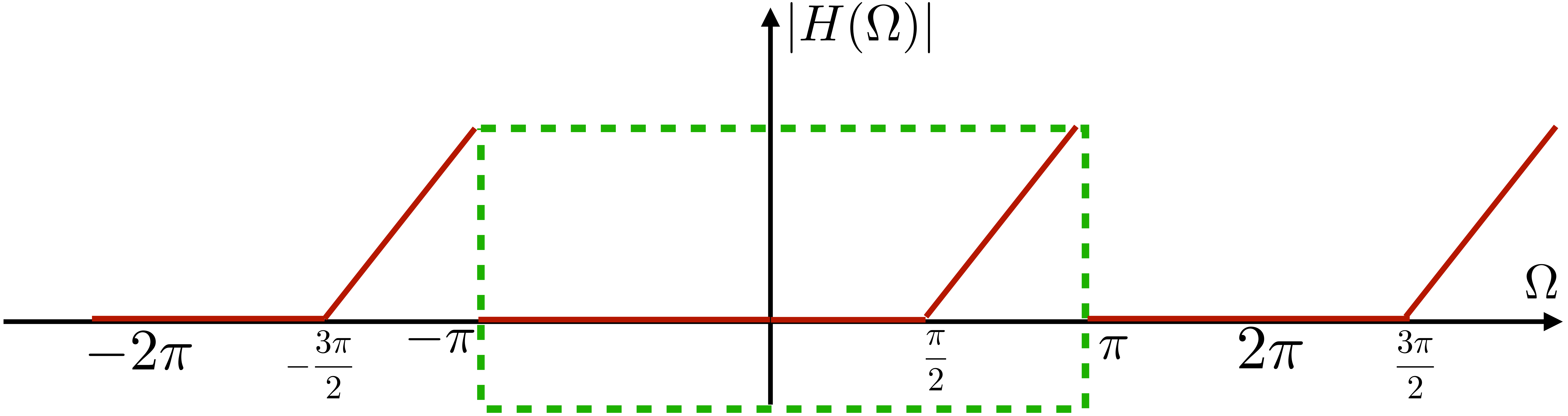


This is a band-pass filter. (proper explanation leave to the student but recall figure below)



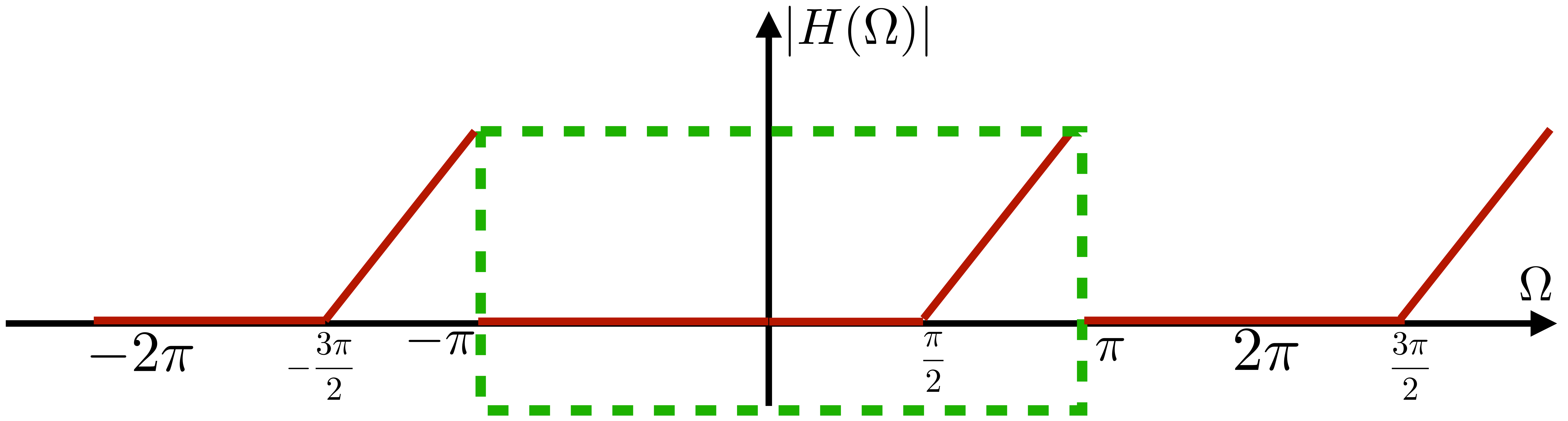
Example 20

Consider the following module of FT:

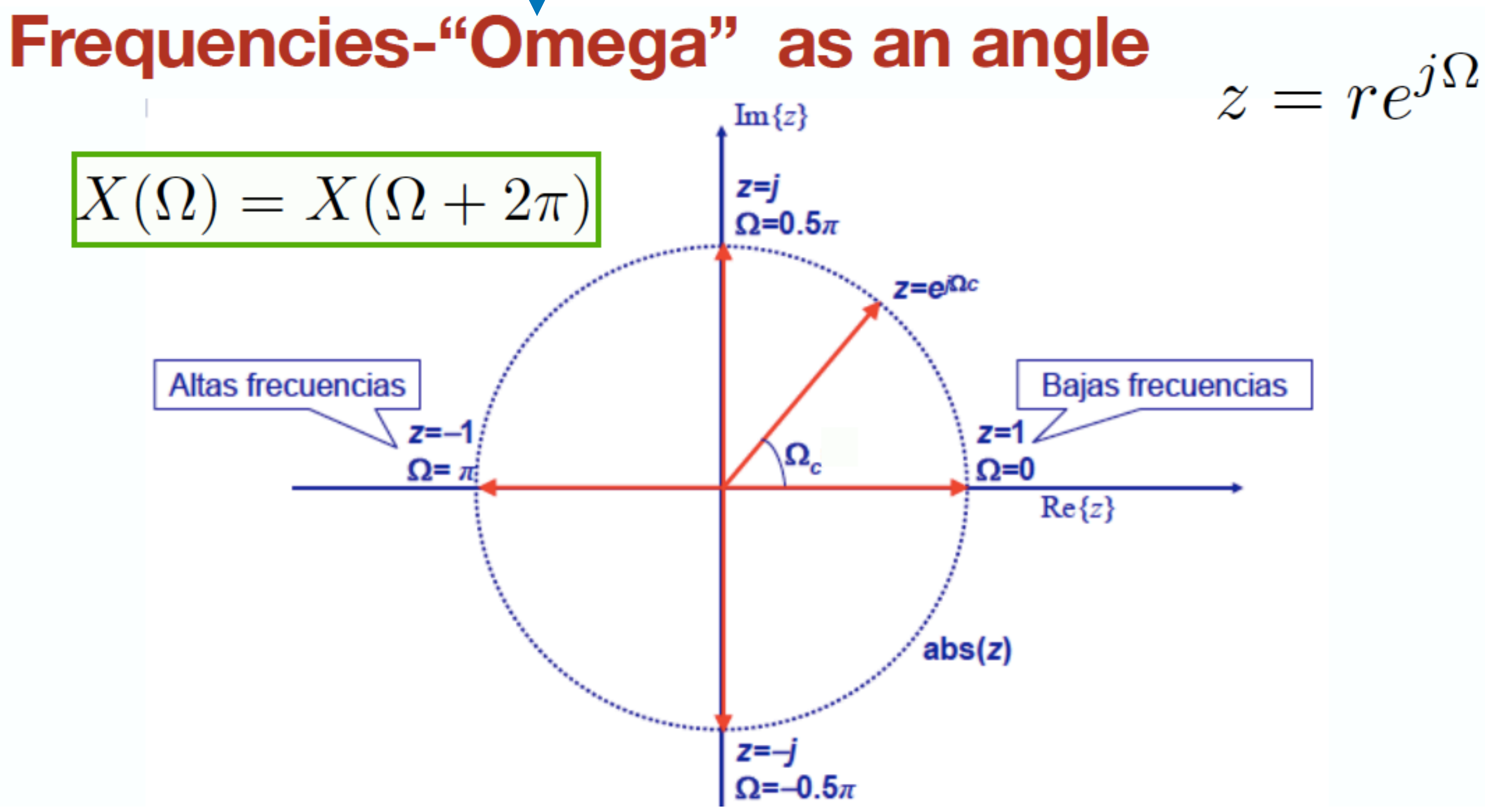


Say if it represents a low-pass filter, high-pass filter or a band-pass filter, and give a proper explanation.

Example 20



This is a high-pass filter. (proper explanation leave to the student but recall figure below)



Example 21

Consider the following FT:

$$X(\Omega) = \frac{e^{-j\Omega} - \frac{1}{3}}{1 - \frac{1}{5}e^{-j\Omega}}$$

Find the corresponding signal x[n]

Example 21

$$X(\Omega) = \frac{e^{-j\Omega} - \frac{1}{3}}{1 - \frac{1}{5}e^{-j\Omega}}$$

$$X(\Omega) = \frac{e^{-j\Omega}}{1 - \frac{1}{5}e^{-j\Omega}} - \frac{1}{3}\frac{1}{1 - \frac{1}{5}e^{-j\Omega}}$$

Recalling the formula of the geometric series or using some Tables of well-known Fourier transformation, we can write:

$$\frac{1}{1 - \frac{1}{5}e^{-j\Omega}} = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n} e^{-j\Omega n}$$

Example 21

Recalling the formula of the geometric series or using some Tables of well-known Fourier transformation, we can write:

$$\frac{1}{1-\frac{1}{5}e^{-j\Omega}} = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n} e^{-j\Omega n}$$

comparing with the definition of FT: $X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$

we can write:

$$z[n] = \left(\frac{1}{5}\right)^{n} u[n] \longleftrightarrow Z(\Omega) = \frac{1}{1-\frac{1}{5}e^{-j\Omega}}$$

Example 21

$$X(\Omega) = \frac{e^{-j\Omega}}{1 - \frac{1}{5}e^{-j\Omega}} - \frac{1}{3}\frac{1}{1 - \frac{1}{5}e^{-j\Omega}} \qquad Z(\Omega) = \frac{1}{1 - \frac{1}{5}e^{-j\Omega}}$$

Then:

$$X(\Omega) = e^{-j\Omega}Z(\Omega) - \frac{1}{3}Z(\Omega)$$

$$e^{-j\Omega}Z(\Omega) \rightarrow z[n-1]$$

Example 21

Then:

$$X(\Omega) = e^{-j\Omega} Z(\Omega) - \frac{1}{3} Z(\Omega)$$

$$x[n] = z[n-1] - \frac{1}{3} z[n]$$

$$z[n] = \left(\frac{1}{5}\right)^n u[n]$$

This is the solution.

$$x[n] = \left(\frac{1}{5}\right)^{n-1} u[n-1] - \frac{1}{3} \left(\frac{1}{5}\right)^n u[n]$$

Questions?