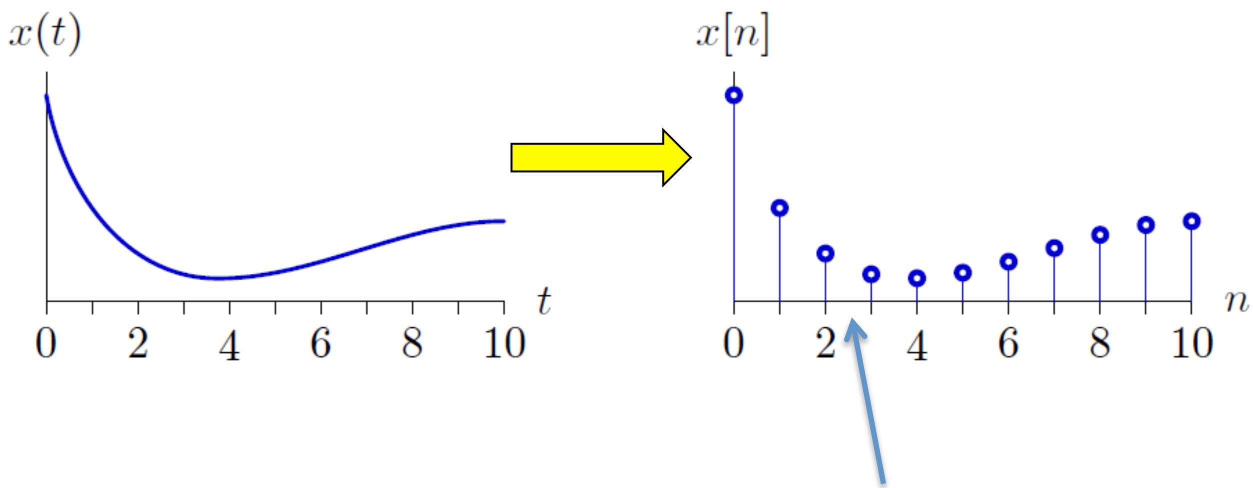
Sampling (in time): from continuous to discrete time

FIRST PART

Sampling ("muestreo")

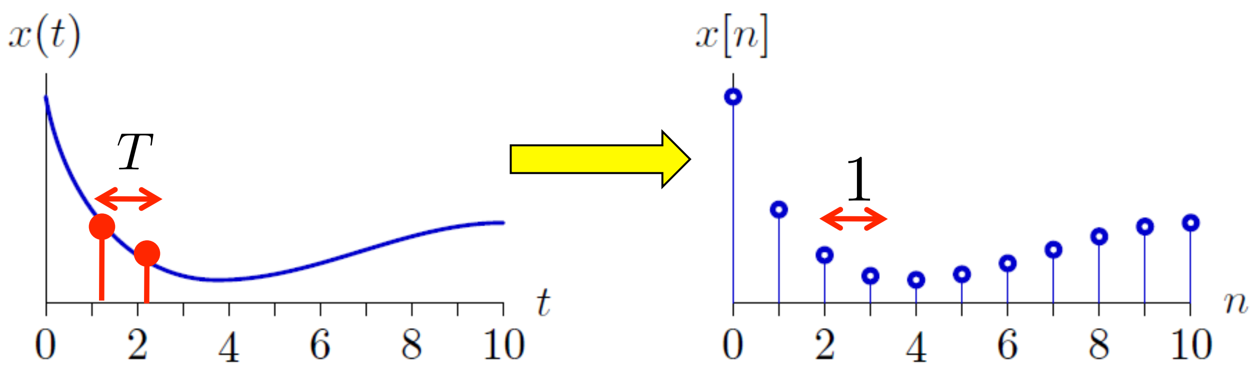
- "Current world": digital information
- from continuous to discrete time: **sampling**



Is defined this signal x[n] between 2 and 3? NOOOOOOOOOO

Regular/Uniform Sampling

- At each time instant T



- Sampling Frequency

$$w_s = \frac{2\pi}{T}$$

Uniform Sampling: fundamental formula

- Given a signal x(t), then we have

$$x[n] = x(nT)$$

$$x[n] = x(t)\Big|_{t=nT}$$

Uniform Sampling: examples

- Given a signal x(t), then we have

Si T=0,25 seg

$n = -1 \Longrightarrow$ x[-1]= x(-0,25)

$n = 0 \Longrightarrow$ x[0] = x(0)

$n = 1 \Longrightarrow$ x[1] = x(0,25)

$n = 2 \Longrightarrow$ x[2] = x(0,50)

Uniform Sampling: examples

- Given a signal x(t), then we have

Si ωs=10π rad/seg

$$T = \frac{2\pi}{10\pi} = \frac{1}{5}$$

$$n = -1 \Longrightarrow x[-1] = x(-0{,}2)$$

$$n = 0 \Longrightarrow x[0] = x(0)$$

$$n = 1 \Longrightarrow x[1] = x(0{,}2)$$

$$n = 2 \Longrightarrow x[2] = x(0{,}4)$$

Uniform Sampling: examples

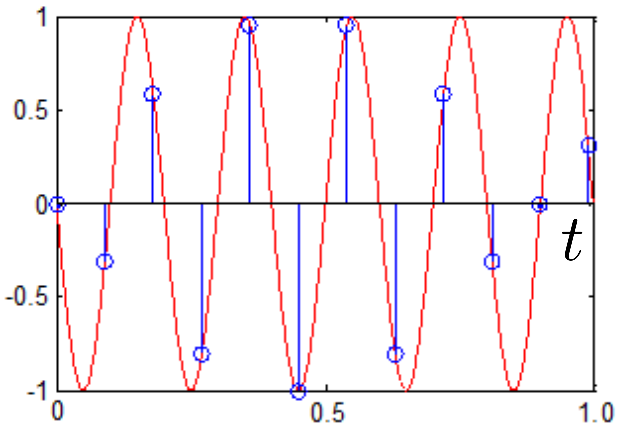
- Given a signal x(t), then we have

Si T=0,5 y **x(t)=(1/4)^{t}u(t)**

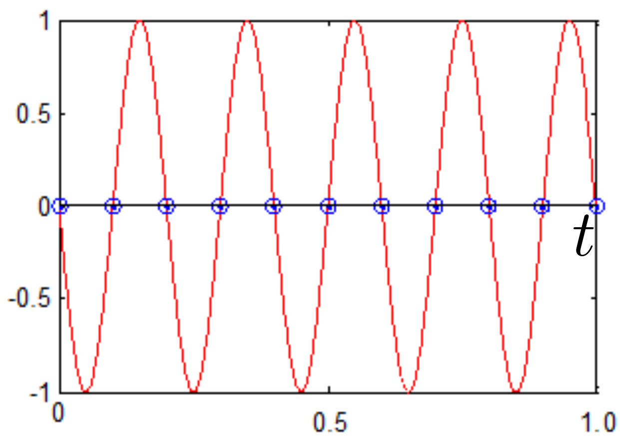
$n = -1 \Longrightarrow$ x[-1] = x(-0,5) = 0
$n = 0 \Longrightarrow$ x[0] = x(0) = 1
$n = 1 \Longrightarrow$ x[1] = x(0,5) = 1/2
$n = 2 \Longrightarrow$ x[2] = x(1) = 1/4

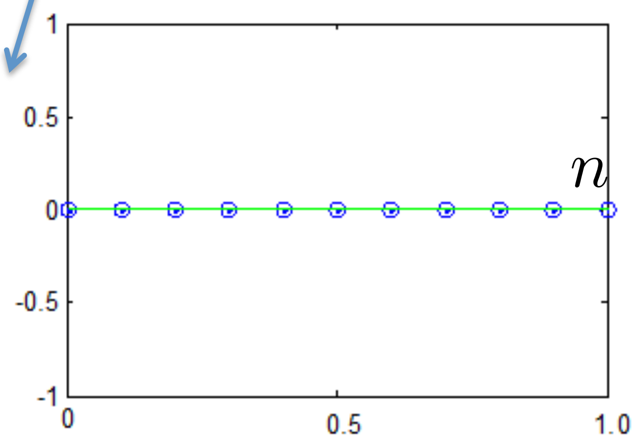
$$x[n] = x(nT) = \left(\frac{1}{4}\right)^{nT} u(nT)$$
$$x[n] = x(0.5n) = \left(\frac{1}{4}\right)^{0.5n} u(0.5n)$$

Uniform Sampling: graphical examples

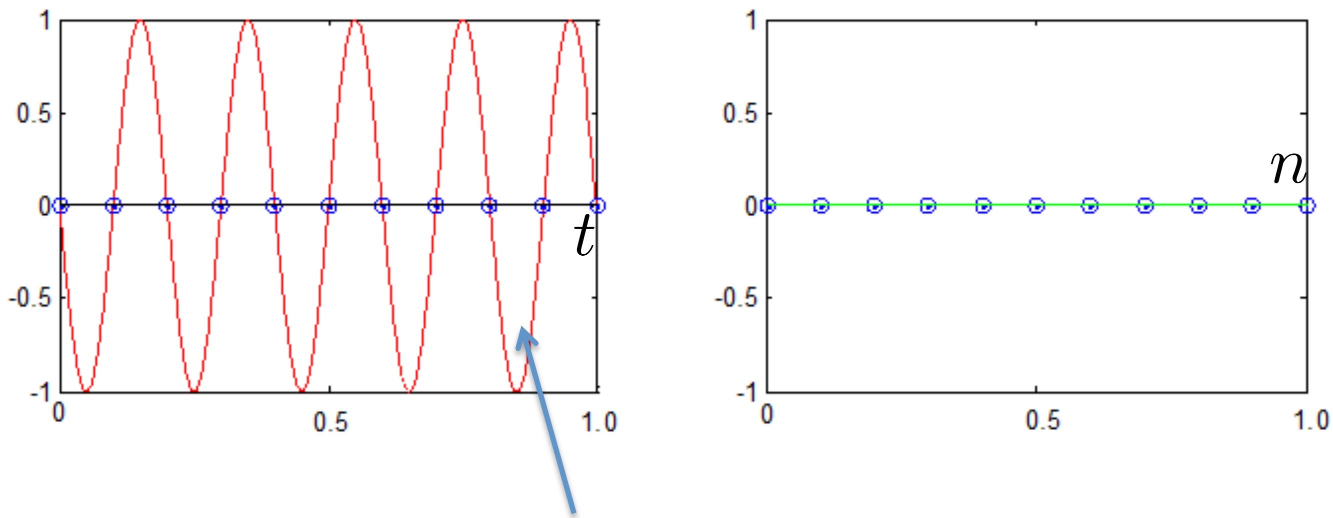


Here something is wrong… we have lost all the information of the x(t); x[n] is always zero…





Very important slide: can we avoid to lose information?



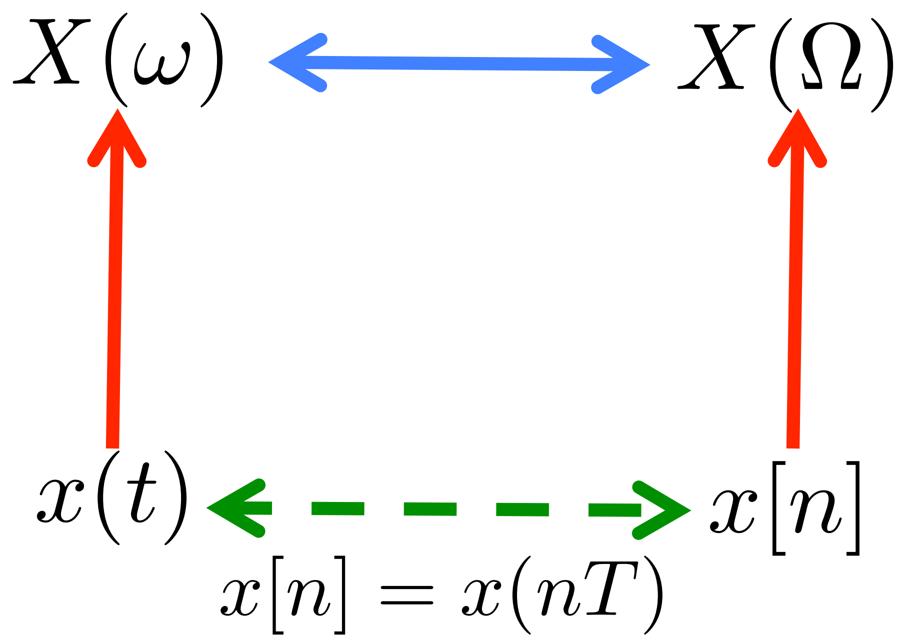
Here the sampling period T is too big …

WE WILL SEE THERE EXISTS AN UPPER BOUND Tn FOR THE SAMPLING PERIOD (T) : if T<Tn WE DO NOT LOSE INFORMATION !

Then, THERE EXISTS A LOWER BOUND wn FOR THE SAMPLING FREQUENCY (ws) : if ws>wn WE DO NOT LOSE INFORMATION !

When sampling, what happens in the frequency domain?

- We study the relationships between



- More specifically, we will study:

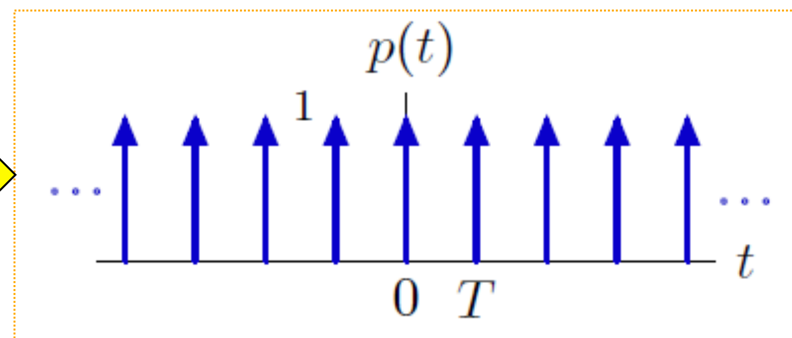
$$X(\omega) \longleftrightarrow X_p(\omega) \longleftrightarrow X(\Omega)$$

$$x(t) \longleftrightarrow x_p(t) \longleftrightarrow x[n]$$

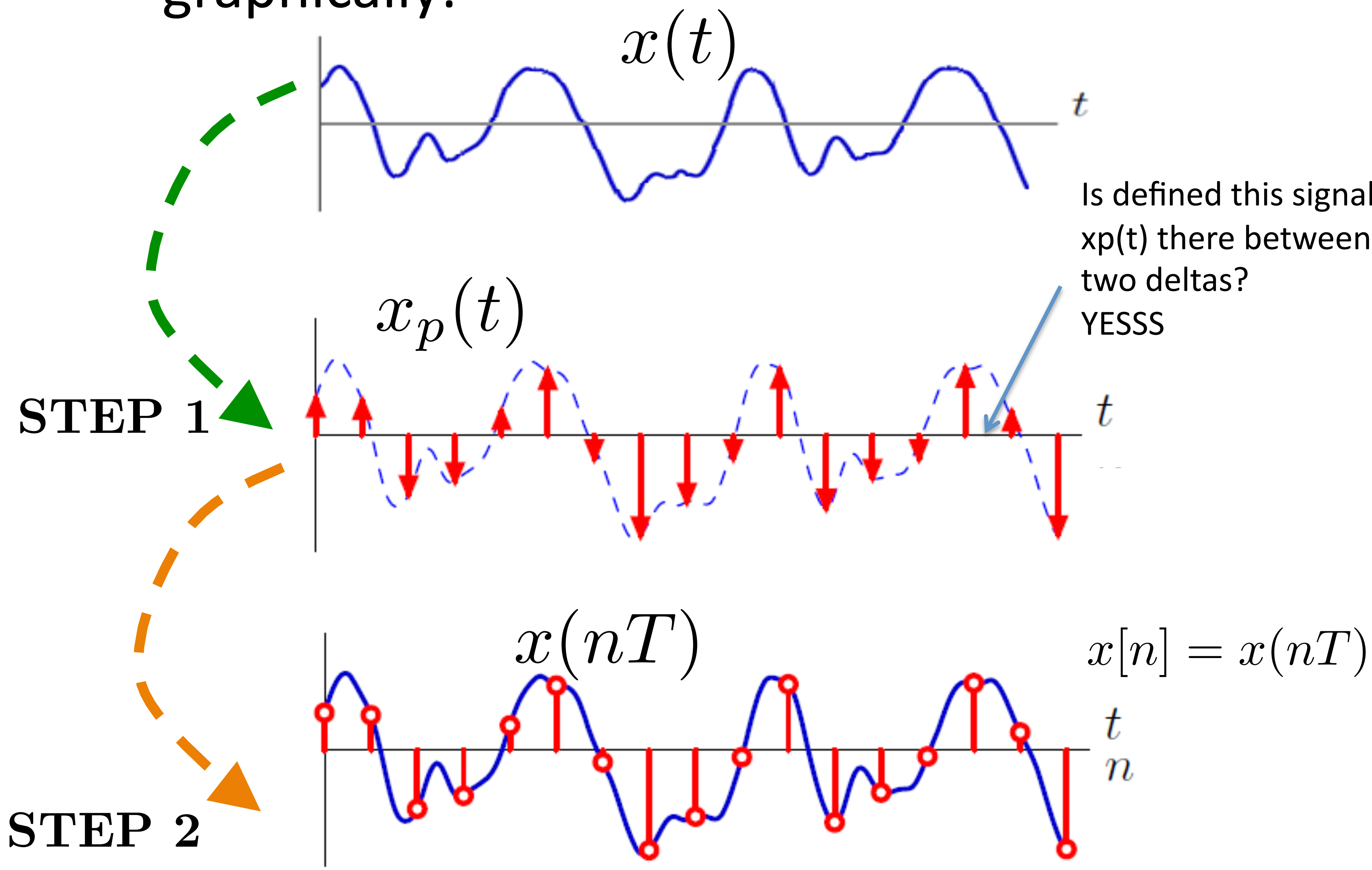
$$x_p(t) = x(t)p(t)$$

$$x[n] = x(nT)$$

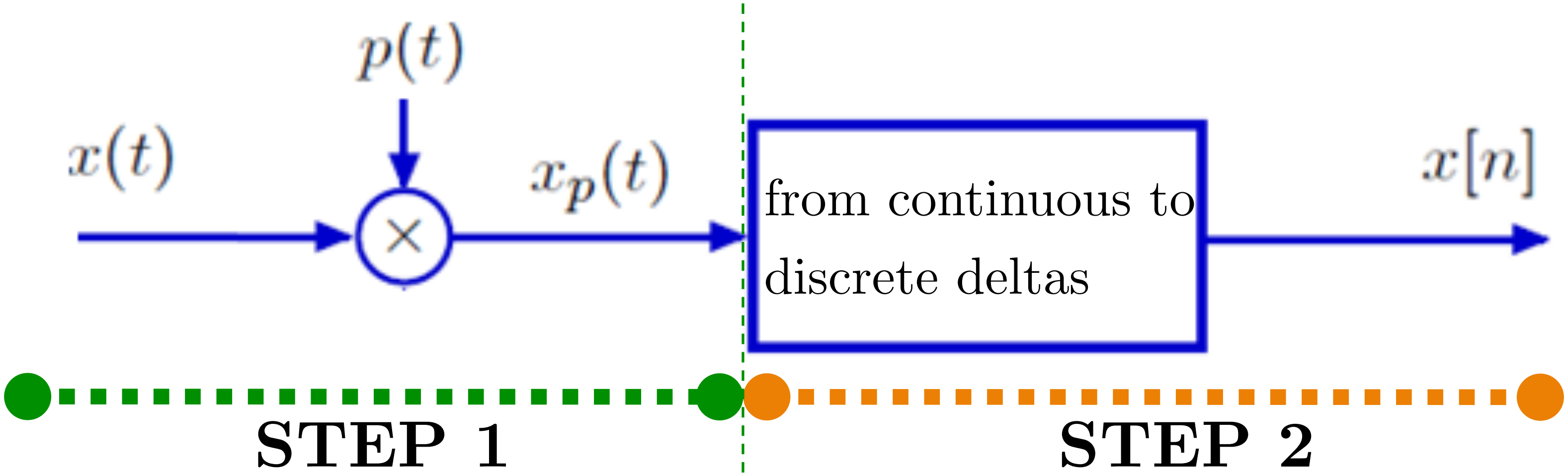
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



- graphically:



- with blocks:



$$x_p(t) = x(t)p(t)$$

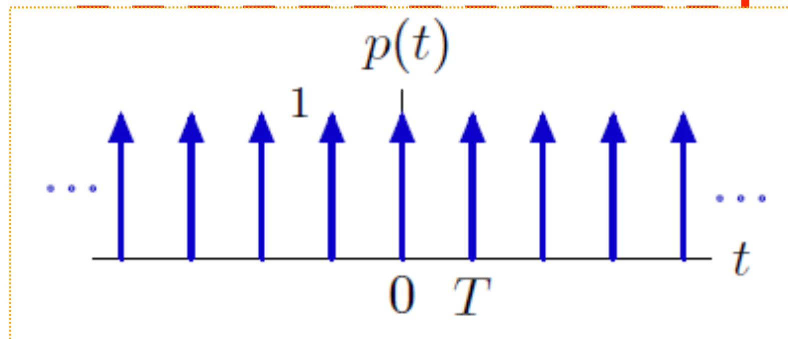
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

From Dirac deltas $\delta(t)$ to Kronecker deltas $\delta[n]$

STEP 1

$$x_p(t) = x(t)p(t)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P_G(\omega)$$

Recalling that ➔
See previous slides

$$P_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

STEP 1

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P_G(\omega)$$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

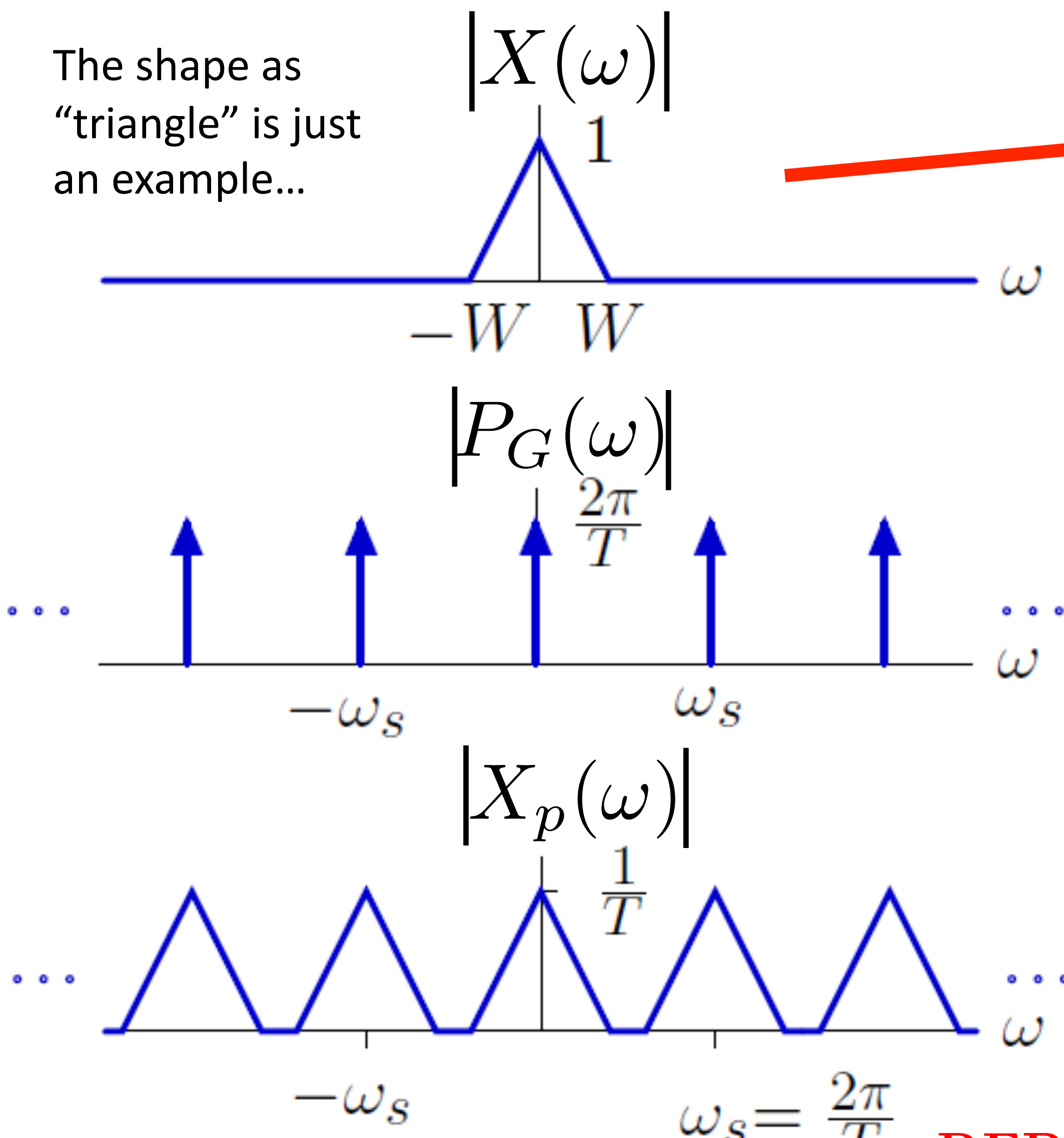
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$

$$w_s = \frac{2\pi}{T}$$

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\omega - k\omega_s\right)$$

STEP 1

The shape as "triangle" is just an example…



<u>Signal with finite bandwidth:</u>

$$X(\omega) > 0$$

$$-W < \omega < W$$

$$P_G(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(\omega - k\omega_s)$$

REPLICAS in FREQUENCY !!

STEP 2

- For step 2, we need another expression of $X_p(\omega)$

$$x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

FT continous time	FT discrete time
$X_p(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) e^{-j\omega t} dt$ $= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt$ $= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega nT}$	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

STEP 2

- Compare the expressions:

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n T}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$\Omega = \omega T$$

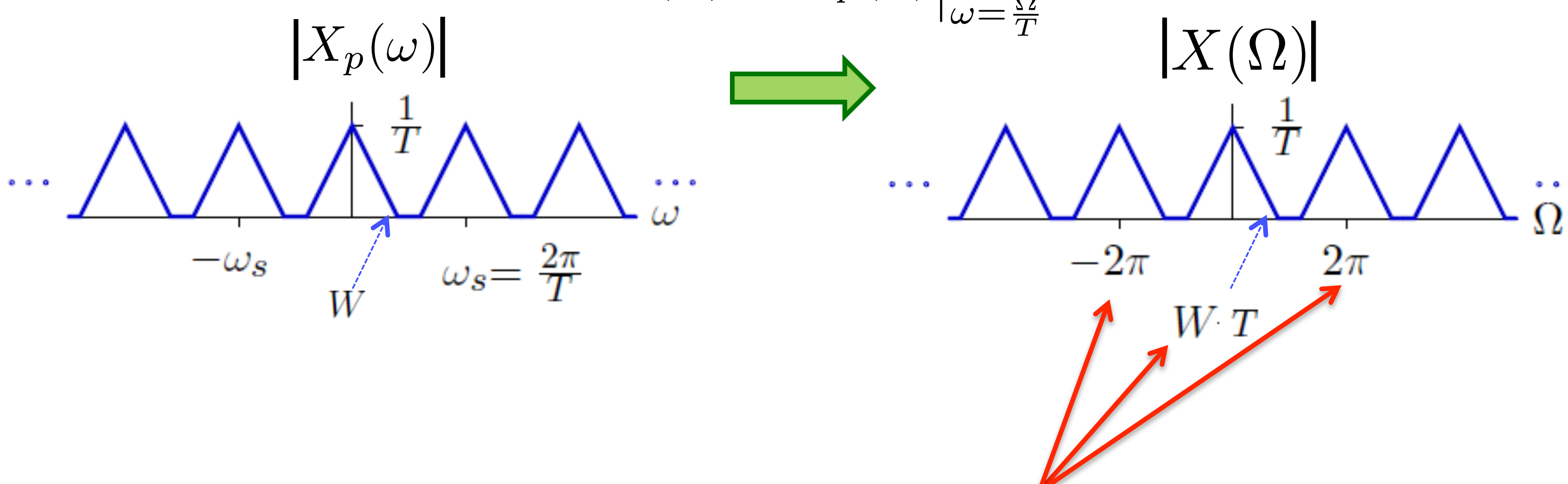
$$X_p(\omega) = X(\Omega)\Big|_{\Omega = \omega T}$$

$$X(\Omega) = X_p(\omega)\Big|_{\omega = \frac{\Omega}{T}}$$

STEP 2

- graphically:

$$X(\Omega) = X_p(\omega)\Big|_{\omega=\frac{\Omega}{T}}$$



We have to multiply the scale by T

STEP 1 + STEP 2

pequeño cambio de notación
X(omega) ==> Xc(omega) --- FT de x(t) (tiempo continuo)
X(Omega) ==> Xd(Omega) --- FT de x[n] (tiempo discreto)

$$x[n] = x(nT)$$

$$x(t)$$

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_C(\omega - k\omega_s)$$

$$X_D(\Omega) = X_p(\omega)\Big|_{\omega = \frac{\Omega}{T}}$$

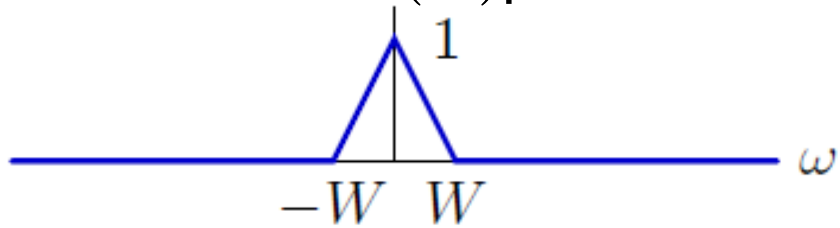
$$X_D(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_C\left(\frac{\Omega}{T} - k\omega_s\right)$$

$$X_D(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)$$

STEP 1 + STEP 2

- graphically:

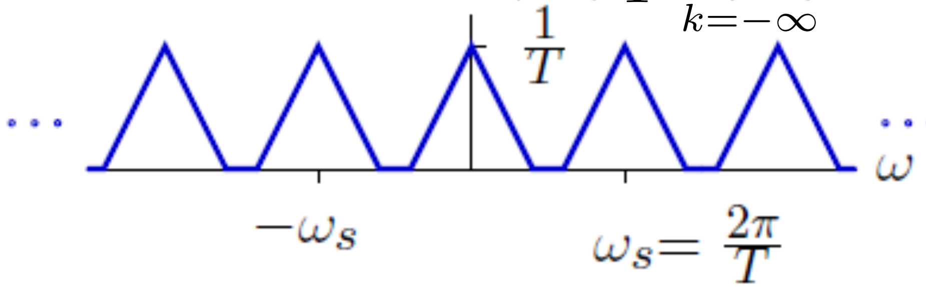
$$\left|X_C(\omega)\right|$$



<u>Effects:</u>

- ➢ REPLICAS in frequency at each ω_s
- ➢ Periodic with period 2π
- ➢ The amplitud is divided by T
- ➢ The scale is "expanded" by T

$$\left|X_p(\omega)\right| = \left|\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_C(\omega - k\omega_s)\right|$$



$$\left|X_D(\Omega)\right| = \left|X_p(\omega)\right|\Big|_{\omega=\frac{\Omega}{T}}$$

$$X_D(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_C\left(\frac{\Omega}{T} - k\omega_s\right)$$