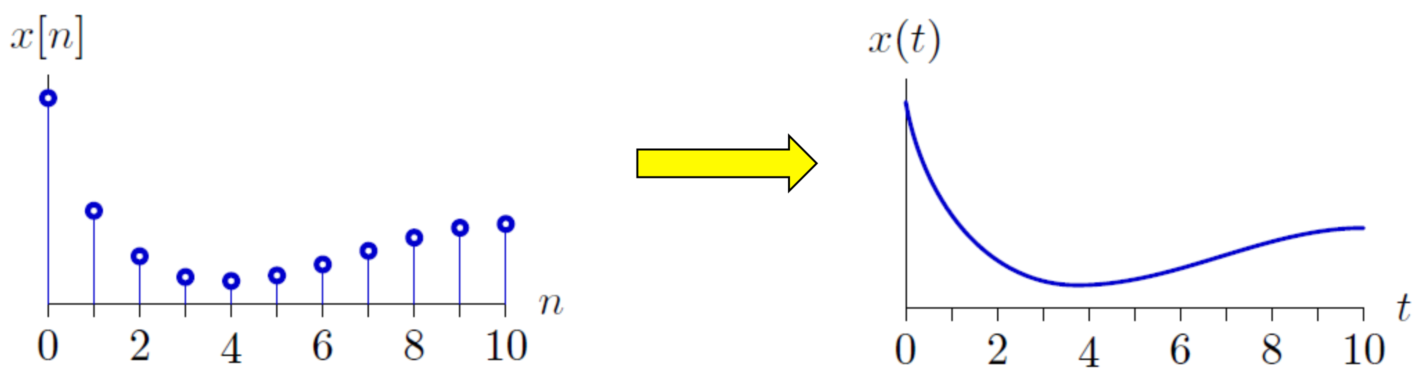
Sampling (in time)

SECOND PART

(come back to continuous time: ideal interpolation)

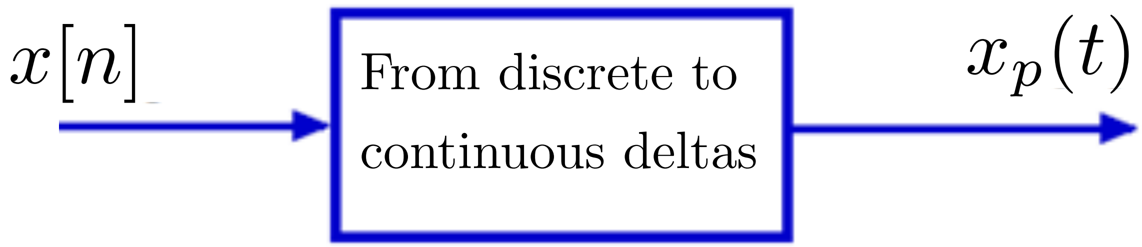
Recovering x(t) given x[n]

- Let consider now to recover x(t) from x[n]



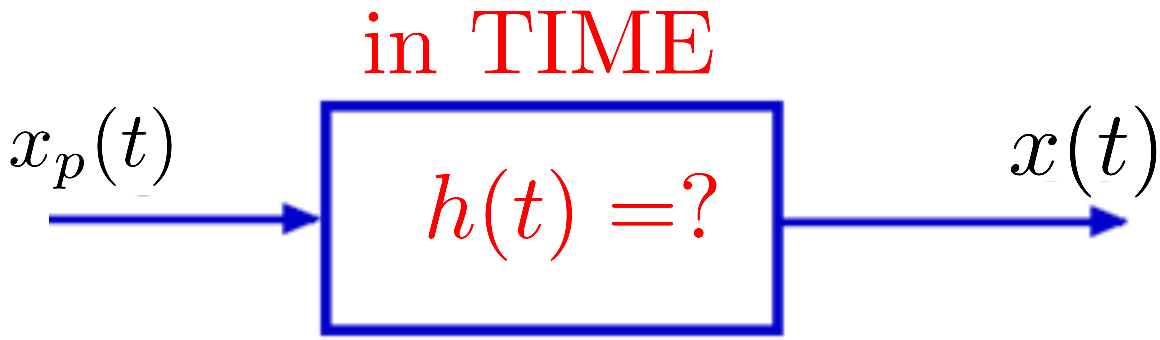
- **Is possible to recover perfectly x(t)? If yes, when is this possible?**

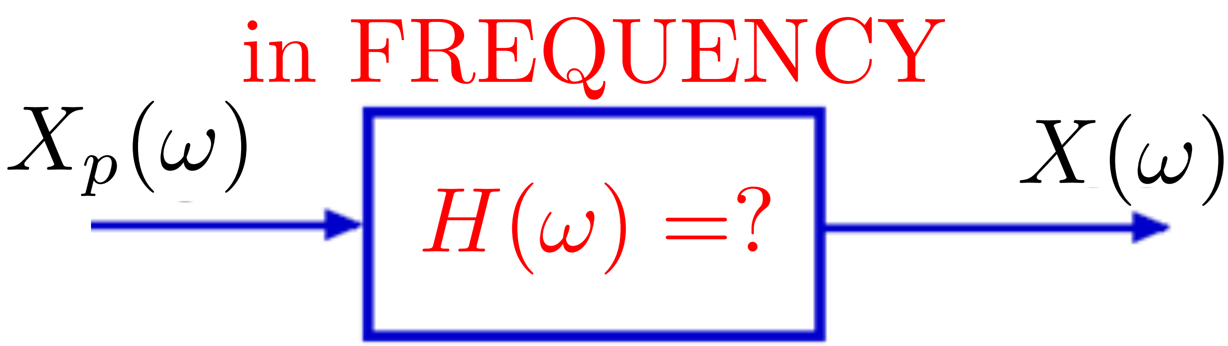
First step "back"



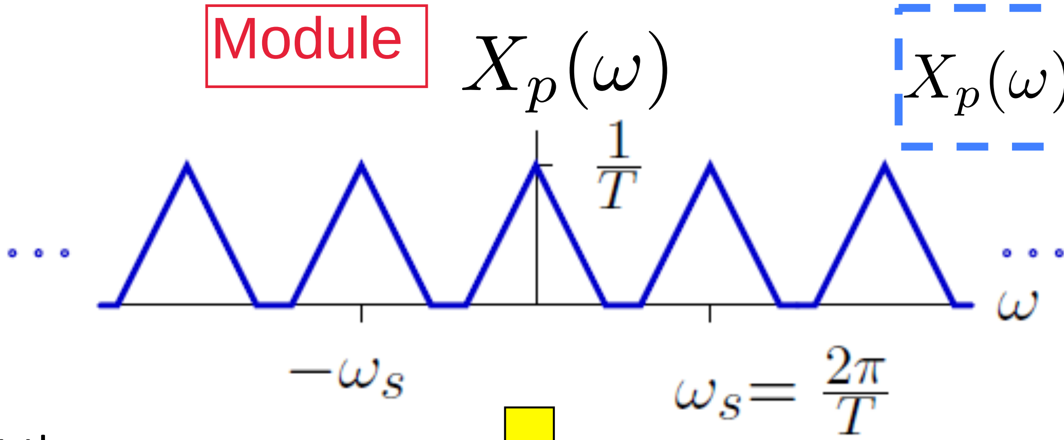
- We are be able to come back to easily to $X_p(\omega)$
- (just rescaling and....)

Second step "back"



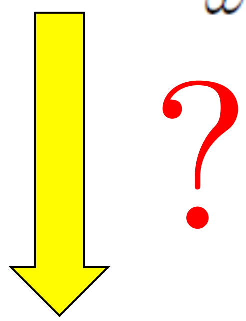


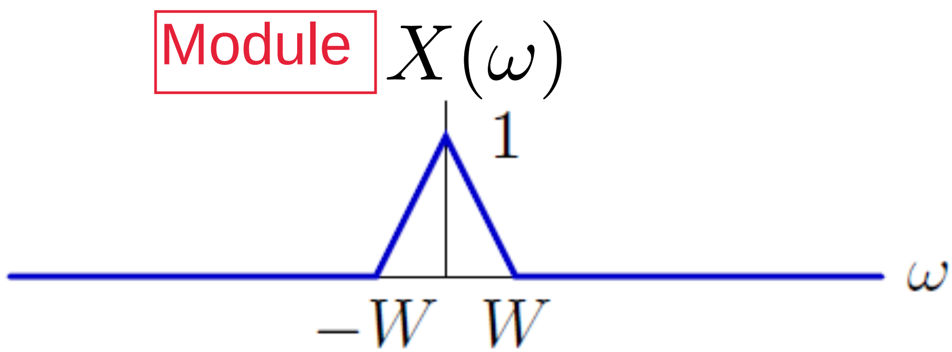
Second step “back”: in frequency



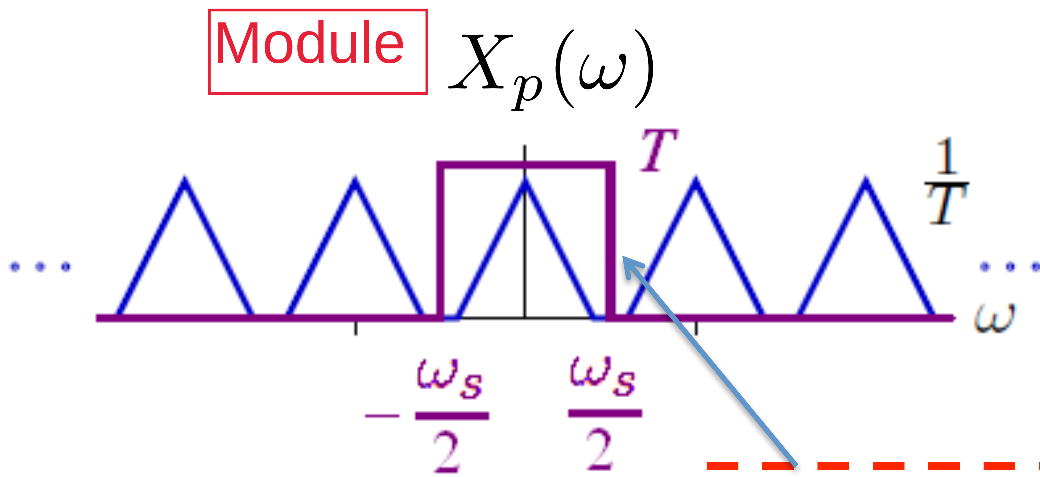
$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P_G(\omega)$$

Recall that the shape as “triangles” is just an example (easy to plot)



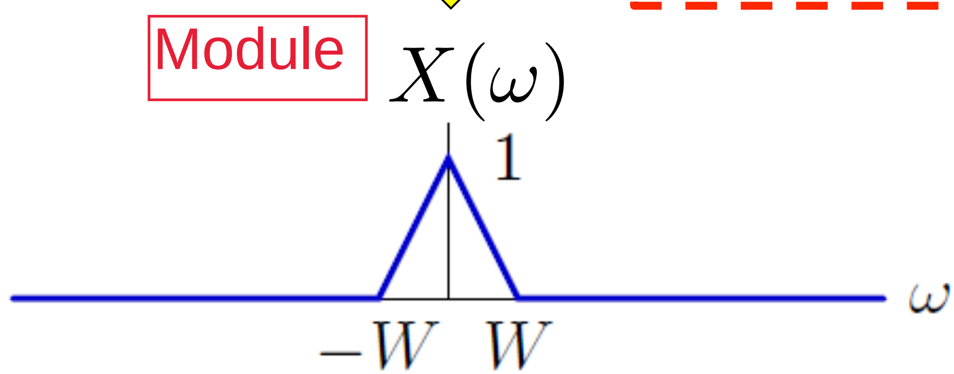


Second step "back": in frequency



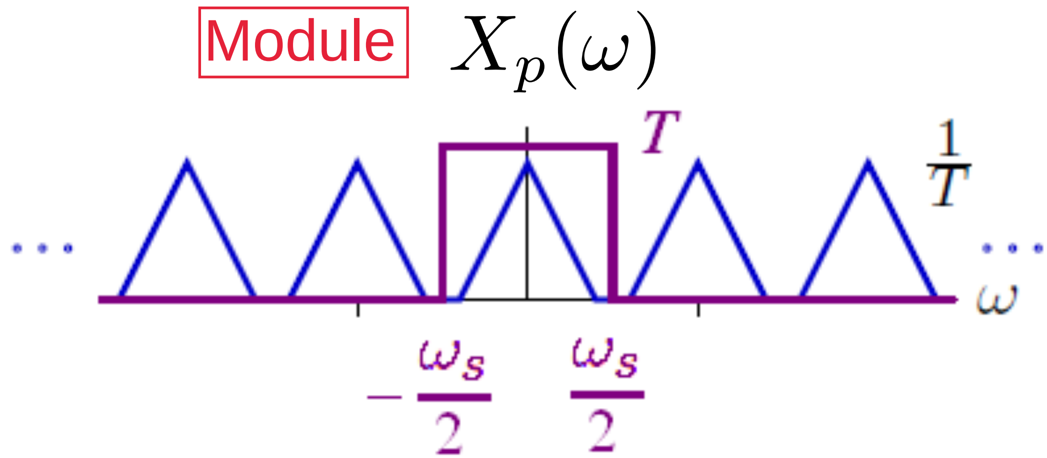
$$X(\omega) = H(\omega) X_p(\omega)$$

$$H(\omega) = \begin{cases} T & \text{si } |\omega| < \omega_s/2 \\ 0 & \text{si } |\omega| \geq \omega_s/2 \end{cases}$$



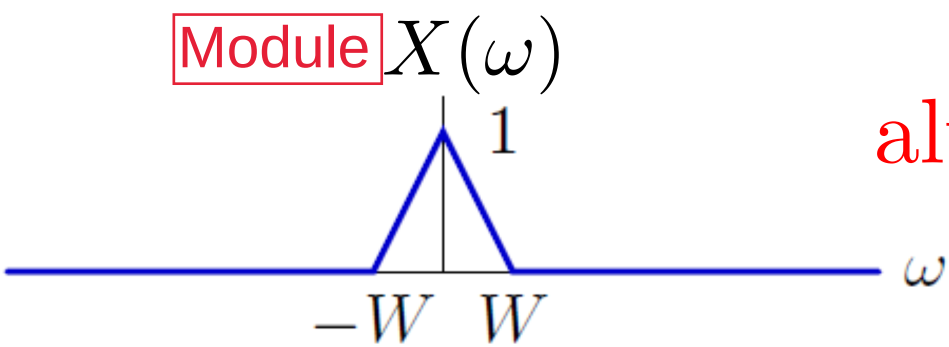
Second step "back": in frequency

PERFECT RECONSTRUCTION !



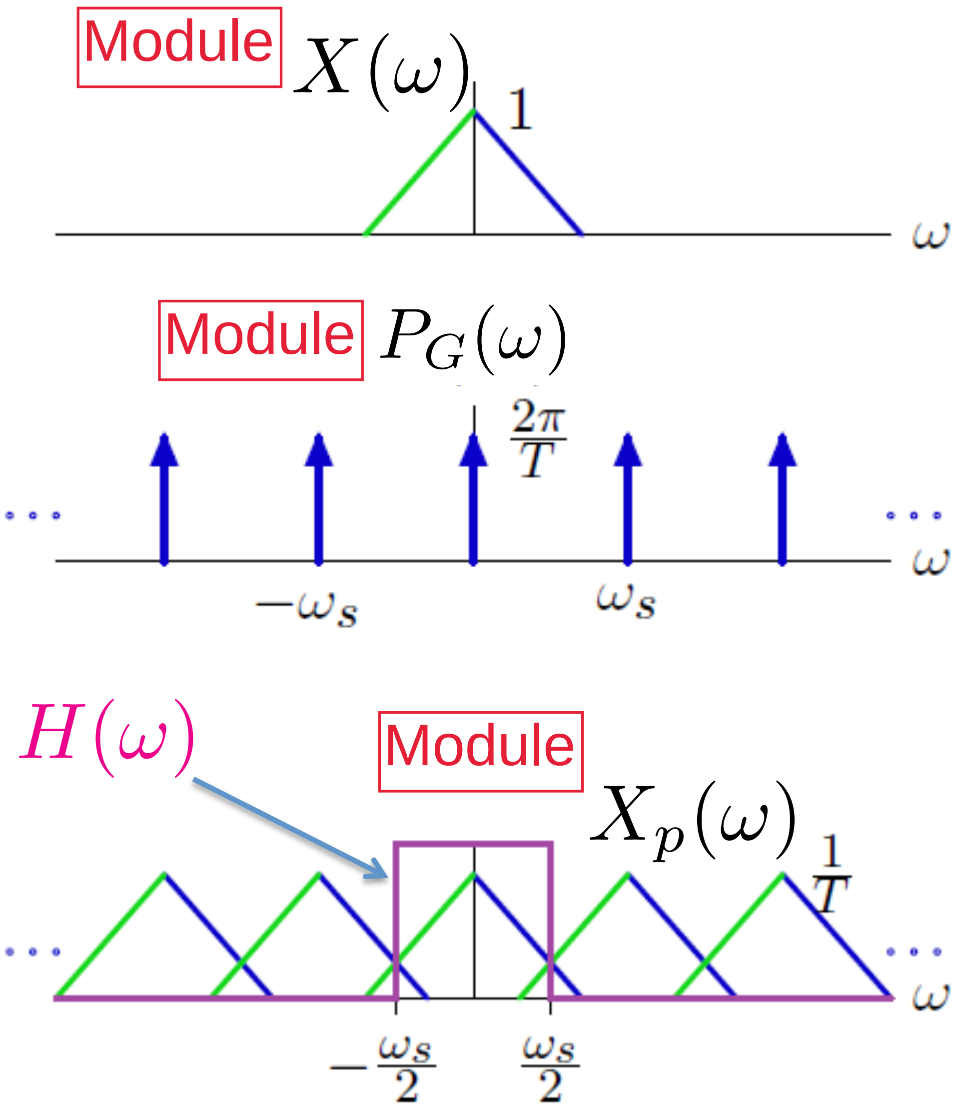
we recover perfectly $X(\omega)$

and then $x(t)$



always possible ???

Not always possible...



Recalling the procedure from continuous to discrete domain....

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P_G(\omega)$$

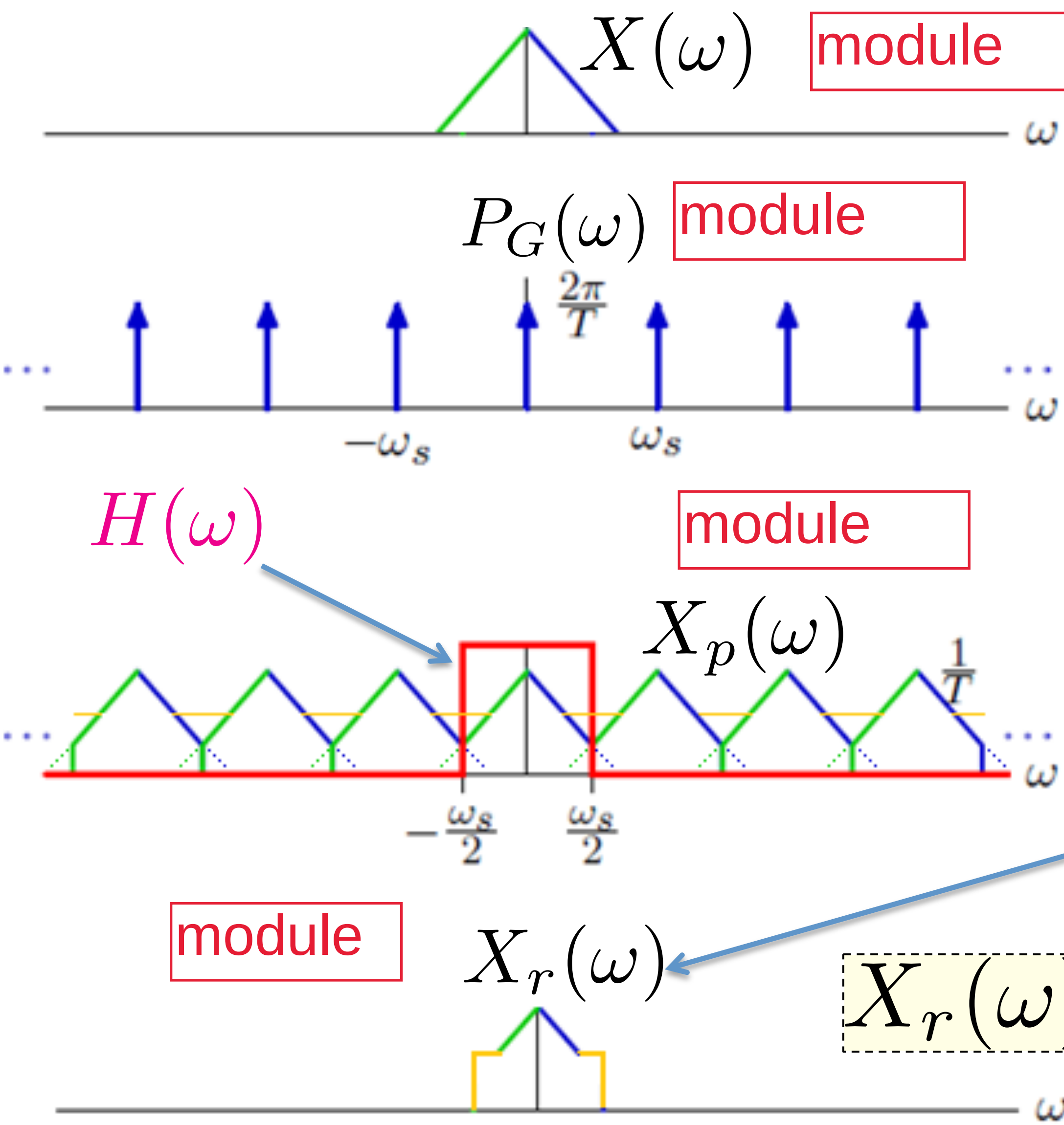
In this case, ω_s is too small !!!

i.e., T is too big !!

$$w_s = \frac{2\pi}{T}$$

THERE IS OVERLAPPING !!!

Not always possible…



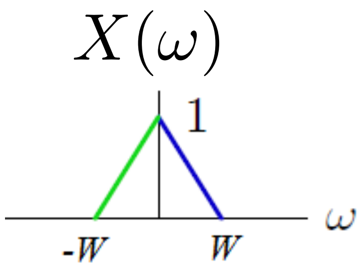
we cannot find a filter $H(\omega)$ such that

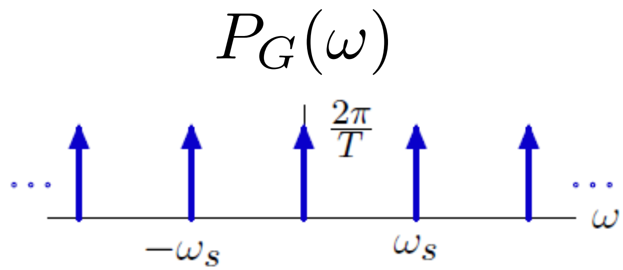
$$X_r(\omega) = X(\omega)$$

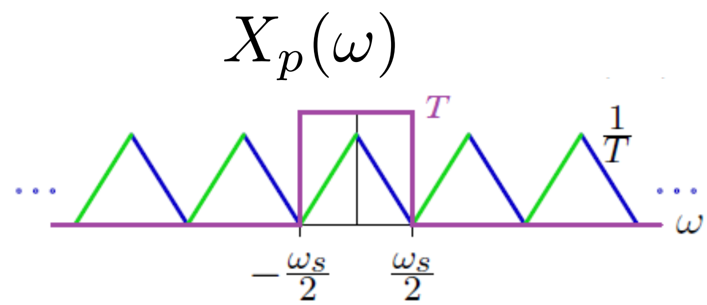
$$X_r(\omega) \neq X(\omega)$$

We cannot recover $X(\omega)$!!!

Nyquist-Shannon sampling theorem







For a bandlimited signal (W)

$$\omega_s = \frac{2\pi}{T}$$

To have a perfect reconstruction we need:

$$\omega_s \geq 2W$$

or equivalently:

$$T \leq \frac{\pi}{W}$$

Nyquist-Shannon sampling theorem

For a bandlimited signal (W)

To have a perfect reconstruction we need:

$$\omega_s \geq 2W$$

or equivalently:

$$T \leq \frac{\pi}{W}$$

$$w_s = \frac{2\pi}{T}$$

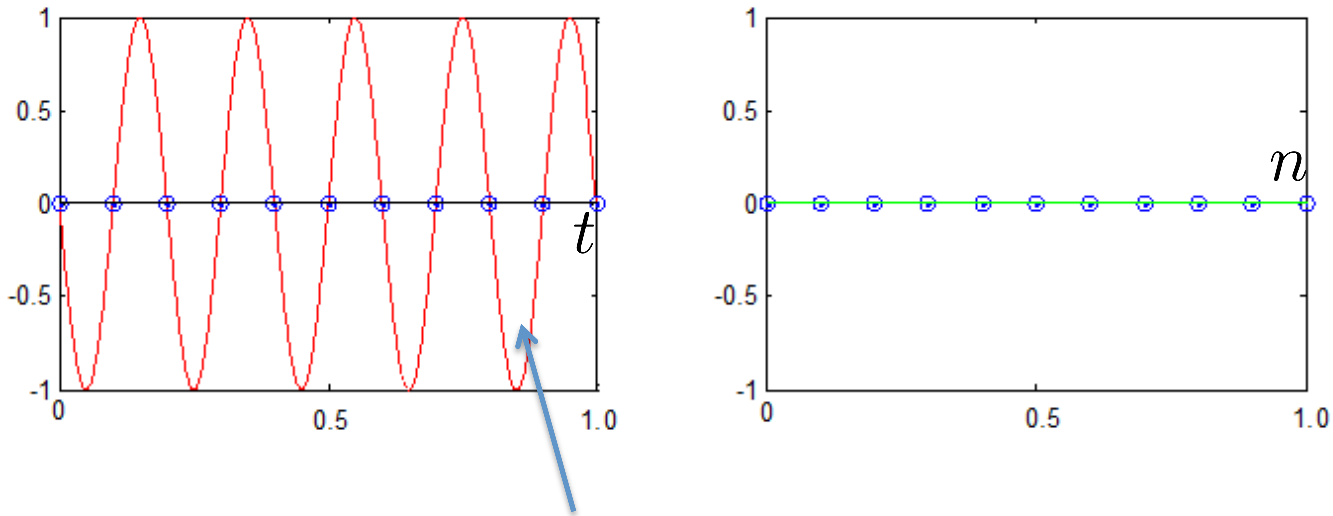
sampling frequency/velocity/rate

We have a lower bound for the sampling velocity !

If the sampling period T is too big (small sampling velocity) then we cannot reconstruct the signal.

Otherwise: we can recover perfectly the signal !!!

Example

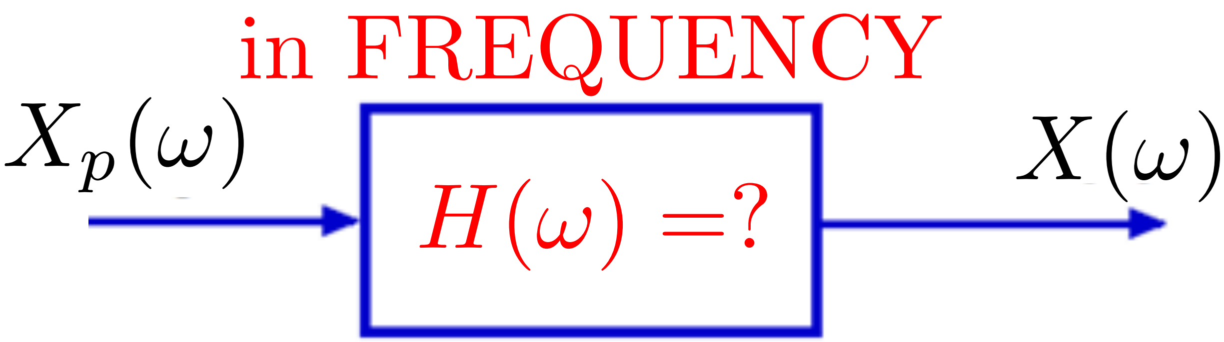


Here the sampling period T is too big …

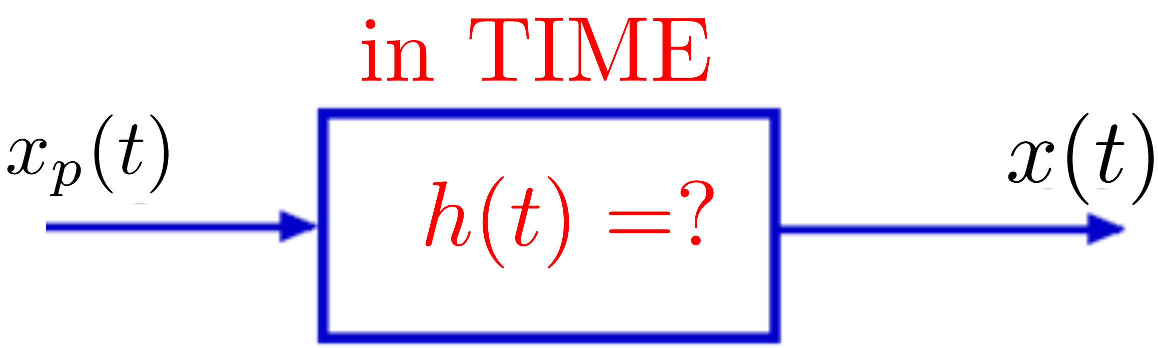
The sampling rate is too slow….

Back to a continuous signal: in time

We have seen:



Now we will see what is the corresponding procedure in time:

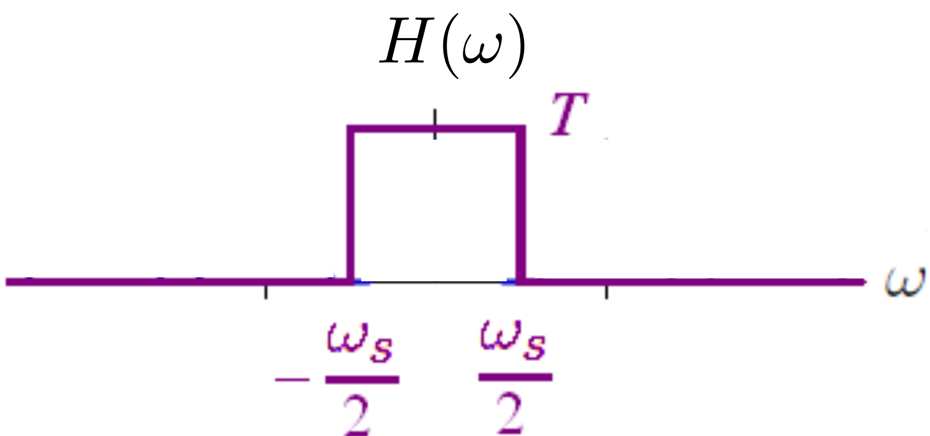


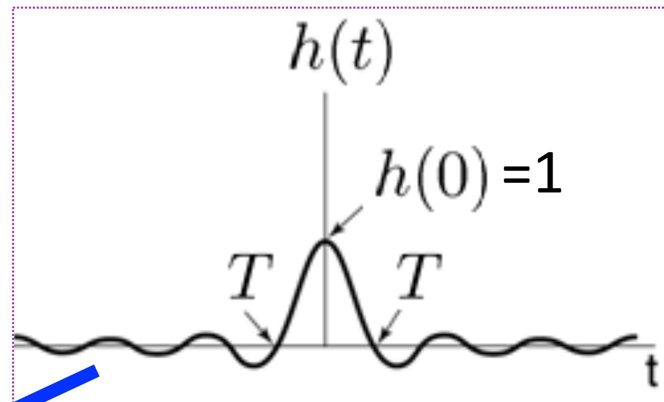
Rectangle in frequency =>> sinc in time

RECALL:

$$H(\omega) = \begin{cases} T & \text{si} \quad |\omega| < \omega_s/2 \\ 0 & \text{si} \quad |\omega| \geq \omega_s/2 \end{cases}$$

$$h(t) = \frac{T \sin(\pi t/T)}{\pi t}$$





Sinc/ "octopus" / "pulpo"

$$h(T) = 0, h(2T) = 0, ..., h(nT) = 0$$

Multiplication in frequency =>> convolution in time

RECALL:

$$X(\omega) = X_p(\omega)H(\omega)$$

$$\Updownarrow$$

$$x(t) = x_p(t) * h(t)$$

Convolution with deltas…

RECALL:

$$h(t) * \delta(t - t_0) = h(t - t_0)$$

Reconstruction: in time

Then:

If we satisfy Nyquist

$$x(t) = x_p(t) * h(t)$$

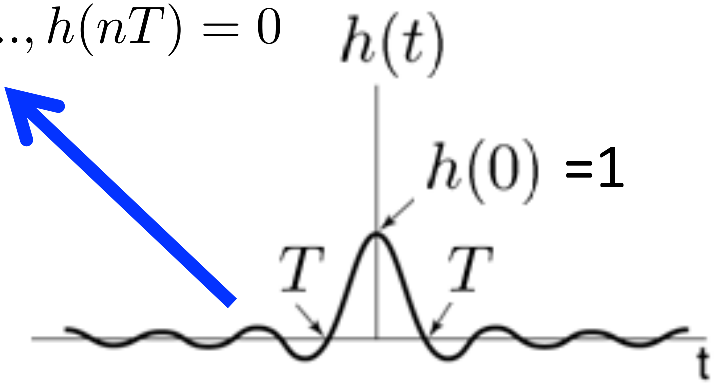
$$h(t) = \frac{T \sin(\pi t/T)}{\pi t}$$

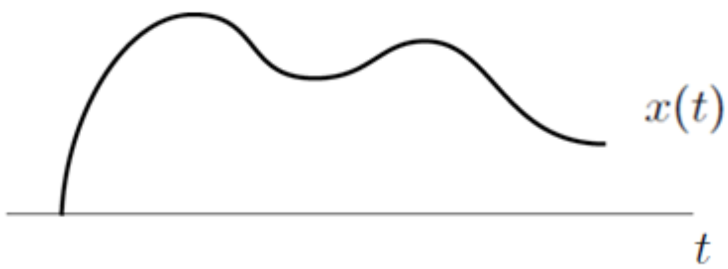
$$= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \right) * h(t)$$

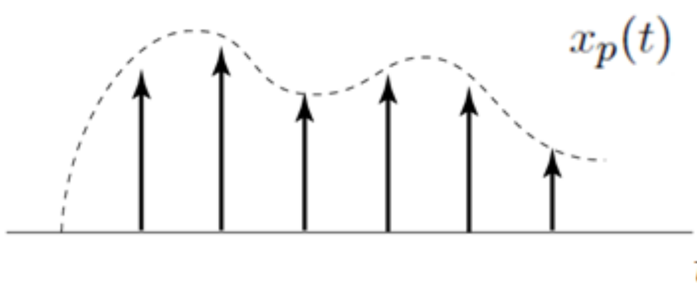
$$= \sum_{n=-\infty}^{\infty} x(nT)h(t - nT) = \sum_{n=-\infty}^{\infty} x[n] \frac{T \sin(\pi(t/T - n))}{\pi(t - nT)}$$

Reconstruction: in time

$$h(T) = 0, h(2T) = 0, ..., h(nT) = 0$$

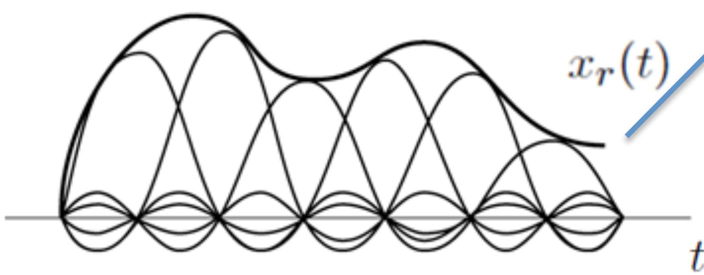






$$x_r(t) = x_p(t) * h(t)$$

RECOVERED SIGNAL



If we satisfy Nyquist:

$$x_r(t) = x(t) \text{ !!!!!}$$

Reconstruction: in time

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[n]\text{sinc}(t - nT)$$

If we satisfy Nyquist:

$$x_r(t) = x(t) \;\; !!!!!$$

$$h(t) = \text{sinc}(t) = \frac{T\sin(\pi t/T)}{\pi t}$$

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[n]\frac{T\sin(\pi\frac{1}{T}(t - nT))}{\pi(t - nT)}$$

Infinite sum of sinc functions ("octopuses"- "pulpos"); linear combination of "octopuses" !!

IDEAL INTERPOLATION

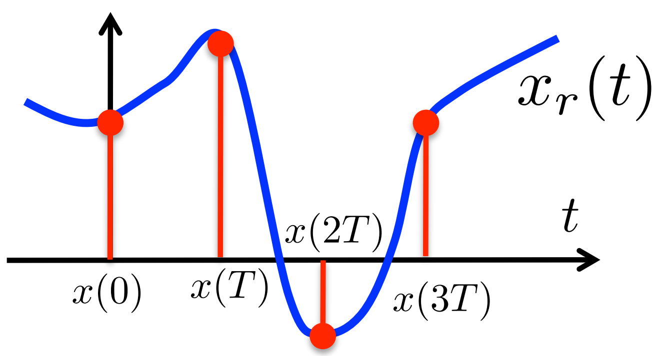
Note that:

$$h(0) = 1 \quad h(T) = 0, h(2T) = 0, ..., h(nT) = 0$$

$$h(t - nT)\Big|_{t=nT} = h(nT - nT) = 1$$

Then, in any case we have:

$$x_r(nT) = x[n] = x(nT)$$



INTERPOLATION:
even if we do not fulfill Nyquist, at least at t=nT we have a perfect reconstruction (we use x[n]).