Standard Fourier Transform: connections with Fourier Series, examples and more…

Discrete Time Systems

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Transformations for signal in discrete time

For Periodic signals	For non-periodic signals	
Fourier Series (FS)	Stand. Fourier Transform (FT)	Zeta Transform (ZT)

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Standard Fourier Transform

Discrete Time Systems

Standard Fourier Transform

DEFINITIONS: (x[n] NO-periodic)

Analysis Equation:

periodic with period 2π

Direct
time ==> freq.

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

Fourier Transform

Syntesis Equation:

Inverse
freq. ==> time

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Inverse Fourier Transform

Standard Fourier Transform

Important property:

$$X(\Omega) = X(\Omega + 2\pi)$$

periodic with period 2π

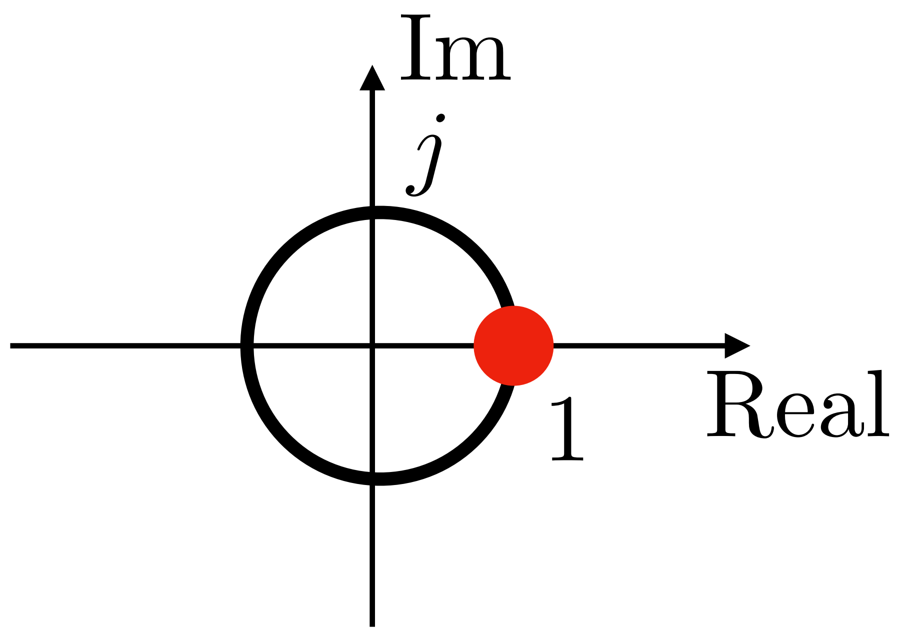
Standard Fourier Transform

$$X(\Omega) = X(\Omega + 2\pi)$$

periodic with period 2π

This is due to:

$$e^{j2\pi n} = 1$$



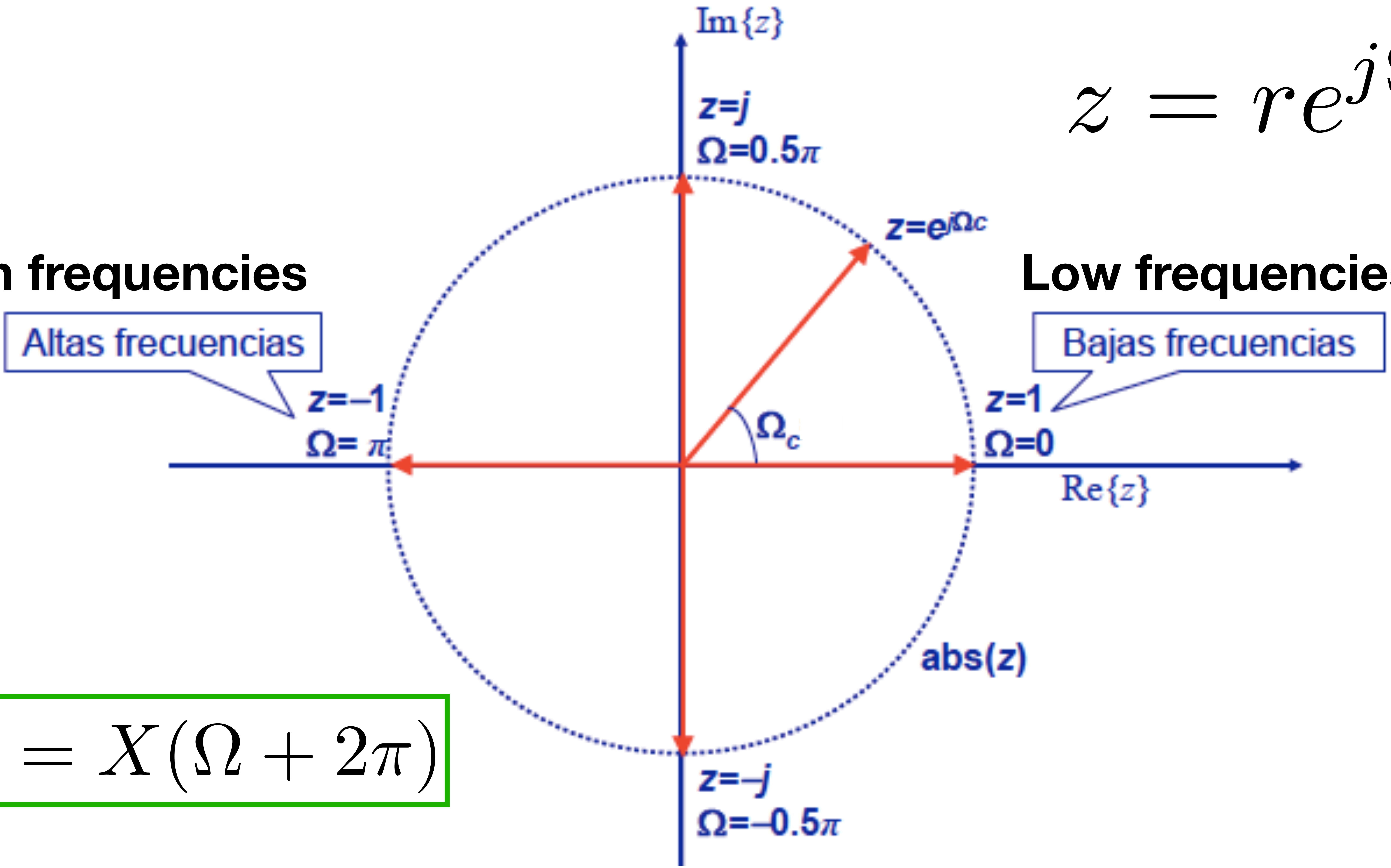
$$e^{j\Omega n} e^{j2\pi n} = e^{j(\Omega + 2\pi)n} = e^{j\Omega n}$$

Frequencies-“Omega” as an angle

$$z = re^{j\Omega}$$

High frequencies

Low frequencies



$$X(\Omega) = X(\Omega + 2\pi)$$

Standard Fourier Transform: example

Let us consider the following signal (signal with finite length):

$$x[-1] = 0.245, \quad x[0] = -3, \quad x[1] = 2, \quad x[2] = -5, \quad x[10] = 2j,$$

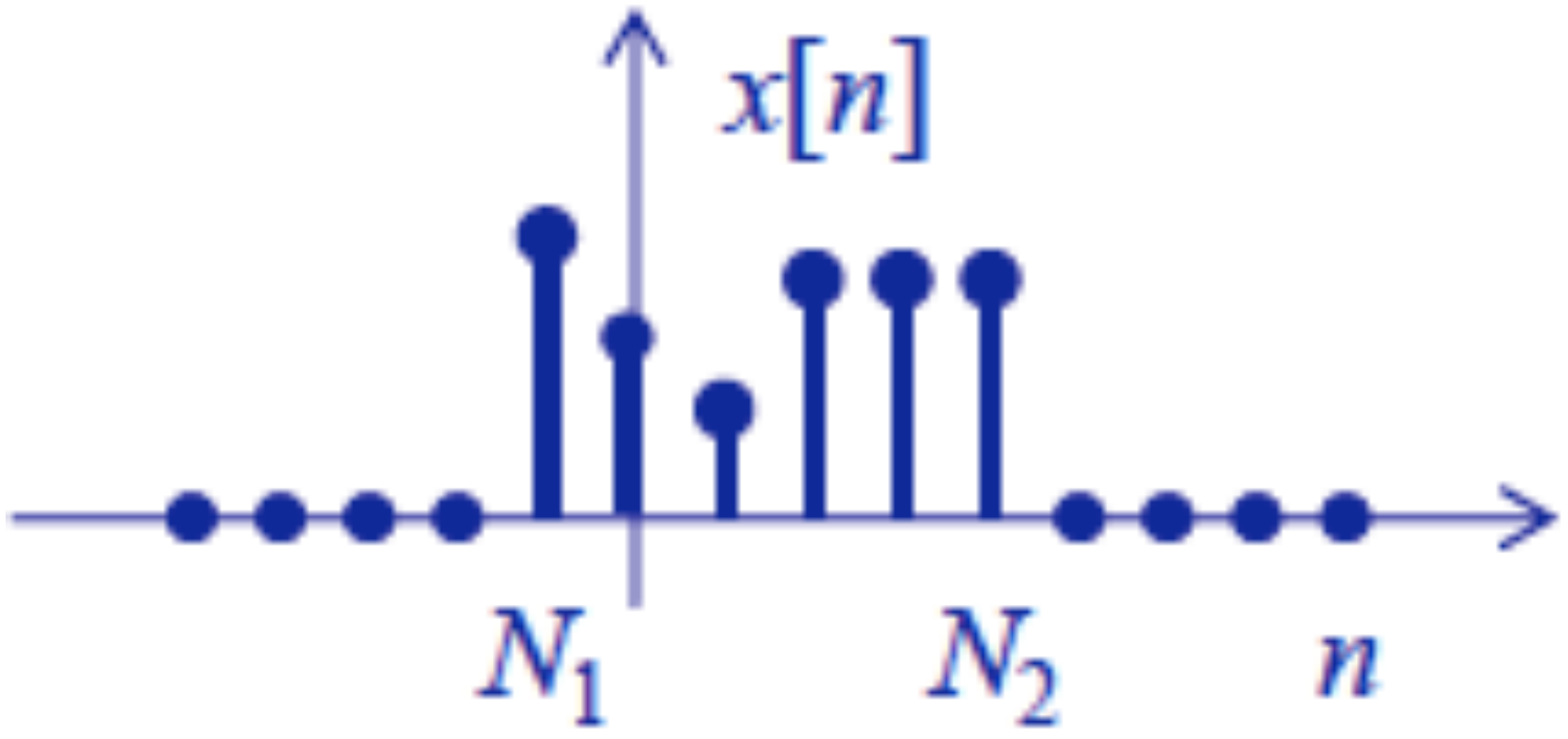
$$x[n] = 0 \qquad \text{for any other } n$$

The FT is:

$$X(\Omega) = 0.245e^{j\Omega} - 3 + 2e^{-j\Omega} - 5e^{-j2\Omega} + 2je^{-j10\Omega}$$

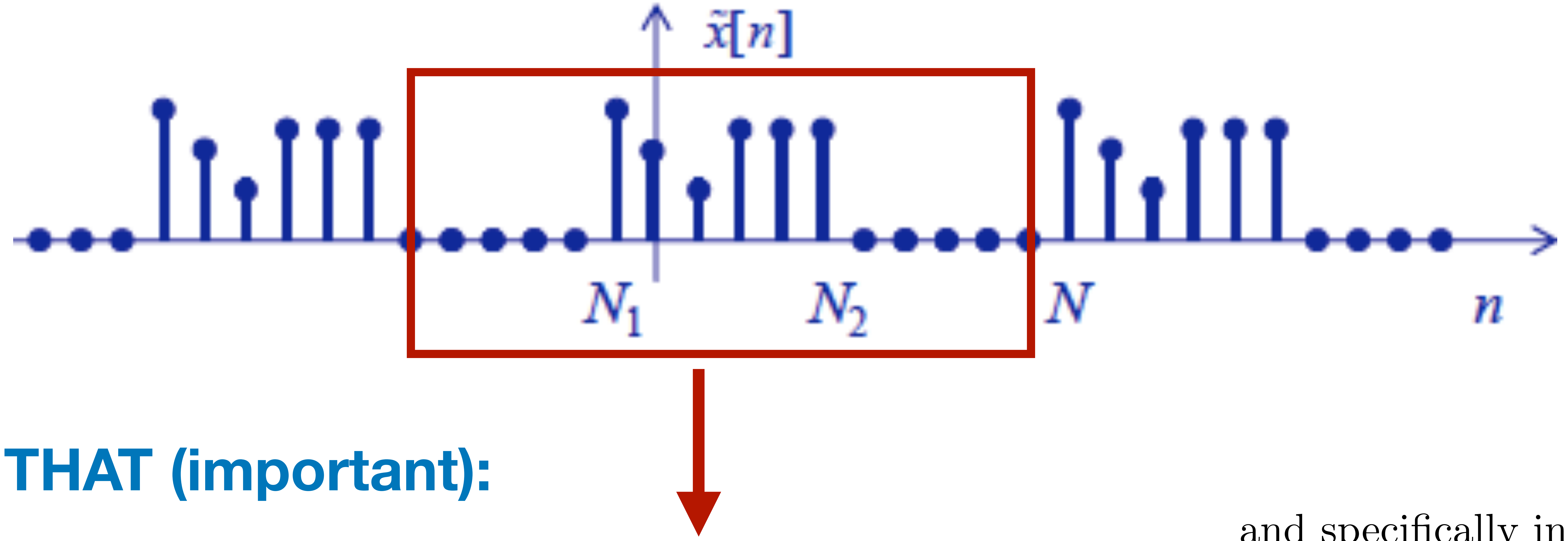
Signal with finite length

Let us consider a generic signal with finite length:



Periodic “brother” signal

We construct the PERIODIC “brother” signal with period N:



NOTE THAT (important):

$$\widetilde{x}[n] = x[n]$$

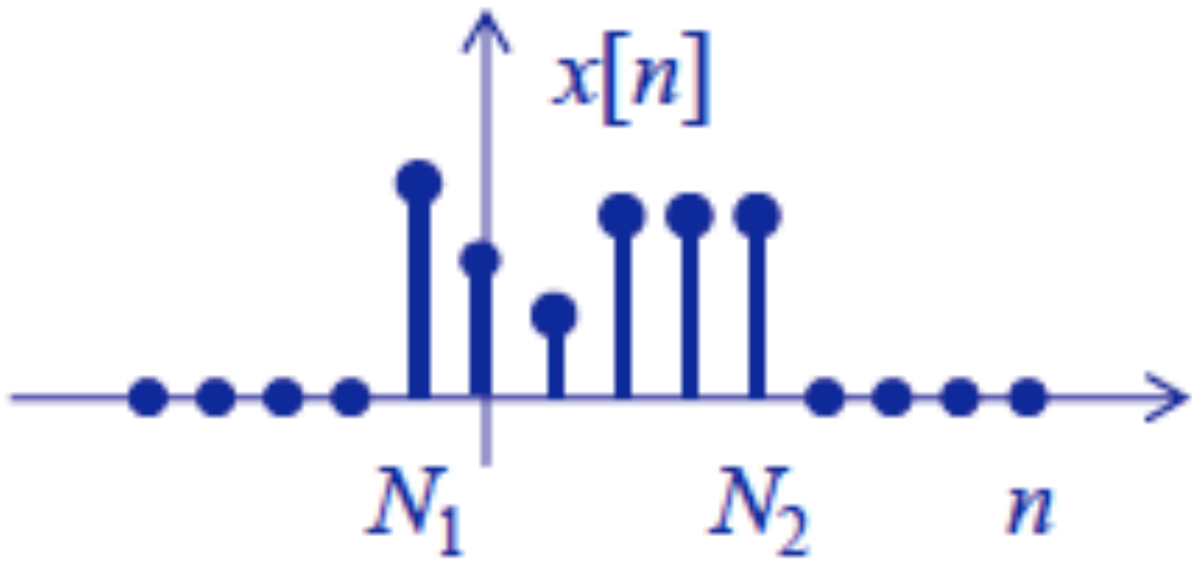
and specifically in

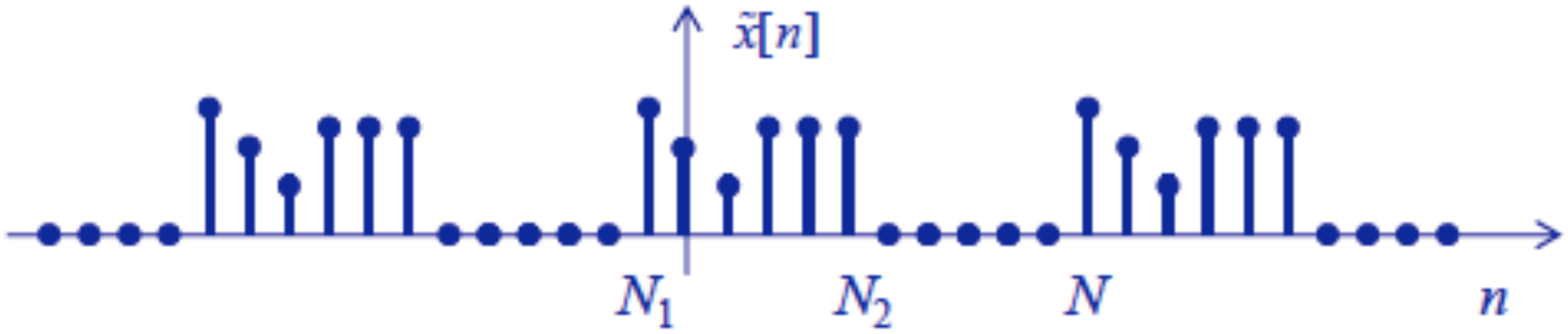
$$N_1 \leq n \leq N_2$$

in the first-central period

We study the connection between FS and FT

We compute the FS of the PERIODIC "brother" signal with period N:





Expresamos la señal $\tilde{x}[n]$ mediante su DTFS:

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k \, e^{jk\frac{2\pi}{N}n} \quad \text{con} \quad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$

En el intervalo $N_1 \leq n \leq N_2$, se cumple $\tilde{x}[n] = x[n]$

$$a_k = \frac{1}{N}\sum_{n=N_1}^{N_2} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

Connection between FS and FT

Compare the two formulas:

$$a_k = \boxed{\frac{1}{N}} \sum_{n=-\infty}^{\infty} x[n] e^{-j\boxed{k\frac{2\pi}{N}}n} = \boxed{\frac{1}{N}} \sum_{n=-\infty}^{\infty} x[n] e^{-j\boxed{k\Omega_0}n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\boxed{\Omega}n}$$

we obtain:

$$a_k = \frac{1}{N} X(k\Omega_0) \qquad \Omega_0 = \frac{2\pi}{N}$$

Connection between FS and FT

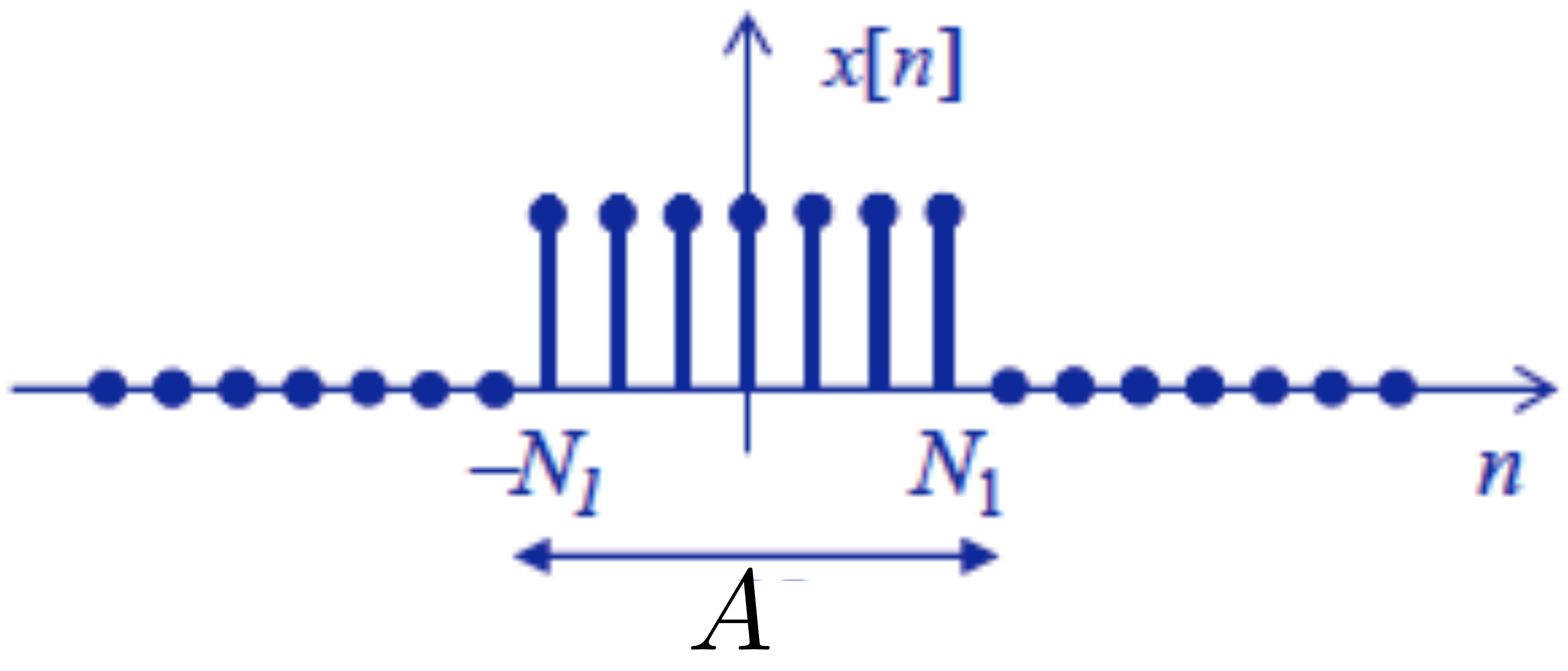
- Los coeficientes a_k son muestras equiespaciadas de la señal $X(\Omega)$

$$a_k = \frac{1}{N} X(\Omega)\Big|_{\Omega = k\Omega_0}$$

donde $\Omega_0 = \frac{2\pi}{N}$

$$a_k = \frac{1}{N} X(k\Omega_0)$$

Example

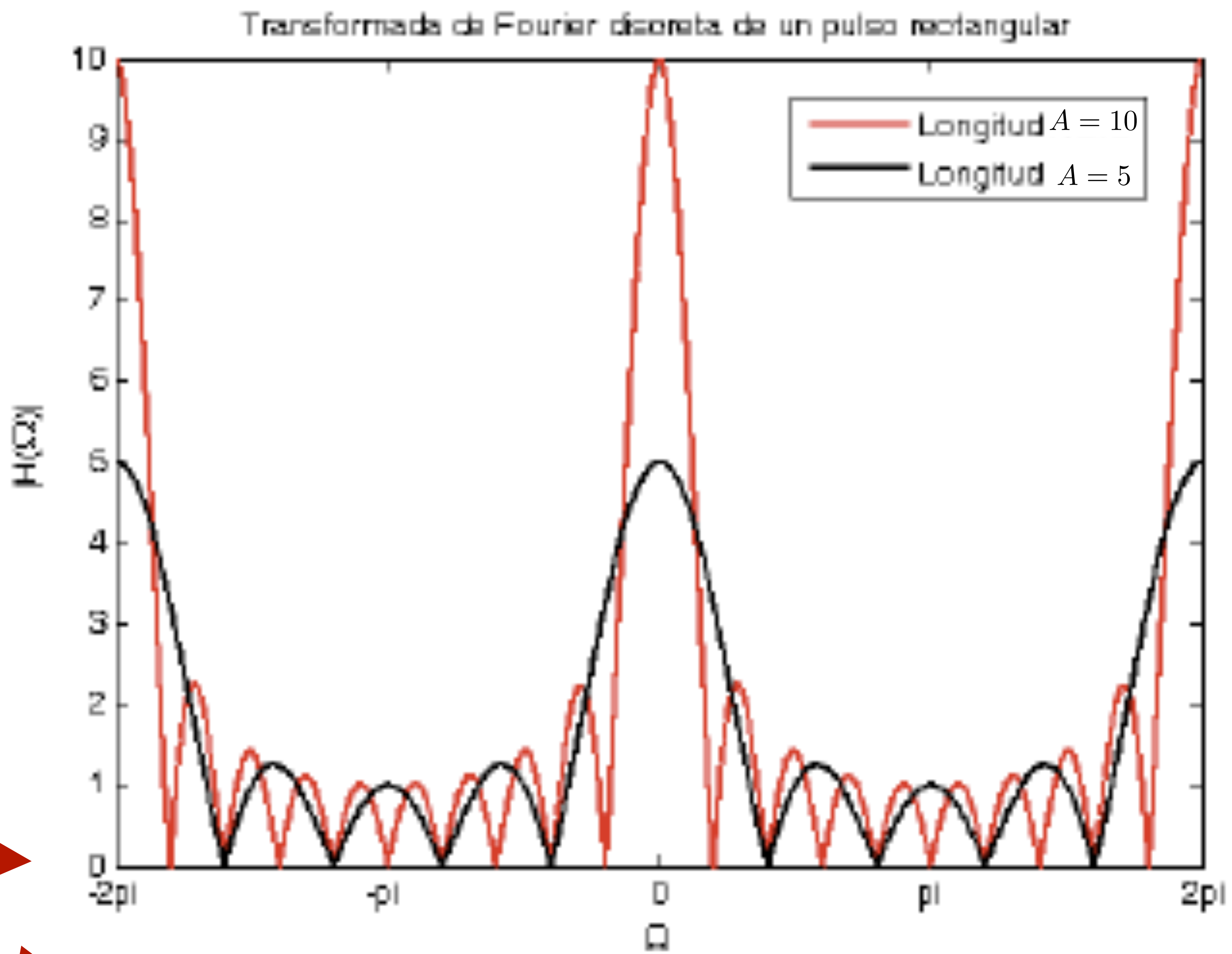


$$X(\Omega) = \sum_{n=-N_1}^{N_1} x[n]e^{-j\Omega n} = \sum_{n=-N_1}^{N_1} 1e^{-j\Omega n}$$

$$= \frac{e^{j\Omega N_1} - e^{-j\Omega(N_1+1)}}{1-e^{-j\Omega}} \frac{e^{+j\frac{\Omega}{2}}}{e^{+j\frac{\Omega}{2}}} =$$

$$= \frac{\operatorname{sen}\left(\Omega(N_1+\frac{1}{2})\right)}{\operatorname{sen}\frac{\Omega}{2}}$$

PERIODIC 2π



"DISCRETE" SINC ("OCTOPUS
- Transform of a discrete rectangle")

Same example: more details of derivation

USEFUL FORMULAS FOR EVERY EXAMPLES/PROBLEMS:

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1}\frac{1-r^{N_2-N_1+1}}{1-r} = \frac{r^{N_1}-r^{N_2+1}}{1-r}$$

N1 HERE IS NOT THE LENGTH
OF THE RECTANGLE;
IT IS JUST A GENERIC CONSTANT

$$\sum_{n=0}^{N_2} r^n = \frac{1-r^{N_2+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{si } |r|<1$$

Same example: more details of derivation

USEFUL FORMULAS FOR EVERY EXAMPLES/PROBLEMS:

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1}\frac{1-r^{N_2-N_1+1}}{1-r} = \frac{r^{N_1}-r^{N_2+1}}{1-r}$$

$$\sum_{n=0}^{N_2} r^n = \frac{1-r^{N_2+1}}{1-r}$$

WE WILL USE THIS FIRST FORMULA

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{si } |r|<1$$

Same example: more details of derivation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$= \sum_{n=-N_1}^{N_1} 1\ e^{-j\Omega n}$$

$$= \sum_{n=-N_1}^{N_1} (e^{-j\Omega})^n$$

$$r = e^{-j\Omega}$$

Same example: more details of derivation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=-N_1}^{N_1} (e^{-j\Omega})^n$$

$$r = e^{-j\Omega}$$

$$= \frac{e^{+j\Omega N_1} - e^{-j\Omega(N_1+1)}}{1 - e^{-j\Omega}}$$

This is already the solution !
We could stop here

Same example: more details of derivation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \frac{e^{+j\Omega N_1} - e^{-j\Omega(N_1+1)}}{1 - e^{-j\Omega}}$$

$$= \frac{e^{+j\Omega N_1} - e^{-j\Omega(N_1+1)}}{1 - e^{-j\Omega}} \frac{e^{j\Omega/2}}{e^{j\Omega/2}}$$

$$= \frac{e^{+j\Omega(N_1+\frac{1}{2})} - e^{-j\Omega(N_1+\frac{1}{2})}}{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}}$$

Same example: more details of derivation

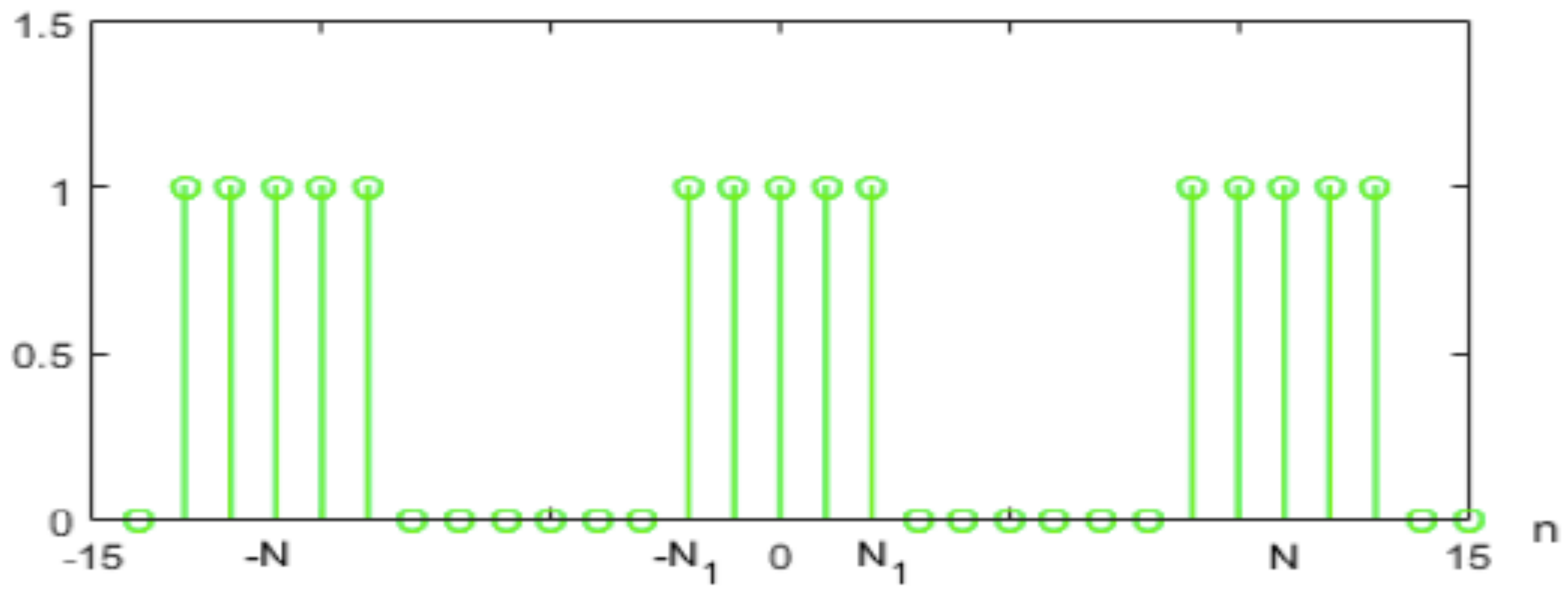
$$X(\Omega) = \frac{e^{+j\Omega(N_1+\frac{1}{2})} - e^{-j\Omega(N_1+\frac{1}{2})}}{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}}$$

$$= \frac{2j\sin\left(\Omega(N_1+\frac{1}{2})\right)}{2j\sin\left(\frac{\Omega}{2}\right)}$$

$$= \frac{\sin\left(\Omega(N_1+\frac{1}{2})\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

Let us consider a periodic “brother”

$\widetilde{x}[n]$



$$\text{with period N}$$

You already study and analyze this signal….

Let us consider a periodic "brother"

Using the analysis equation of the **Fourier Series**:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk\Omega_0 n} \rightarrow \quad \text{Analysis eq.}$$

For k multiple of N (i.e. k=0):

$$a_0 = a_{-N} = a_{6N} = \frac{1}{N}\sum_{n=-N_1}^{N_1} \tilde{x}[n]\cdot 1 = \frac{2N_1+1}{N}$$

Here we use the analysis formula

Para k≠multiple of N:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\Omega_0(m-N_1)}$$

$$= \frac{1}{N}e^{jk\Omega_0 N_1}\sum_{m=0}^{2N1}(e^{-jk\Omega_0})^m = \frac{1}{N}e^{jk\Omega_0 N_1}\frac{1-e^{-jk\Omega_0(2N_1+1)}}{1-e^{j}k\Omega_0}$$

$$= \frac{1}{N}\frac{\sin[k(N_1+1/2)\Omega_0]}{\sin(k\Omega_0/2)} = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$

Let us consider a periodic "brother"

$$a_k = \frac{1}{N} \frac{\sin\left[k\Omega_0\left(N_1 + \frac{1}{2}\right)\right]}{\sin\left(k\frac{\Omega_0}{2}\right)} \qquad k \neq 0, N, 2N, \ldots$$

We can obtain it in other way?

Considering the non-periodic "brother"

Using the formula: $$a_k = \frac{1}{N} X(k\Omega_0)$$

$$X(\Omega) = \frac{\sin\left(\Omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$$a_k = \frac{1}{N} X(k\Omega_0) = \frac{1}{N} \frac{\sin\left(k\Omega_0\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{k\Omega_0}{2}\right)}$$

Questions?