Solved Problems - Zeta Transform
part 2

Linear systems and circuit applications
Discrete Time Systems

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Example 10

Consider the following signal:

$$x[n] = (-1)^n u[n]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 10

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{+\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{+\infty} (-z^{-1})^n$$

ONLY IF $|-z^{-1}| < 1$

$$X(z) = \frac{1}{1-(-z^{-1})} = \frac{1}{1+z^{-1}}$$

$$= \frac{z}{z+1}$$

then the ROC is: $|z| > 1$!!!!

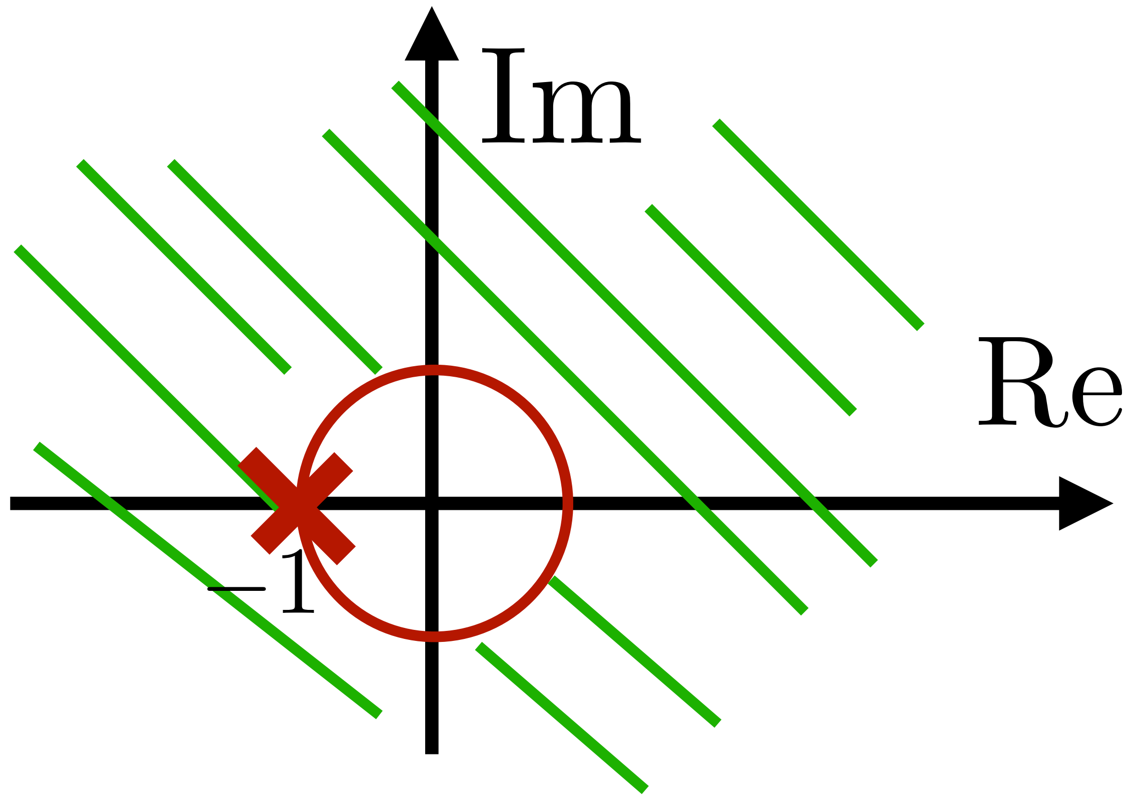
Example 10

summary:

$$X(z) = \frac{z}{z+1} \qquad \text{ROC: } |z| > 1$$

(b) zeros: one zero at z=0.

(c) poles and ROC: one pole at z=-1, and $\text{ROC: } |z| > 1$



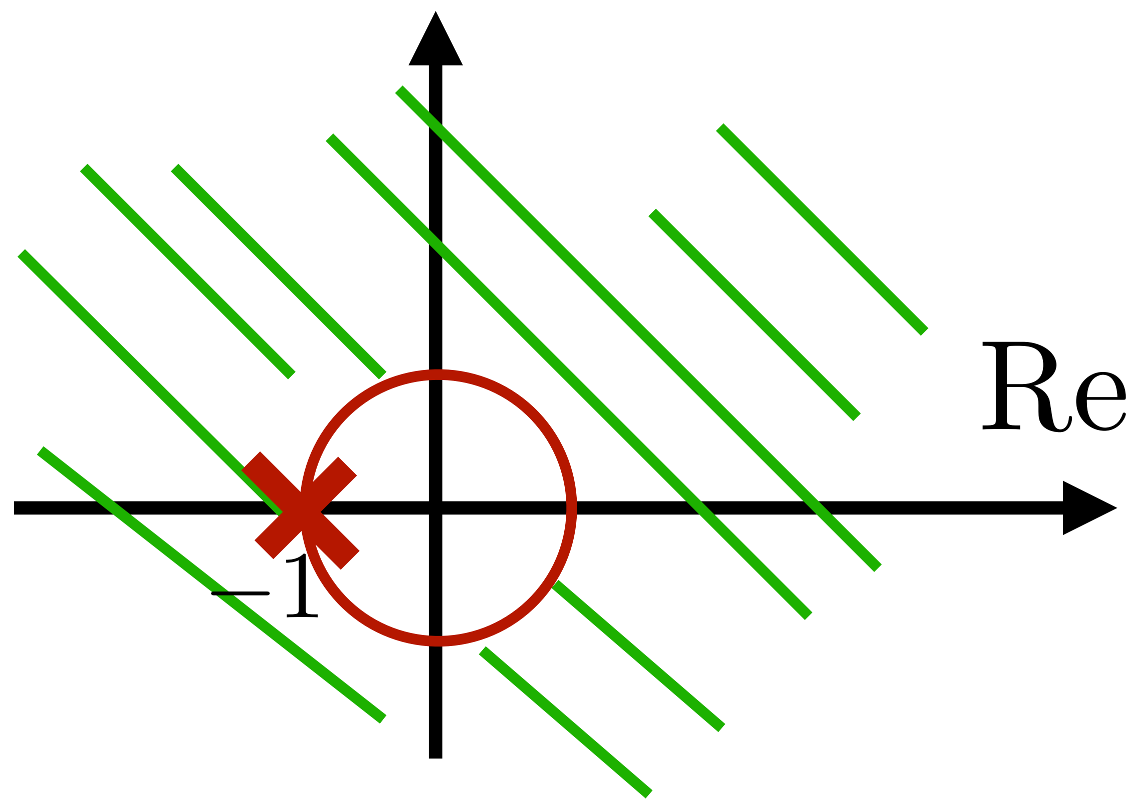
Example 10

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has **infinite energy, hence the standard FT does not exist!!**

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? **No, then standard FT DOES NOT exists.**

$$\text{ROC: } |z| > 1$$



The circle of radius 1 is not included in the ROC.

Example 11

Consider the following signal:

$$x[n] = \left(\frac{1}{2}\right)^{n+1} u[n+3]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 11

(a) Using the direct definition:

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \\
&= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{n+1} u[n+3]z^{-n} \\
&= \sum_{n=-3}^{+\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} \\
&= \frac{1}{2}\sum_{n=-3}^{+\infty} \left(\frac{1}{2}\right)^{n} z^{-n} = \frac{1}{2}\sum_{n=-3}^{+\infty} \left(\frac{1}{2}z^{-1}\right)^{n}
\end{aligned}
$$

Example 11

We have to recall:

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1}\frac{1-r^{N_2-N_1+1}}{1-r} = \frac{r^{N_1}-r^{N_2+1}}{1-r}$$

$$\sum_{n=0}^{N_2} r^n = \frac{1-r^{N_2+1}}{1-r}$$

We will use the formula above with N1=-3 y N2=Infinity

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{si } |r|<1$$

Example 11

Then, the formula is:

$$\sum_{n=N_1}^{\infty} r^n = r^{N_1} \frac{1}{1-r} \qquad \text{ONLY IF } |r| < 1$$

Since $\lim_{N_2 \to \infty} r^{N_2+1} = 0$ if $|r| < 1$.

Otherwise $\lim_{N_2 \to \infty} r^{N_2+1} = \infty$ if $|r| \geq 1$.

Example 11

... and coming back to our previous formula apply the previous formulas:

$$X(z) = \frac{1}{2}\sum_{n=-3}^{+\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

In our case r=1/2*z^(-1)

$$= \frac{1}{2}\left(\left(\frac{1}{2}\right)^{-3} z^3 \frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

ONLY IF $\left|\frac{1}{2}z^{-1}\right| < 1$

$$= \frac{1}{2}8\frac{z^3}{1-\frac{1}{2}z^{-1}} = 4\frac{z^4}{z-\frac{1}{2}}$$

$$X(z) = \frac{4z^4}{z-\frac{1}{2}}$$

ROC: $\left|\frac{1}{2z}\right| < 1 \Rightarrow \left|\frac{1}{2}\right| < |z|$ then $|z| > \frac{1}{2}$.

Example 11

summary:

$$X(z) = \frac{4z^4}{z - \frac{1}{2}}$$

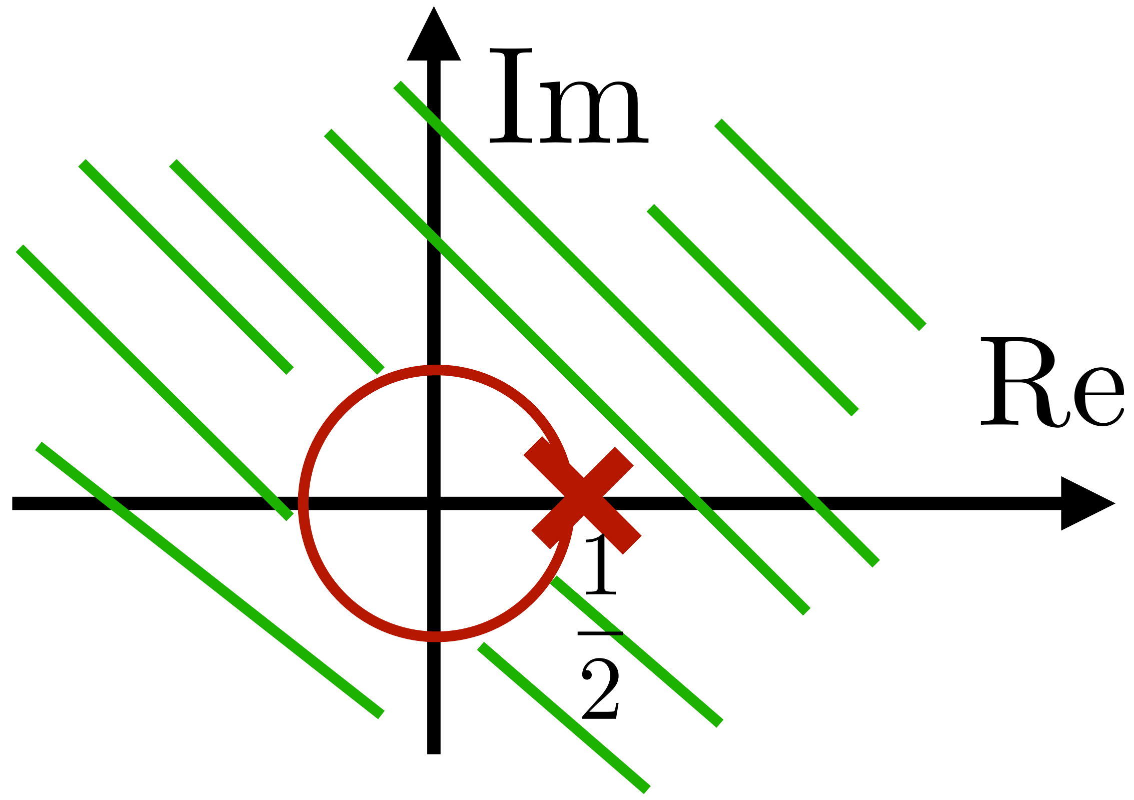
(b) zeros: a multiple zero at z=0 (of order 4, i.e., 4 coincident zeros at z=0). Infinity is not a zero (but is a pole) since

$$\lim_{z\to\infty} X(z) \approx 4\frac{4z^4}{z} = 4z^3 = \infty$$

(c) poles and ROC: one pole at z=1/2, and a multiple pole at z=Infinity (order 3 - two coincident poles at z=Infinity)

$$\text{ROC: } |z| > \frac{1}{2}$$

$$\text{except } \{\infty\} \text{ !!!}$$



Example 11

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{4z^4}{z - \frac{1}{2}} \Longleftrightarrow \frac{4e^{j4\Omega}}{e^{j\Omega} - \frac{1}{2}}$$

Example 12

Consider the following signal:

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-2]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 12

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left(-\frac{1}{3}\right)^{-n} u[-n-2]z^{-n}$$

$$= \sum_{n=-\infty}^{-2} \left(\frac{1}{3}\right)^{-n} z^{-n}$$

ONLY TRUE IF $\left|\frac{1}{3}z\right| < 1 \Longrightarrow |z| < 3$

In our case r=1/3*z

$$= \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^{k} z^{k}$$

$$= \sum_{k=2}^{\infty} \left(\frac{1}{3}z\right)^{k} = \left(\frac{1}{3}z\right)^{2} \frac{1}{1-\frac{1}{3}z}$$

$$X(z) = \frac{z^2}{3(3-z)}$$

We use the same formula of Ex. 11

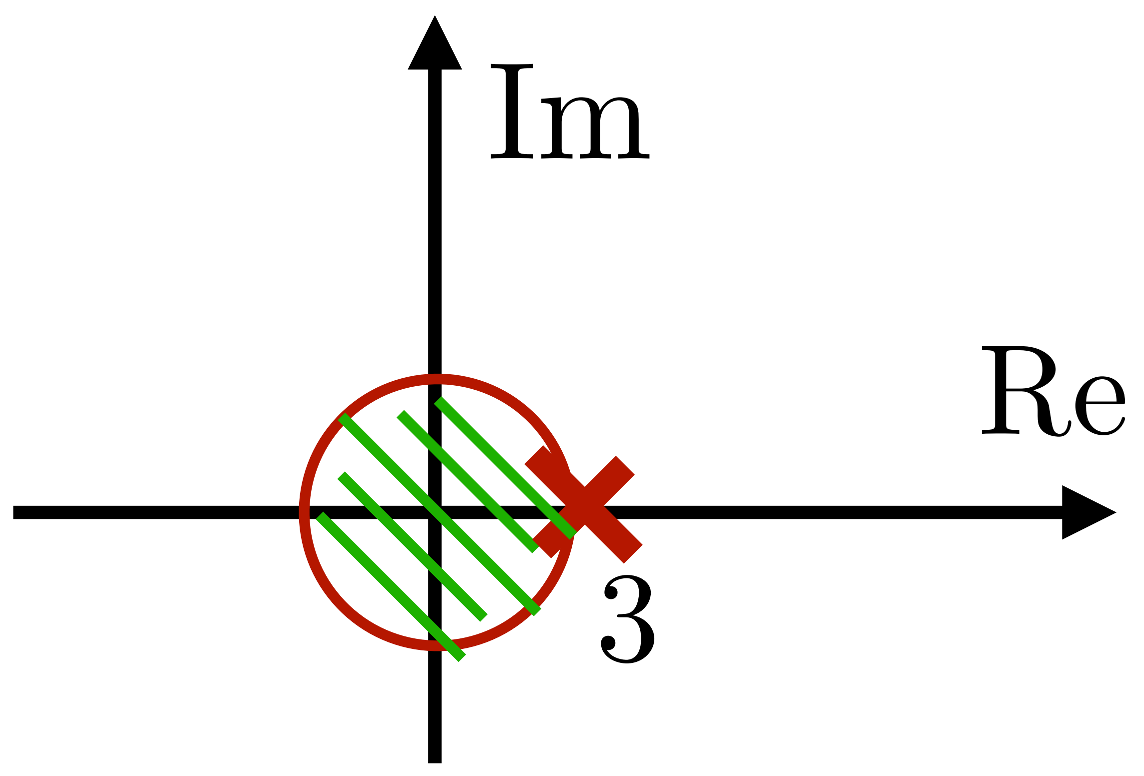
Example 12

summary: $$X(z) = \frac{z^2}{3(3-z)}$$ $$\text{ROC: } |z| < 3$$

(b) zeros: a multiple zero at z=0 (of order 2, i.e., 2 coincident zeros at z=0).

(c) poles and ROC: one pole at z=3, and a one pole at z=Infinity

$$\text{ROC: } |z| < 3$$



Example 12

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{z^2}{3(3-z)} \Longleftrightarrow \frac{e^{j2\Omega}}{3(3-e^{j\Omega})}$$

Example 13

Consider the following signal:

$$x[-1] = 1.5, x[0] = -3, x[6] = 10, \text{ and for the rest of } n, \ x[n] = 0.$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 13

(a) We can use the definition. However, we can also consider the property:

$$x[n-n_0] \Longleftrightarrow z^{-n_0}X(z)$$
$$\delta[n-n_0] \Longleftrightarrow z^{-n_0}$$

and since that our signal can be written as:

$$x[n] = 1.5\delta[n+1] - 3\delta[n] + 10\delta[n-6]$$

We can write:

$$X(z) = 1.5z - 3 + 10z^{-6} = \frac{1.5z^7 - 3z^6 + 10}{z^6}$$

Example 13

(b) Zeros of X(z):

$$X(z) = \frac{1.5z^7 - 3z^6 + 10}{z^6}$$

The zeros are:

- **7 different zeros which are the solutions in the complex plane of the equation:**

$$1.5z^7 - 3z^6 + 10 = 0$$

Example 13

$$X(z) = \frac{1.5z^7 - 3z^6 + 10}{z^6}$$

Why infinity is not a zero? since

$$\lim_{z \to +\infty} X(z) \approx \frac{1.5z^7}{z^6} = 1.5z = \infty$$

Example 13

(c) Poles of X(z):

$$X(z) = \frac{1.5z^7 - 3z^6 + 10}{z^6}$$

We have a multiple pole (of order 6) at z=0 (6 coincident poles at z=0) and a single pole (i.e., order 1) at z=Infinity, since

$$\lim_{z \to +\infty} X(z) \approx \frac{1.5z^7}{z^6} = 1.5z = \infty$$

The the ROC is all the complex plane except zero and Infinity!!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C} \backslash \{\infty\} \cup \{0\}$$

Example 13

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{1.5z^7 - 3z^6 + 10}{z^6} \Longleftrightarrow X(\Omega) = \frac{1.5e^{j7\Omega} - 3e^{j6\Omega} + 10}{e^{j6\Omega}}$$

Questions?