Solved Problems - Zeta Transform part 4

Linear systems and circuit applications

Discrete Time Systems

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Example 24

Consider the following Zeta Transform:

$$X(z) = \frac{z^2}{(z - 1/3)(z - 2)}$$

Find all the possible ROCs and obtain the corresponding signals x[n].

Example 24

The poles are

$$z = \frac{1}{3}$$

$$z = 2$$

Possible ROCs:

$$\text{ROC}: \quad |z| < \frac{1}{3}$$

$$\text{ROC}: \quad \frac{1}{3} < |z| < 2$$

$$\text{ROC}: \quad |z| > 2$$

Example 24

For finding all x[n]'s, first of all:

$$X(z) = \frac{z^2}{(z-1/3)(z-2)} = \frac{1}{(1-1/3z^{-1})(1-2z^{-1})}$$

$$X(z) = \frac{1}{(1-1/3z^{-1})(1-2z^{-1})} = \frac{A}{(1-1/3z^{-1})} + \frac{B}{(1-2z^{-1})}$$

We have to find the value A and B.

Example 24

We have to find the value A and B:

$$X(z) = \frac{1}{(1-1/3z^{-1})(1-2z^{-1})} = \frac{A}{(1-1/3z^{-1})} + \frac{B}{(1-2z^{-1})}$$

$$= \frac{1}{(1-1/3z^{-1})(1-2z^{-1})} = \frac{A(1-2z^{-1}) + B(1-1/3z^{-1})}{(1-1/3z^{-1})(1-2z^{-1})}$$

$$\Downarrow$$

$$A(1-2z^{-1}) + B(1-1/3z^{-1}) = 1$$

Example 24

$$A(1-2z^{-1})+B(1-1/3z^{-1})=1$$

$$A+B-\left(2A+\frac{1}{3}B\right)z^{-1}=1$$

Then:

$$A+B=1$$

$$2A+\frac{1}{3}B=0$$

$$B=\frac{6}{5}$$

$$A=-\frac{1}{5}$$

Example 24

Thus, we arrive:

$$X(z) = \frac{-1/5}{(1 - 1/3z^{-1})} + \frac{6/5}{(1 - 2z^{-1})}$$

Now, we have to consider the 3 different possible ROCs

$$\text{ROC}: \quad |z| < \frac{1}{3}$$

$$\text{ROC}: \quad \frac{1}{3} < |z| < 2$$

$$\text{ROC}: \quad |z| > 2$$

Example 24

If we consider the ROC |z|<1/3 and using the "tables" (i.e., the transformation that we already known), we can come back in time. Note that if the complete ROC is |z|<1/3, it means that we have |z|<1/3 for the first sequence (first part of the sum) and |z|<2 for the second sequence (second part of the sum). Namely,

$$\text{ROC:} \qquad \{|z| < 1/3\} = \{|z| < 1/3\} \cup \{|z| < 2\}$$

left-sided signal   left-sided signal

Similar arguments can be used the other ROCs…and find "right-sided" and "left-sided" signals…

Example 24

Then with the ROC |z|<1/3:

$$X(z) = \frac{-1/5}{(1 - 1/3z^{-1})} + \frac{6/5}{(1 - 2z^{-1})}$$

See other examples or tables!

$$x[n] = -\frac{1}{5}\left(-\left(\frac{1}{3}\right)^n u[-n-1]\right) + \frac{6}{5}\left(-2^n u[-n-1]\right)$$

Example 24

Then with the ROC 1/3<|z|<2:

$$X(z) = \frac{-1/5}{(1 - 1/3z^{-1})} + \frac{6/5}{(1 - 2z^{-1})}$$

See other examples or tables!

$$x[n] = -\frac{1}{5}\left(\left(\frac{1}{3}\right)^n u[n]\right) + \frac{6}{5}\left(-2^n u[-n-1]\right)$$

Example 24

Then with the ROC |z|>2:

$$X(z) = \frac{-1/5}{(1 - 1/3z^{-1})} + \frac{6/5}{(1 - 2z^{-1})}$$

See other examples or tables!

$$x[n] = -\frac{1}{5}\left(\left(\frac{1}{3}\right)^n u[n]\right) + \frac{6}{5}\left(2^n u[n]\right)$$

Example 25

Consider the following Zeta Transform:

$$X(z) = \frac{z-2}{z^2-1}$$

Find all the possible ROCs and say how many different signals x[n] can have this X(z).

Example 25

$$X(z) = \frac{z-2}{z^2-1}$$

The poles are $z = 1, \quad z = -1$

Then we have two possible ROCs (and then only two possible signals):

$$\text{ROC:} \quad |z| < 1,$$

$$\text{ROC:} \quad |z| > 1.$$

Example 26

Consider the following Zeta Transform:

$$X(z) = \frac{z-2}{z^2-1}$$

Try to find all the possible x[n] as in Example 24 (and using the information in Example 25).

Example 26

$$X(z) = \frac{z-2}{z^2-1} = \frac{z-2}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$= \frac{(A+B)z + A - B}{z-1}$$

$$\begin{aligned} A + B &= 1 \\ A - B &= -2 \end{aligned} \iff A = -\frac{1}{2}, B = \frac{3}{2}$$

Example 26

$$X(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z+1}$$

$$X(z) = -\frac{1}{2}\frac{z^{-1}}{1-z^{-1}} + \frac{3}{2}\frac{z^{-1}}{1+z^{-1}}$$

$z^{-1}X_1(z)$ $\qquad$ $z^{-1}X_2(z)$

$$X_1(z) = \frac{1}{1-z^{-1}} \qquad X_2(z) = \frac{1}{1+z^{-1}}$$

Example 26

Solution for the case |z|>1:

$$x[n] = -\frac{1}{2}(1)^{n-1}u[n-1] + \frac{3}{2}(-1)^{n-1}u[n-1]$$

$$x[n] = -\frac{1}{2}u[n-1] + \frac{3}{2}(-1)^{n-1}u[n-1]$$

Solution for the case |z|<1:

$$x[n] = \frac{1}{2}u[-n-2] - \frac{3}{2}(-1)^{n-1}u[-n-2]$$

Example 27

Un sistema LIT discreto causal está caracterizado por la siguiente ecuación en diferencias,

$$y[n] - \frac{1}{2}y[n-1] - \frac{1}{2}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(a) Obtenga la función de transferencia del sistema, $H(z)$.

(b) Obtenga y represente la región de convergencia (ROC) y el diagrama de polos y ceros de $H(z)$. Indique justificadamente si el sistema es estable o no.

(c) Determine la respuesta al impulso del sistema, $h[n]$.

(d) Calcule la salida del sistema, $y[n]$, cuando la entrada es $x[n] = 2^n$.

(e) Obtenga el sistema inverso de $H(z)$, $H_{inv}(z)$. ¿Podría ser $H_{inv}(z)$ un sistema anticausal?. Justifique la respuesta.

Example 27

Solución:

(a) Aplicando la transformada Z a la ecuación en diferencias obtenemos,

$$Y(z)\left(1-\tfrac{1}{2}z^{-1}-\tfrac{1}{2}z^{-2}\right)=X(z)\left(1-\tfrac{1}{2}z^{-1}\right)$$

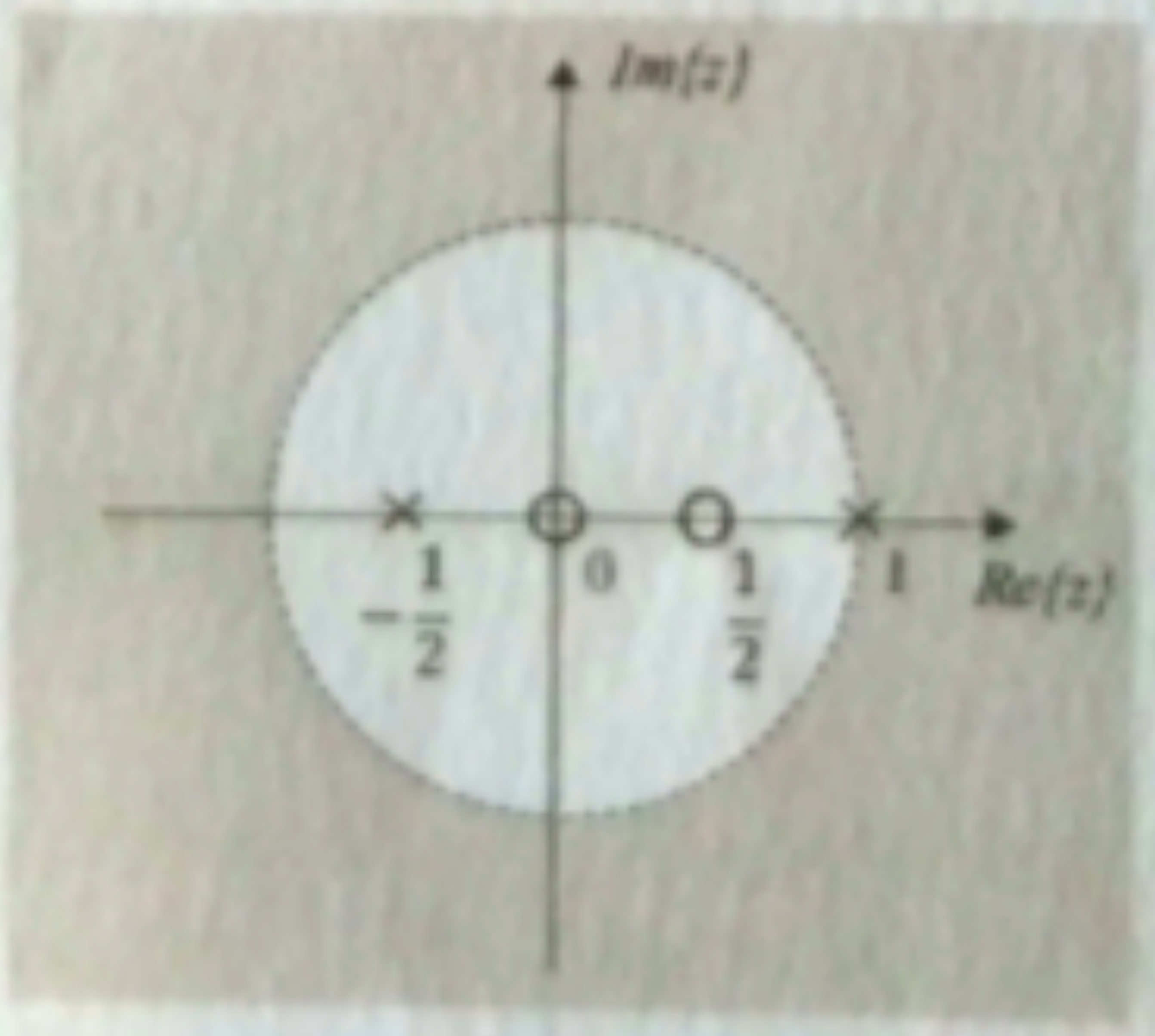
Por tanto concluimos que,

$$H(z)=\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}=\frac{z^2-\frac{1}{2}z}{z^2-\frac{1}{2}z-\frac{1}{2}}$$

(b) Como puede observarse, hay un cero en $z_{c1}=0$ y otro en $z_{c2}=\frac{1}{2}$. Además, el sistema tiene dos polos en $z_{p1}=1$ y otro en $z_{p2}=-\frac{1}{2}$. Teniendo en cuenta que se trata de un sistema causal, su ROC se extiende desde el polo más exterior hasta infinito. En nuestro caso, $\text{ROC}=\{|z|>1\}$.

Example 27

El diagrama de polos y ceros está representado en la figura siguiente,



El sistema no es estable ya que la ROC no incluye el círculo unidad.

(a causal system is not stable only if all the poles has a module smaller than 1)

(and a non-causal system? answer as an homework)

Example 27

(c) Para obtener la respuesta al impulsode $H(z)$, realizamos la descomposición en fracciones simples,

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{(1 - z^{-1})(1 + \frac{1}{2}z^{-1})}$$
$$= \frac{A}{1 - z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}}$$
$$= \frac{A(1 + \frac{1}{2}z^{-1}) + B(1 - z^{-1})}{(1 - z^{-1})(1 + \frac{1}{2}z^{-1})}$$

Identificando numeradores concluímos que,

$$A\left(1 + \frac{1}{2}z^{-1}\right) + B\left(1 - z^{-1}\right) = 1 - \frac{1}{2}z^{-1}.$$

Igualando los términos de igual orden en esta expresión despejamos los valores de A y B,

$$A + B = 1$$
$$\frac{1}{2}A - B = -\frac{1}{2} \quad \Rightarrow \quad A = \frac{1}{3} \quad B = \frac{2}{3}$$

Example 27

Entonces,

$$H(z) = \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{2}{3}}{1 + \frac{1}{2}z^{-1}}.$$

Teniendo en cuenta que la ROC es $\{|z| > 1\}$ y utilizando la tabla de pares de transformada Z obtenemos,

$$h[n] = \frac{1}{3} \cdot u[n] + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n u[n]$$

Example 27

(d) Dado que $x[n] = 2^n$ es una autosecuencia de los sistemas LIT discretos, su salida se puede calcular como,

$$y[n] = x[n]H(z = 2) = \frac{6}{5}2^n$$

(e) El sistema inverso de $H(z)$ es,

$$H_{inv}(z) = \frac{1}{H(z)} = \frac{z^2 - \frac{1}{2}z - \frac{1}{2}}{z^2 - \frac{1}{2}z}$$

Como puede observarse, $H_{inv}(z)$ tiene un polo en $z = 0$, por lo que $z = 0$ no pertenece a su ROC. Por lo tanto, no puede ser un sistema anticausal, ya que la ROC de los sistemas anticausales es el interior de la circunferenica delimitada por el polo más interno incluyendo el punto $z = 0$.

Example 28

Consider two cascade-connected LTI systems. The first system is **causal** and is characterised by the following differential equation

$$y_1[n] - \frac{1}{2}y_1[n-1] = x_1[n].$$

The second system is **stable** with frequency response

$$H_2(z) = \frac{5}{(1-2z^{-1})(1+3z^{-1})}$$

(a) Obtain the Z-transform of the first system, $H_1(z)$. Determine its region of convergence (ROC) together with its poles-zeros diagram.

(b) Determine the ROC of the second system ($H_2(z)$). Plot it schematically together with its poles-zeros diagram. Determine the impulse response in time domain ($h_2[n]$).

(c) Plot the ROC and the poles-zeros diagram of the join system $H(z) = H_1(z)H_2(z)$.

(d) Obtain the differential equation in the time domain describing the system $H(z)$. Does it depend on the ROC of $H(z)$?

(e) Determine the output $y[n]$ of the system $H(z)$ for the input signal $x[n] = \frac{3}{5}(\delta[n] + 3\delta[n-1])$. Does it depend on the ROC of $H(z)$?

Example 28

Solution:

(a) Applying the Z-transform to the differential equation we have

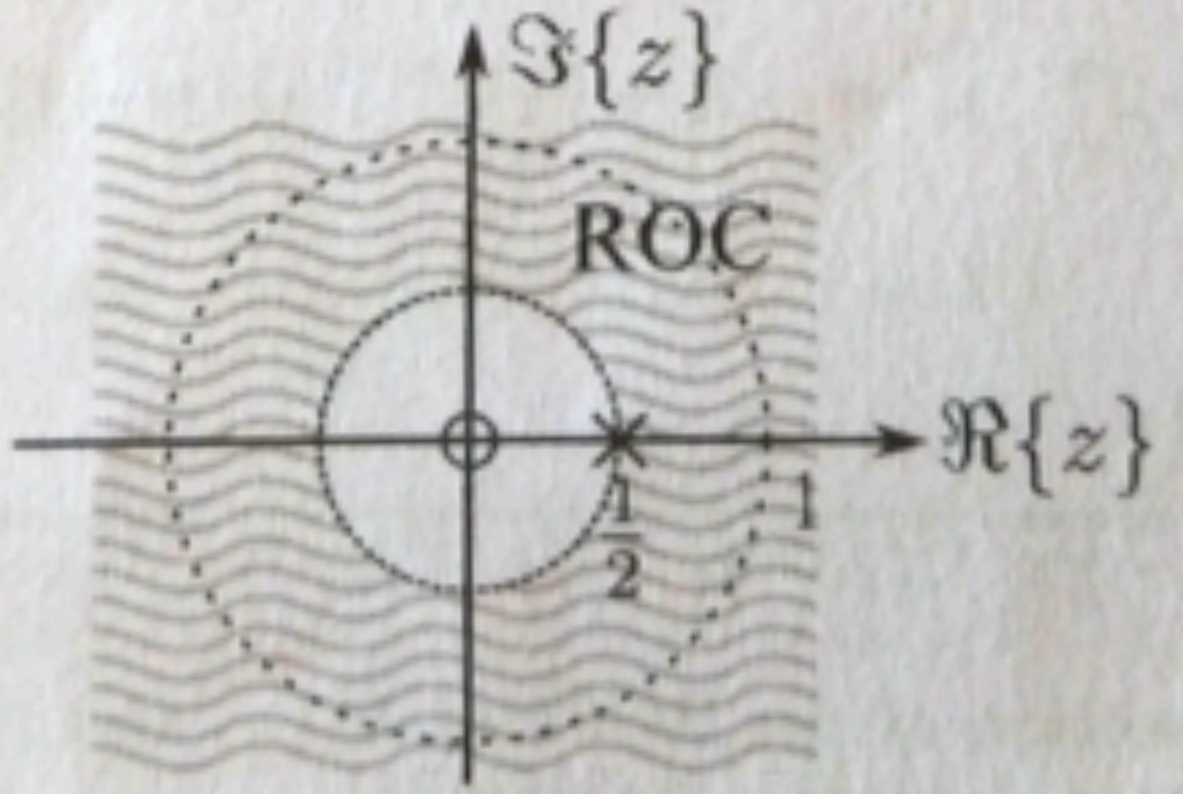
$$Y_1(z)\left(1-\tfrac{1}{2}z^{-1}\right)=X_1(z) \Rightarrow Y_1(z)=\frac{1}{1-\frac{1}{2}z^{-1}}X_1(z).$$

Therefore, we may conclude that

$$H_1(z)=\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{z}{z-\frac{1}{2}}$$

There is a zero in $z=0$ and a pole in $z=\frac{1}{2}$. Since it is a causa system, its response to the impulse signal is right-sided. Thus, its ROC must extend from the outer pole to the infinity. For our system, $\text{ROC}_{H_1}=\left\{|z|>\frac{1}{2}\right\}$.

The poles-zeros diagram is:



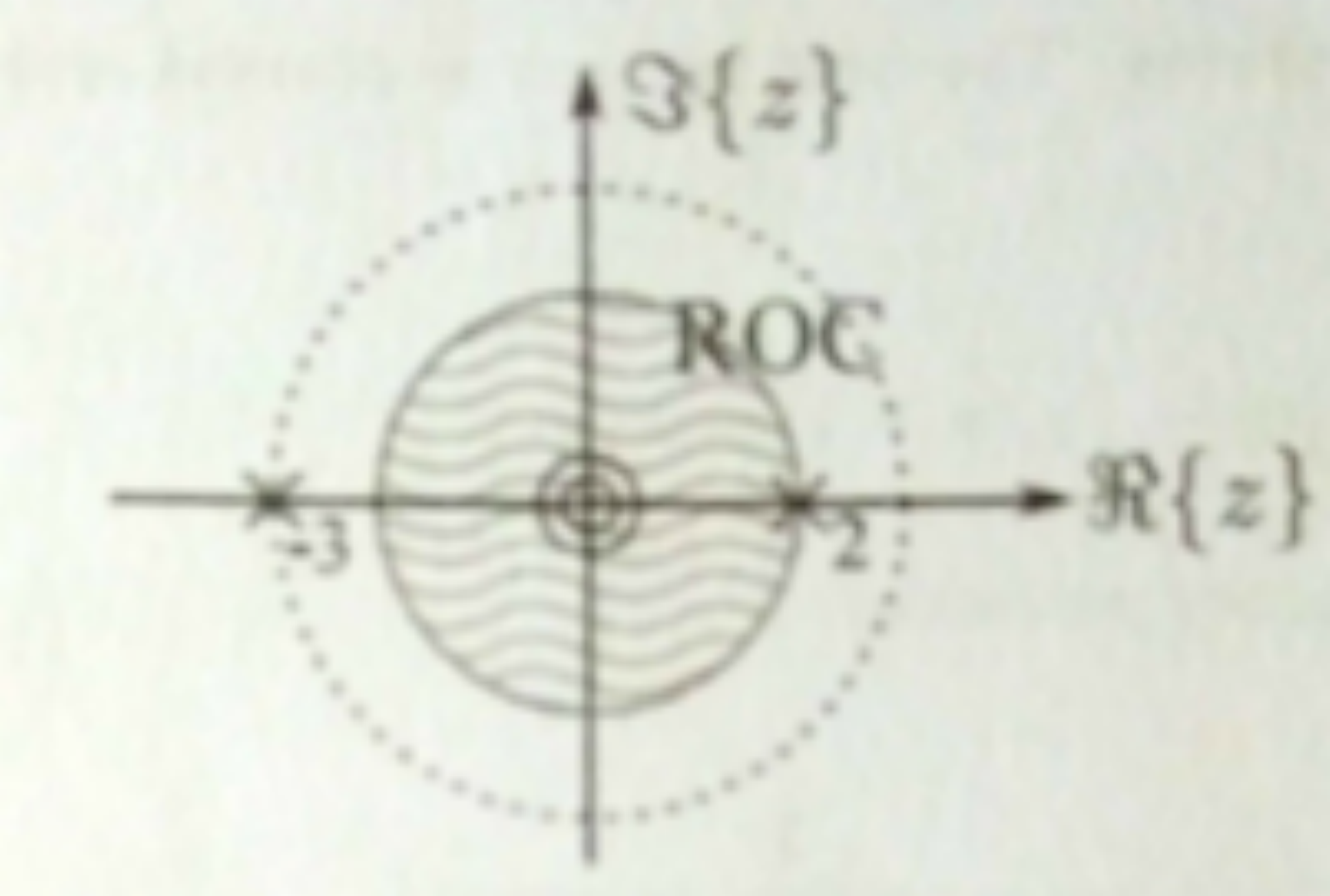
Example 28

(b) We have

$$H_2(z) = \frac{5}{(1-2z^{-1})(1+3z^{-1})} = \frac{5z^2}{(z-2)(z+3)}.$$

So, we can see that there is a double zero at $z = 0$, and two poles at $z = 2$ and $z = -3$, respectively. Since the system is stable, its ROC must contains the unitary circumference. Therefore, the ROC has to be extended from the pole at $z = 2$ inwards. $ROC_{H_2} = \{|z| < 2\}$.

The poles-zeros diagram is:



Example 28

To obtain the impulse response in the time domain, we first need to extend $H_2(z)$ as simple fractions:

$$\begin{aligned} H_2(z) &= \frac{5}{(1-2z^{-1})(1+3z^{-1})} \\ &= \frac{A}{1-2z^{-1}} + \frac{B}{1+3z^{-1}} \\ &= \frac{A(1+3z^{-1}) + B(1-2z^{-1})}{(1-2z^{-1})(1+3z^{-1})}. \end{aligned}$$

Identifying the two numerators,

$$A(1+3z^{-1}) + B(1-2z^{-1}) = 5.$$

This relation must hold for any value of z. Substituting $z = -3$ and $z = 2$, we obtain

$$\begin{aligned} z = -3 &\Rightarrow A \cdot 0 + B(1 + \frac{2}{3}) = 5 \Rightarrow B = 3. \\ z = 2 &\Rightarrow A(1 + \frac{3}{2}) + B \cdot 0 = 5 \Rightarrow A = 2. \end{aligned}$$

Thus,

$$H_2(z) = \frac{2}{1-2z^{-1}} + \frac{3}{1+3z^{-1}}.$$

Example 28

The ROC of $H_2(z)$ is $\text{ROC}_{H_2} = \{|z| < |2| < |-3|\}$. Using the table of Z-transform pairs,

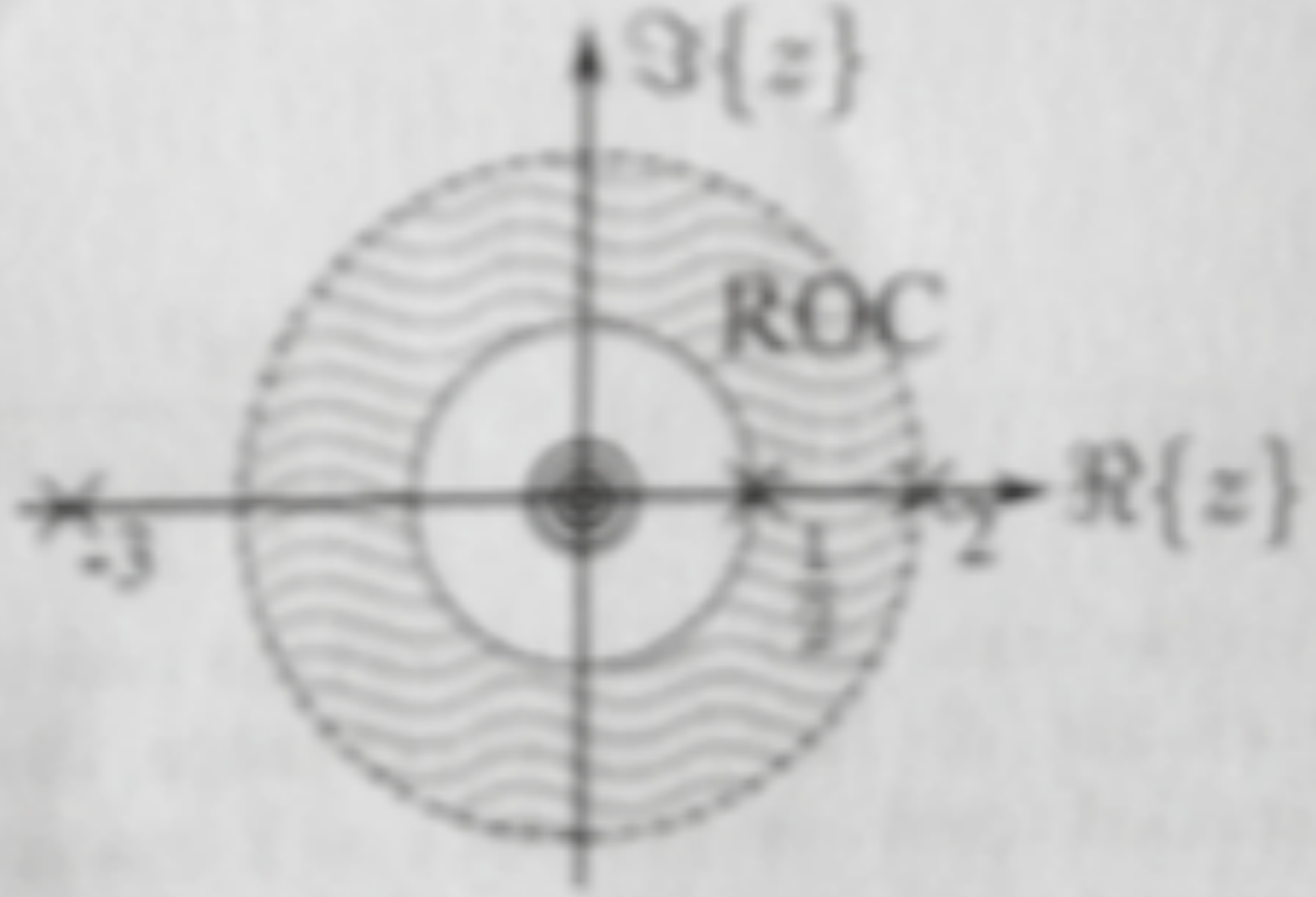
$$\begin{aligned} h_2[n] &= -2\cdot 2^n u[-n-1] - 3\cdot(-3)^n u[-n-1] \\ &= -2^{n+1} u[-n-1] + (-3)^{n+1} u[-n-1]. \end{aligned}$$

Example 28

(c) The join system is

$$H(z) = H_1(z)H_2(z) = \frac{5}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1+3z^{-1})} = \frac{z^3}{(z-\frac{1}{2})(z-2)(z+3)}$$

and its ROC must be the intersection of the ROCs of $H_1(z)$ and $H_2(z)$. This is, $ROC_H = \{\frac{1}{2} < |z| < 2\}$. The corresponding poles-zeros diagram is:



Example 28

(d) To obtain the differential equation describing $H(z)$, it is not necessary knowing its ROC. First, we perform the operation in the denominator

$$H(z) = \frac{5}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 + 3z^{-1})} = \frac{5}{1 + \frac{1}{2}z^{-1} - \frac{14}{2}z^{-2} + 3z^{-3}}.$$

Using $Y(z) = H(z)X(z)$, we obtain

$$Y(z) = \frac{5}{1 + \frac{1}{2}z^{-1} - \frac{14}{2}z^{-2} + 3z^{-3}} X(z),$$

$$Y(z)(1 + \frac{1}{2}z^{-1} - \frac{13}{2}z^{-2} + 3z^{-3}) = 5X(z),$$

and in the time domain

$$y[n] + \frac{1}{2}y[n-1] - \frac{14}{2}y[n-2] + 3y[n-3] = 5x[n].$$

Example 28

(e) The Z-transform of $x[n]$ is

$$x[n] = \frac{3}{5}(\delta[n] + 3\delta[n-1]) \quad \overset{\mathcal{Z}}{\leftrightarrow} \quad X(z) = \frac{3}{5}(1 + 3z^{-1})$$

Taking the relation $Y(z) = H(z)X(z)$ into account and simplifying te expression,

$$\begin{aligned} Y(z) &= \frac{5 \cdot \frac{3}{5}(1 + 3z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 + 3z^{-1})} \\ &= \frac{3}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}. \end{aligned}$$

To obtain $y[n]$ we first expand in simple fractions of $Y(z)$:

$$\begin{aligned} Y(z) &= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - 2z^{-1}} \\ &= \frac{A(1 - 2z^{-1}) + B(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}. \end{aligned}$$

Matching denominators,

$$A(1 - 2z^{-1}) + B(1 - \frac{1}{2}z^{-1}) = 3.$$

Example 28

Form the roots of the equation we can determine A and B:

$$z = 2 \Rightarrow A \cdot 0 + B(1 - \tfrac{1}{4}) = 3 \Rightarrow B = 4.$$
$$z = \tfrac{1}{2} \Rightarrow A(1 - 4) + B \cdot 0 = 3 \Rightarrow A = -1.$$

Then,

$$Y(z) = \frac{-1}{1 - \frac{1}{2}z^{-1}} + \frac{4}{1 - 2z^{-1}}.$$

Now, to obtain $y[n]$ in the time domain, we need to know the ROC of $Y(z)$. Form the properties of the Z-transform, $ROC_Y \supseteq ROC_H \cap ROC_X$. ROC_X is the entire z plane ($|z| > 0$). As we have not cancel any pole in $H(z)$, $ROC_Y = ROC_H = \{\frac{1}{2} < |z| < 2\}$. Using this ROC and the tables,

$$y[n] = -(\tfrac{1}{2})^n u[n] - 4 \cdot 2^n u[-n-1].$$

Questions?