Standard Fourier Transform: properties and examples…

Discrete Time Systems

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Transformations for signal in discrete time

For Periodic signals

For non-periodic signals

Fourier Series (FS)

Stand. Fourier Transform (FT)

Zeta Transform (ZT)

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Standard Fourier Transform

Discrete Time Systems

Standard Fourier Transform

DEFINITIONS: (x[n] NO-periodic)

Analysis Equation:

periodic with period 2π

Direct
time ==> freq.

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

Fourier Transform

Syntesis Equation:

Inverse
freq. ==> time

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Inverse Fourier Transform

Standard Fourier Transform

Important property:

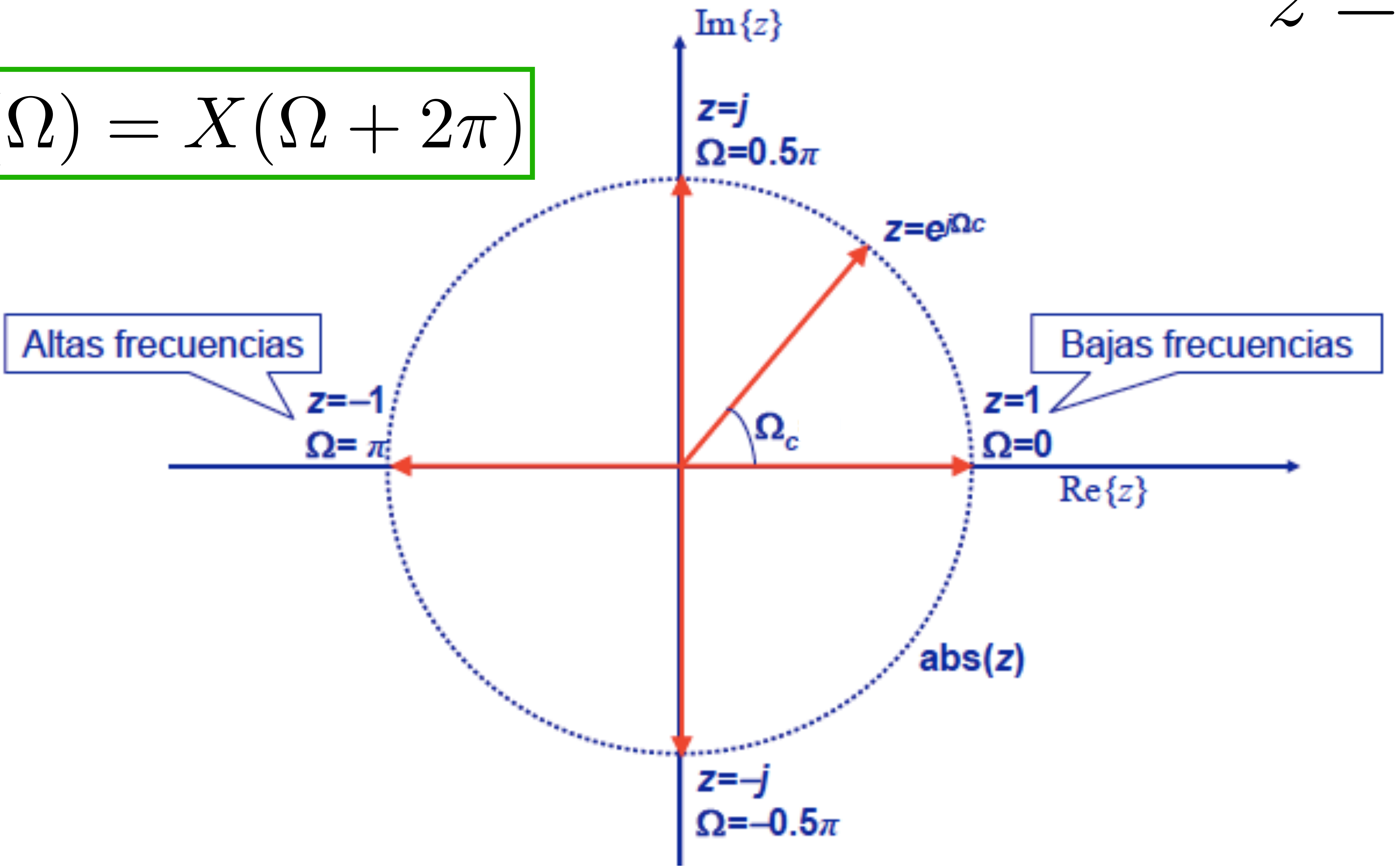
$$X(\Omega) = X(\Omega + 2\pi)$$

periodic with period 2π

Frequencies-“Omega” as an angle

$$z = re^{j\Omega}$$

$$X(\Omega) = X(\Omega + 2\pi)$$



Existence (convergence of series of stand. FT)

- Ec. síntesis: $$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

  - Integramos sobre un intervalo finito → no hay problemas de convergencia
  - Si los valores de la TF son finitos, la integral (el área) es finita

- Ec. análisis: $$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

  - Tenemos una suma infinita, esto es "peligroso" porque aunque los valores de la secuencia sean finitos, sumamos infinitos términos, con lo cual la suma sí puede dar infinito
  - Conclusión, no todas las señales discretas tienen TF
  - Para poder garantizar que la TF existe, necesitamos exigir a la señal unas condiciones análogas a las que pedíamos en CT, por ejemplo:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \text{ -Energía finita}$$

o

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Examples (1) of standard FT

DELTA DE KRONECKER:

$$x[n] = \delta[n]$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} \delta[n]e^{-j\Omega n} = e^{-j\Omega 0} = 1$$

$$\delta[n] \xleftrightarrow{\mathcal{TF}} 1$$

Examples (2) of standard FT

DELTA DE KRONECKER "moved/shifted":

$$x[n] = \delta[n - n_0]$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\Omega n} = e^{-j\Omega n_0}$$

$$\boxed{\delta[n - n_0] \overset{\mathcal{TF}}{\longleftrightarrow} e^{-j\Omega n_0}}$$

Examples (3) of standard FT

$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=0}^{+\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{+\infty} (ae^{-j\Omega})^n$$

Examples (3) of standard FT

Recalling:

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1}\frac{1-r^{N_2-N_1+1}}{1-r} = \frac{r^{N_1}-r^{N_2+1}}{1-r}$$

$$\sum_{n=0}^{N_2} r^n = \frac{1-r^{N_2+1}}{1-r}$$

We will use this one:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{si } |r| < 1$$

Examples (3) of standard FT

$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X(\Omega) = \sum_{n=0}^{+\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{+\infty} (ae^{-j\Omega})^n$$

since: $r = ae^{-j\Omega}$

$$|r| = |ae^{-j\Omega}| = |a| < 1$$

Examples (3) of standard FT

since: $$r = ae^{-j\Omega}$$

$$|r| = |ae^{-j\Omega}| = |a| < 1$$

$$X(\Omega) = \sum_{n=0}^{+\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{+\infty} (ae^{-j\Omega})^n$$

$$X(\Omega) = \sum_{n=0}^{+\infty} (ae^{-j\Omega})^n = \frac{1}{1 - ae^{-j\Omega}}$$

Examples (3) of standard FT

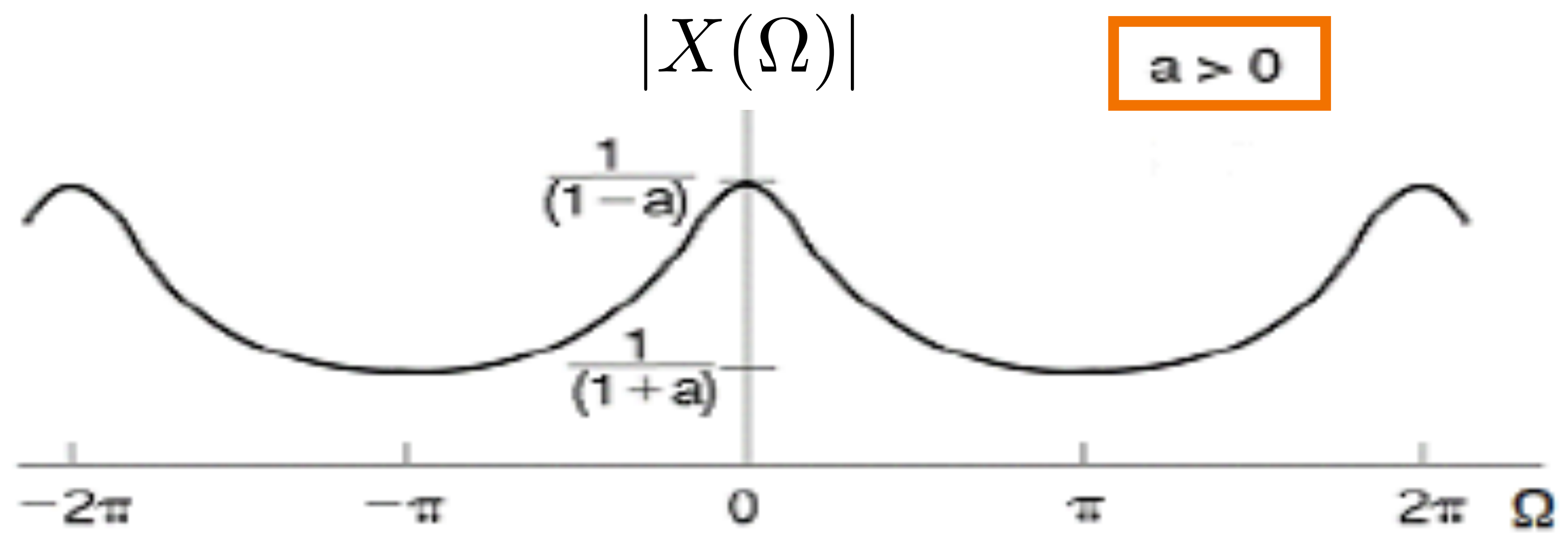
$$a^n u[n], \quad |a| < 1 \quad \stackrel{\mathcal{TF}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\Omega}}$$

Examples (3) of standard FT

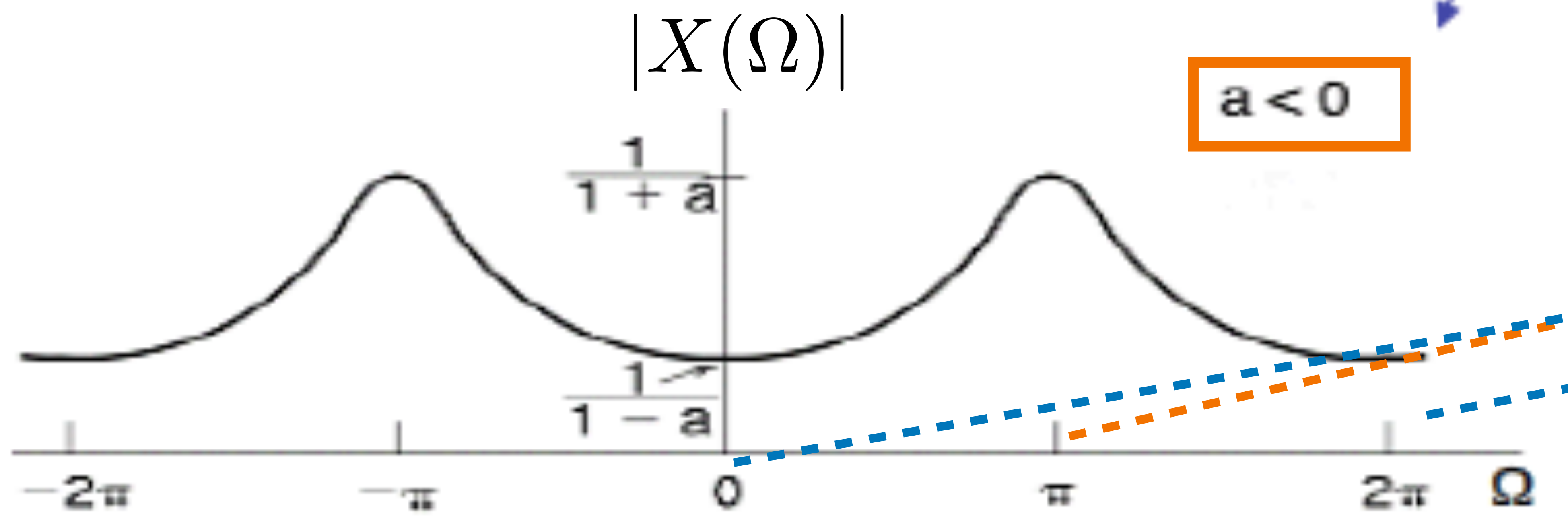
SUMMARY

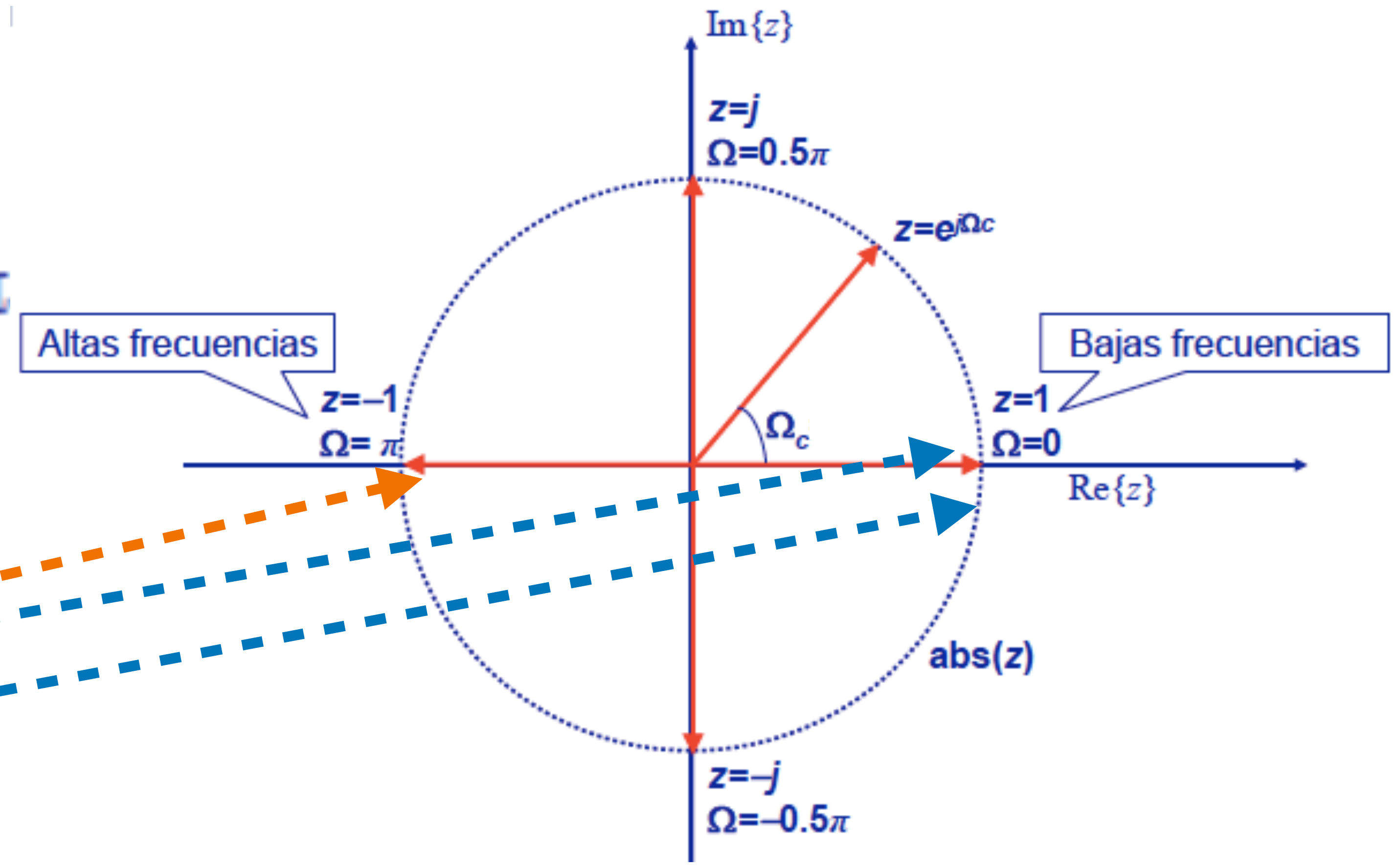
Ejemplo: exponencial real causal

- ❑ $x[n] = a^n u[n], \; |a| < 1 \qquad TF \rightarrow X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}$

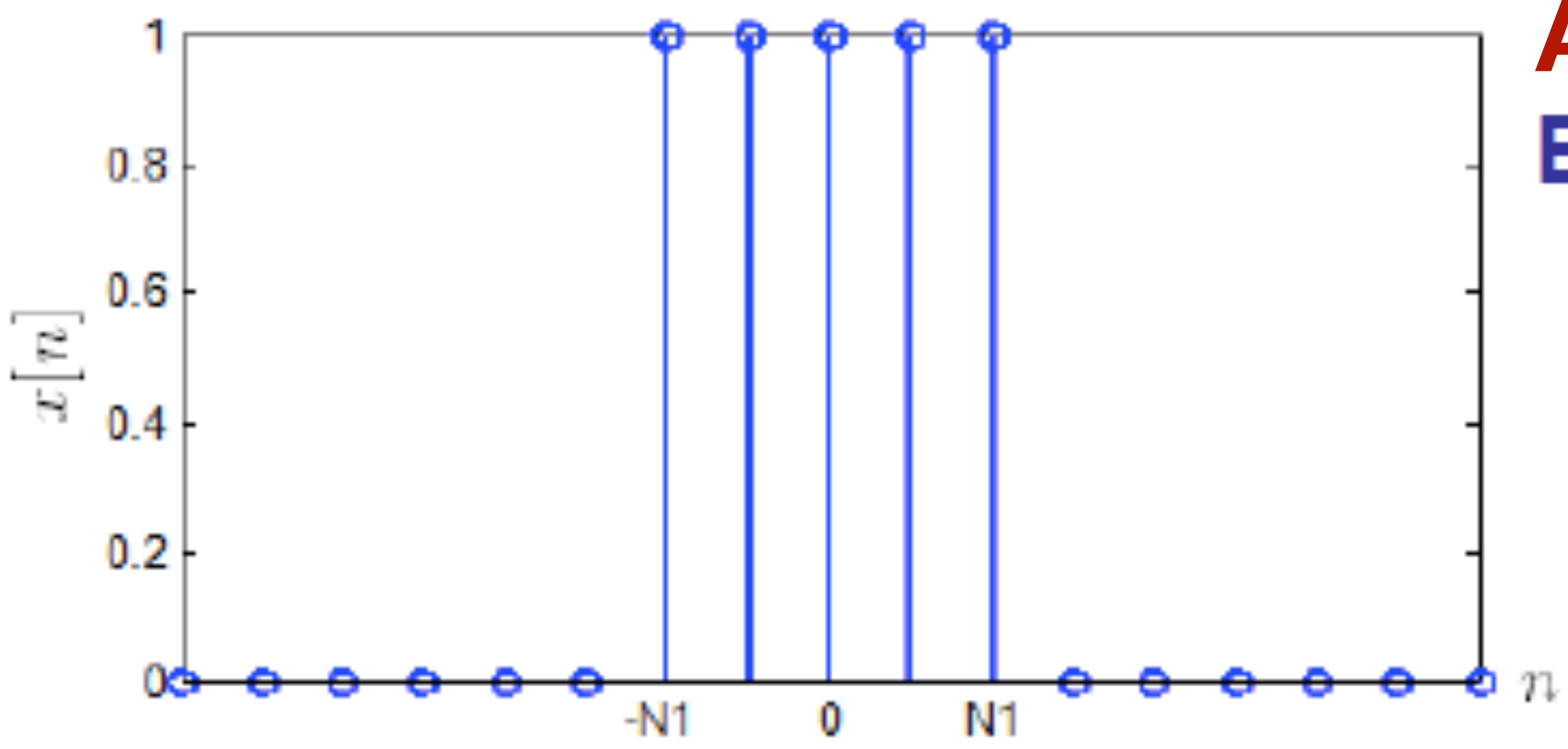


periódica, con periodo 2π





Examples (4) of standard FT



Already done in other slides

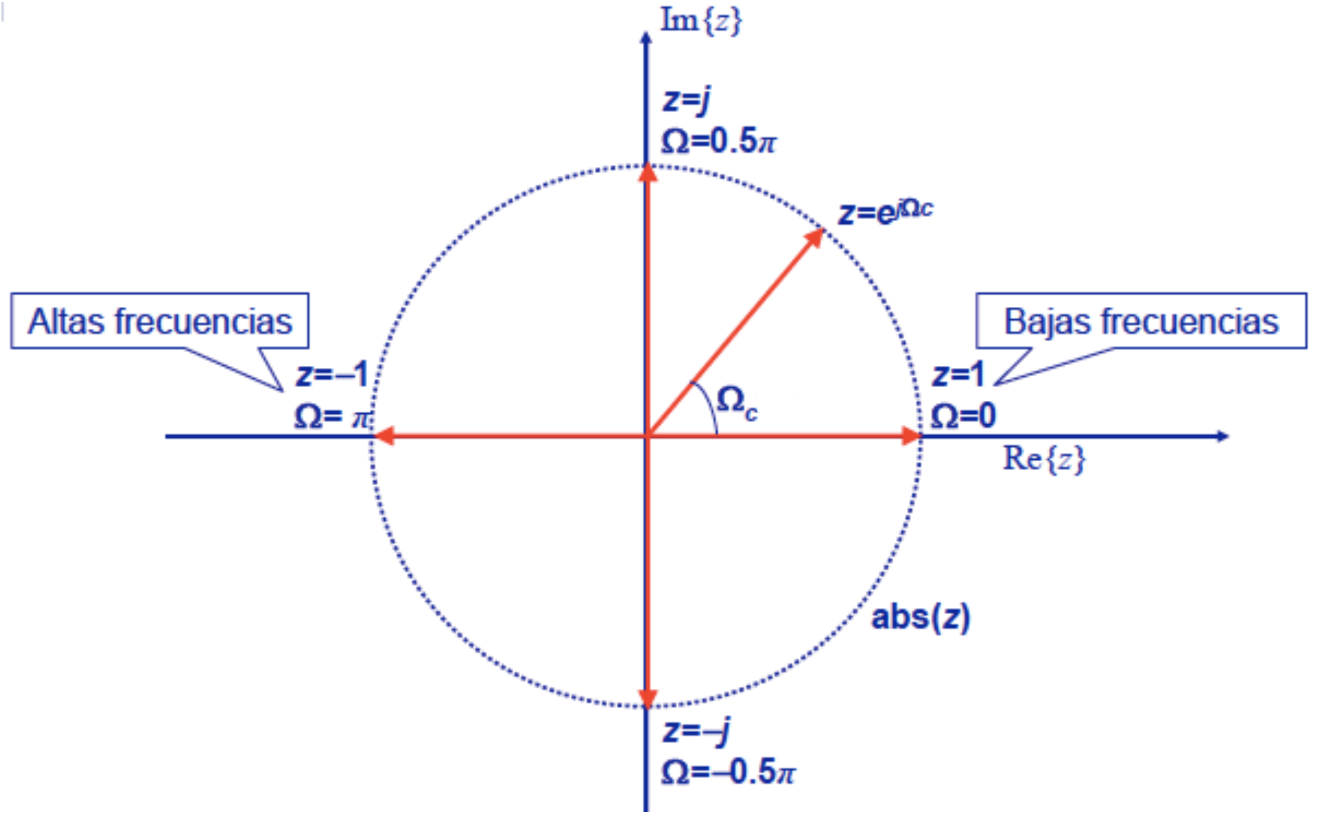
Ejemplo: pulso rectangular

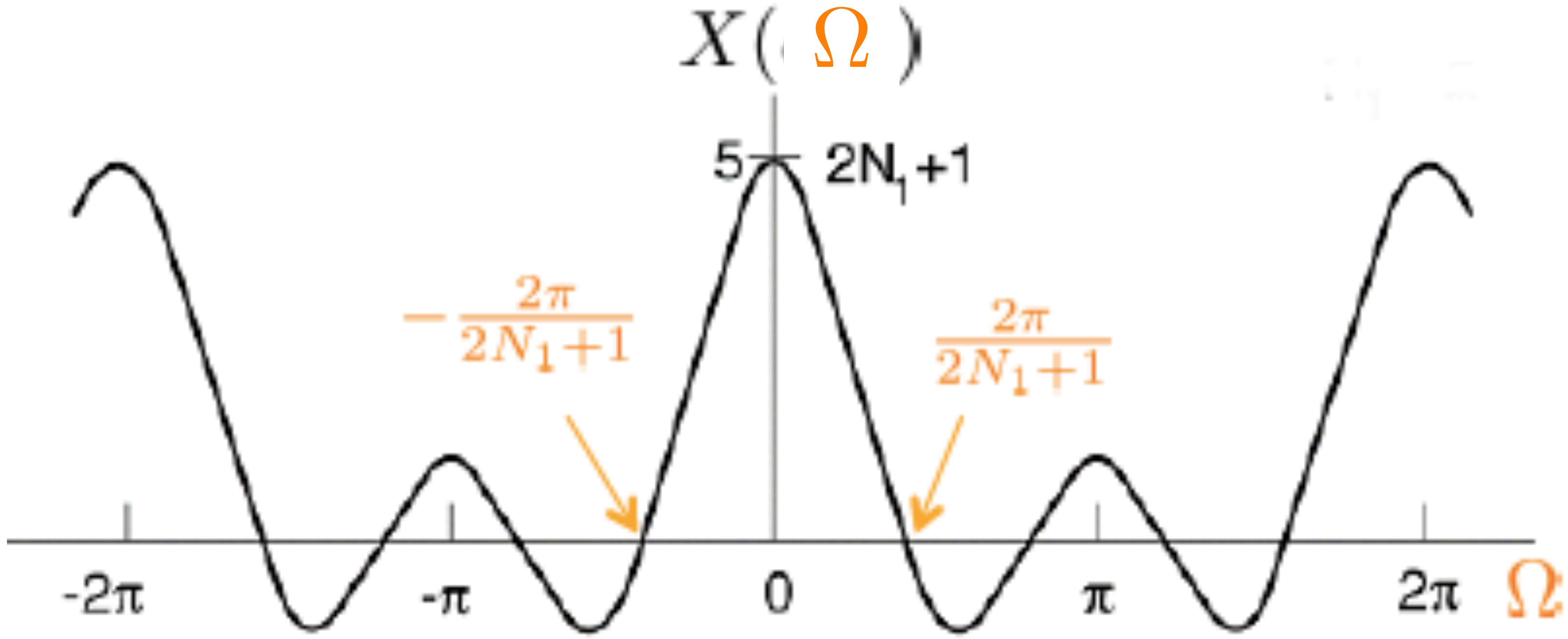
$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$TF$$

$$\Downarrow$$

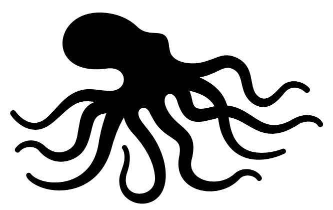
$$\frac{\sin[\Omega(N_1 + \frac{1}{2})]}{\sin(\frac{\Omega}{2})}$$





Periódica con periodo 2π

"discrete" octopus

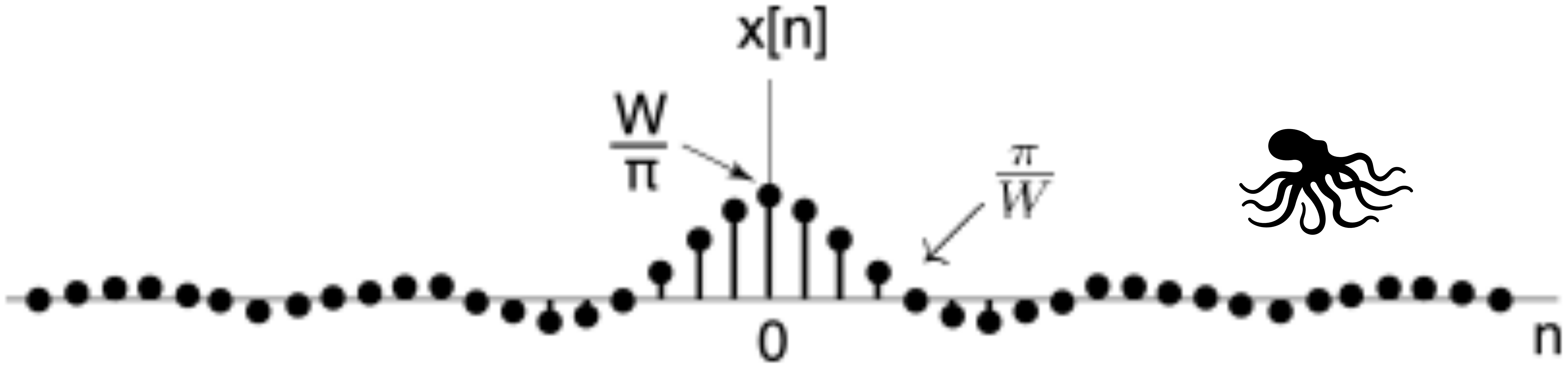


"discrete" Sinc

Examples (5) of standard FT

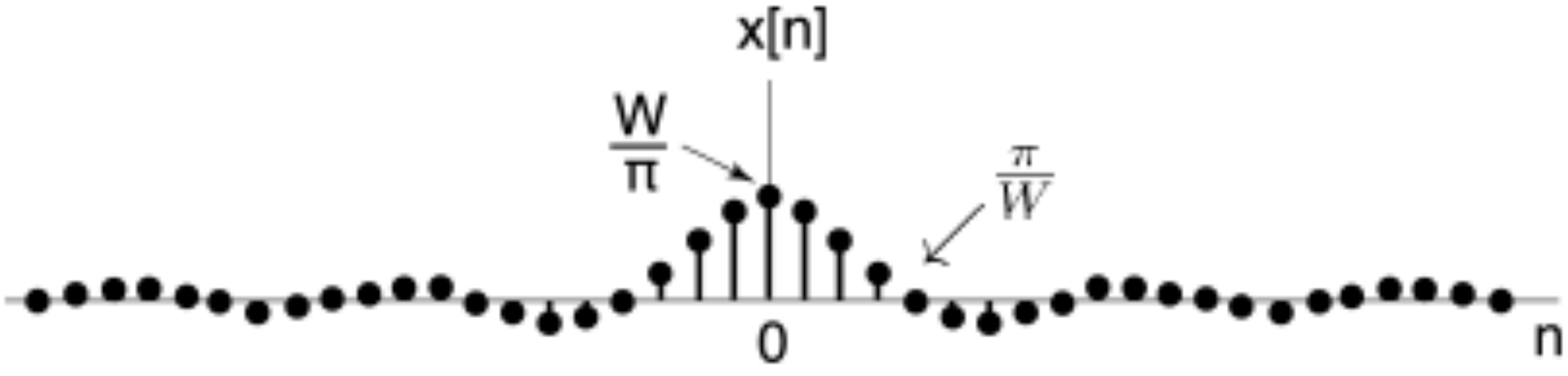
Sinc function in discrete time
(in the previous slide we saw a Sinc in frequency - continuous- periodic)

$$x[n] = \frac{\sin Wn}{\pi n}$$



Examples (5) of standard FT

$$x[n] = \frac{\sin Wn}{\pi n}$$



In that case, it is difficult to use the direct definition:

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} \frac{\sin Wn}{\pi n} e^{-j\Omega n}$$

Examples (5) of standard FT

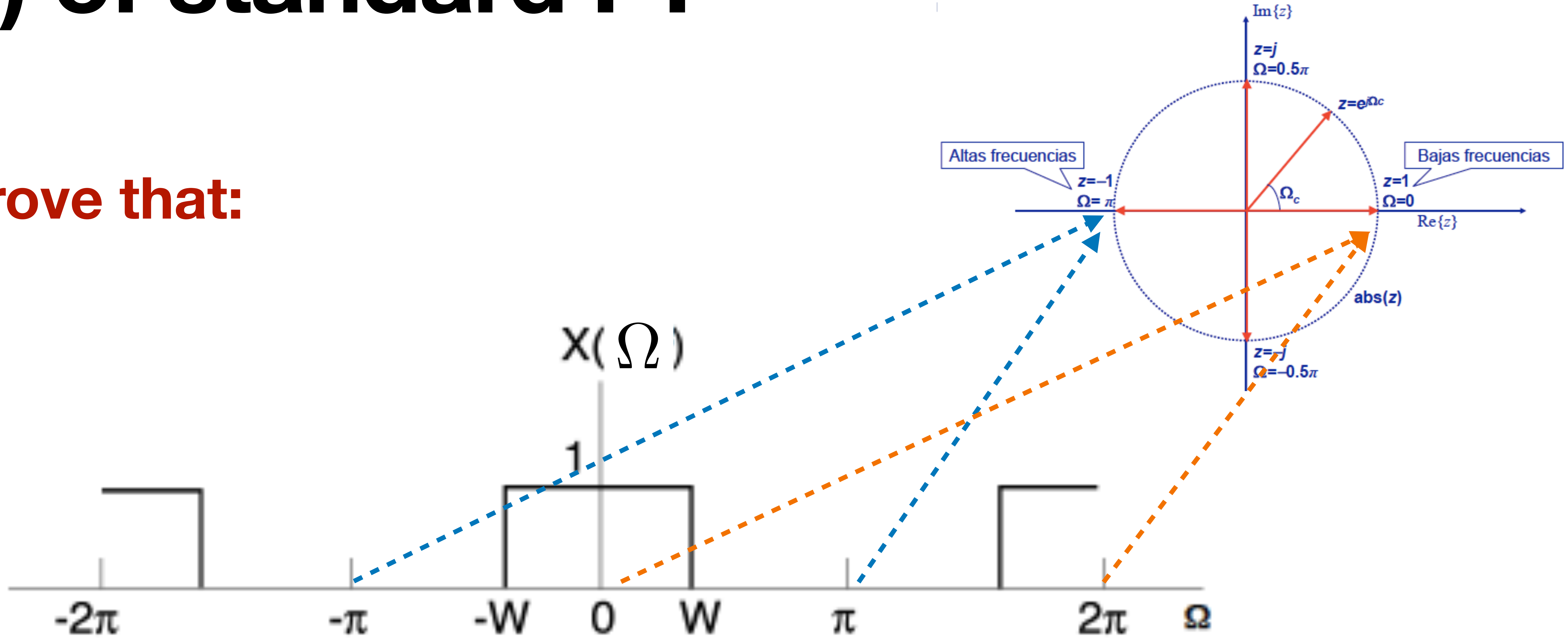
In that case, it is difficult to use the direct definition:

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} \frac{\sin Wn}{\pi n} e^{-j\Omega n}$$

This series converges but it is not easy to find the "result" of the convergence.

Examples (5) of standard FT

However, we can prove that:



Inverse Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega = \frac{\sin Wn}{\pi n}$$

Properties

Señal	Transformada
$x[n]$ $y[n]$	$\left.\begin{matrix} X(\Omega) \\ Y(\Omega) \end{matrix}\right\}$ Periódicas de peridodo 2π
$ax[n]+by[n]$	$aX(\Omega)+bY(\Omega)$
$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
$e^{j\Omega_0 n}x[n]$	$X(\Omega-\Omega_0)$
$x^*[n]$	$X^*(-\Omega)$
$x[-n]$	$X(-\Omega)$

Properties

very important:

$x[n] * y[n]$	$X(\Omega)Y(\Omega)$
$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(\theta)Y(\Omega-\theta)d\theta$

Properties

$$nx[n] \rightarrow j\frac{d}{d\Omega}X(\Omega)$$

$$x[n]-x[n-1] \rightarrow \left(1-e^{-j\Omega}\right)X(\Omega)$$

differences are like "derivative" in discrete time….

differential equations (continuous) ===> difference equations (discrete)

Properties

$x[n]$ real	$\begin{cases} X(\Omega) = X^*(-\Omega) \\ \text{Re}\{X(\Omega)\} = \text{Re}\{X(-\Omega)\} \\ \text{Im}\{X(\Omega)\} = -\text{Im}\{X(-\Omega)\} \\ \lvert X(\Omega) \rvert = \lvert X(-\Omega) \rvert \\ \angle X(\Omega) = -\angle X(-\Omega) \end{cases}$

Relación de Parseval para secuencias no periódicas

$$\sum_{k=-\infty}^{\infty} \lvert x[k] \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(\Omega) \rvert^2 d\Omega$$

Questions?