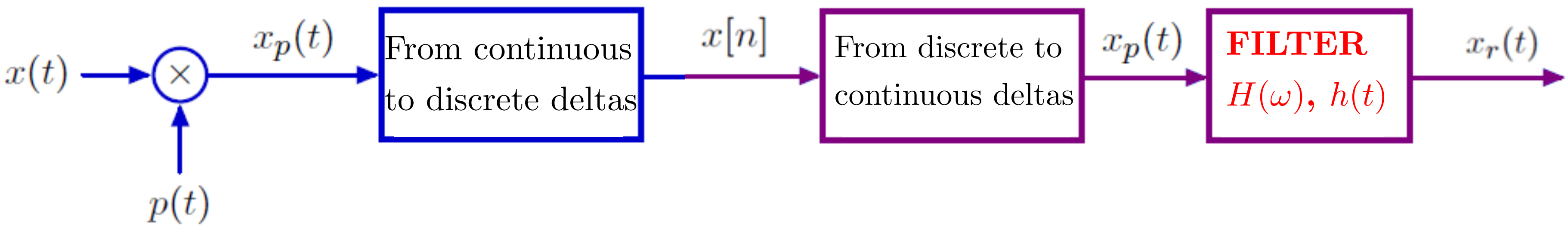
Sampling (in time)

THIRD PART

(suboptimal interpolation)

Summary: From x(t) to x[n] and back…

- Complete scheme studied so far:



- If we satisfy the Nyquist condition:

$$x_r(t) = x(t) \;\; !!!!!$$

- Nyquist condition:

$$\omega_s \geq 2W$$

or equivalently:

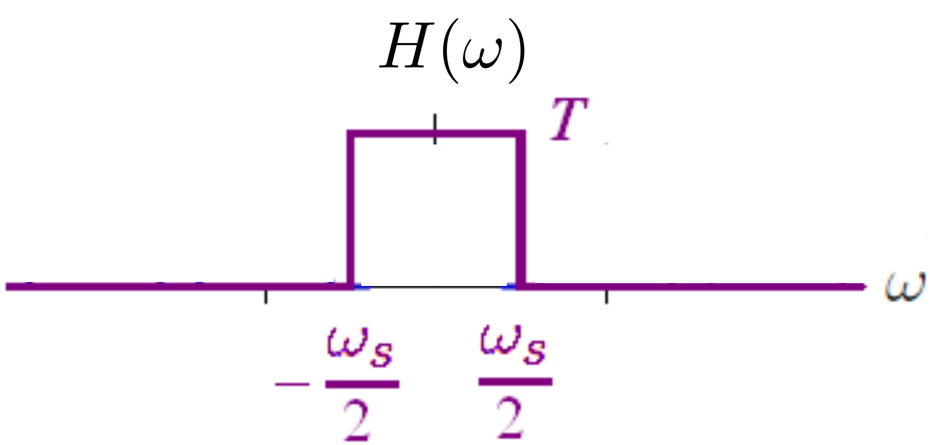
$$T \leq \frac{\pi}{W}$$

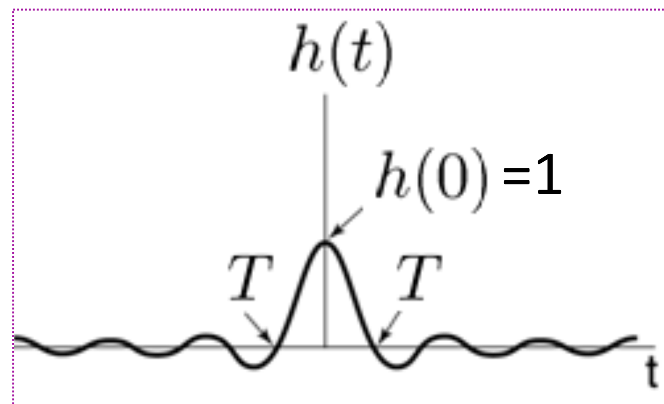
Ideal Filter - Ideal interpolation

RECALL:

$$H(\omega) = \begin{cases} T & \text{si} \quad |\omega| < \omega_s/2 \\ 0 & \text{si} \quad |\omega| \geq \omega_s/2 \end{cases}$$

$$h(t) = \frac{T\sin(\pi t/T)}{\pi t}$$

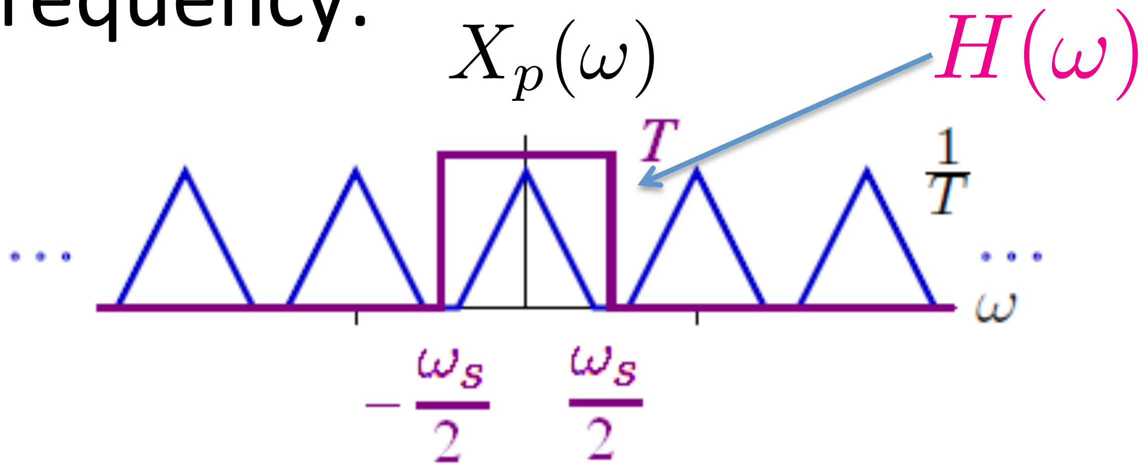




Sinc/ “octopus” / “pulpo”

Ideal Filter - Ideal interpolation

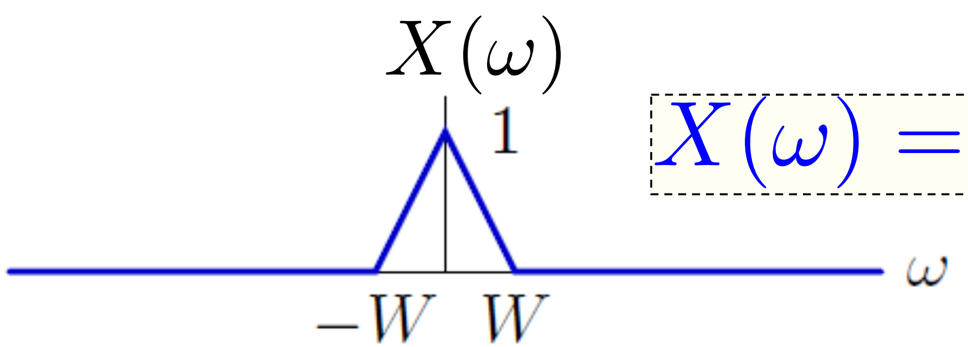
In frequency:



Recall that we are plotting the module of the FTs

If we satisfy Nyquist:

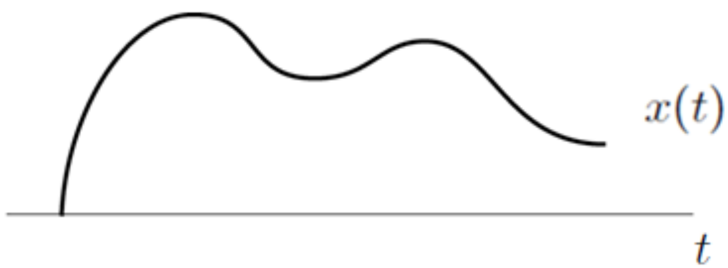
We can recover perfectly the signal $x(t)$!!

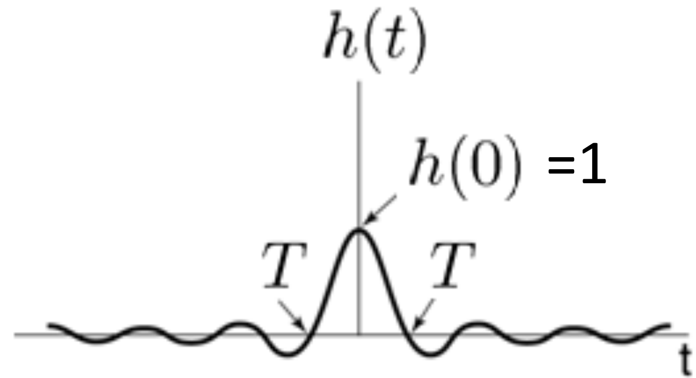


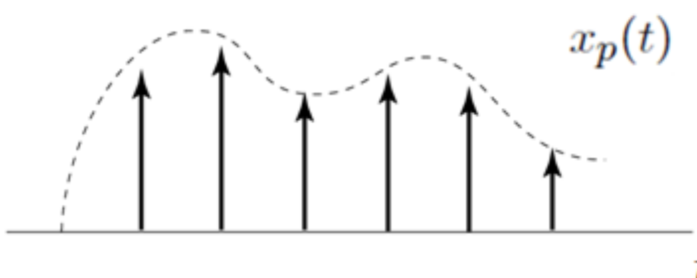
$$X(\omega) = H(\omega)X_p(\omega)$$

Ideal Filter - Ideal interpolation

In TIME:

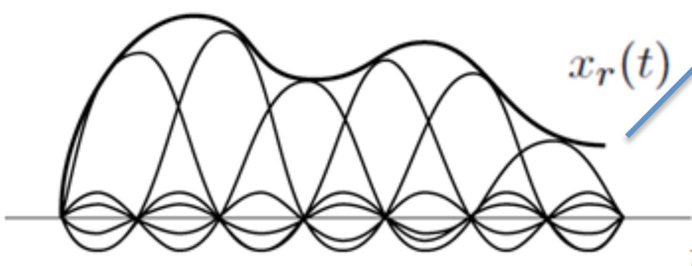






$$x_r(t) = x_p(t) * h(t)$$

RECOVERED SIGNAL

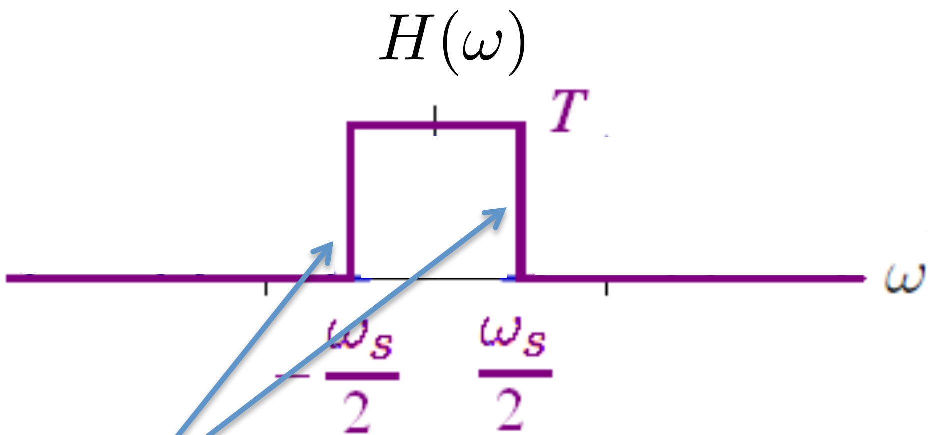


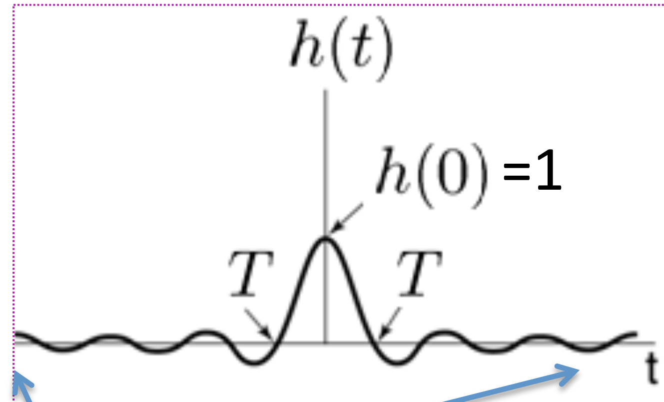
If we satisfy Nyquist:

$$x_r(t) = x(t) \text{ !!!!!}$$

Ideal Filter - Ideal interpolation

Why ideal?





Infinite slopes

Infinite temporal length

Difficult (impossible) practical implementation (good approximations are very costly)

What is an INTERPOLATOR ?

$$x_r(t) = x_p(t) * h(t)$$

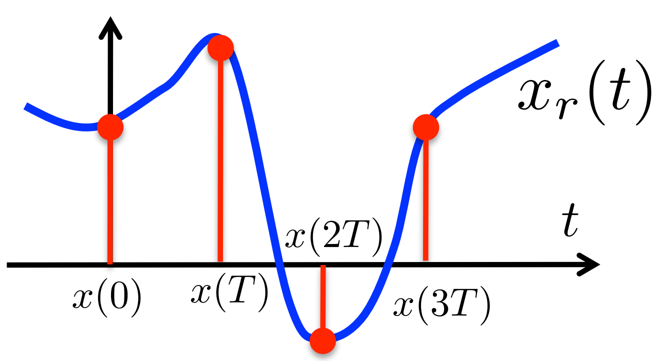
It is a filter such that:

$$h(0) = 1$$

$$h(t - nT)\Big|_{t=nT} = h(nT - nT) = 1$$

Then, in any case we have:

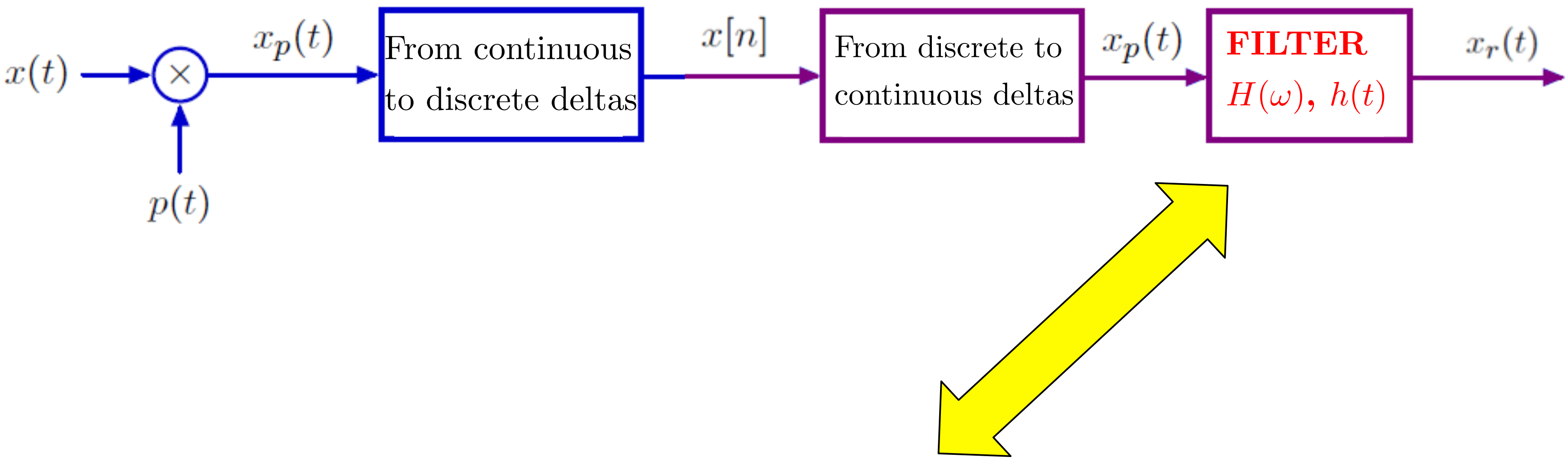
$$x_r(nT) = x[n] = x(nT)$$



INTERPOLATION:
(at least) at t=nT we have a perfect reconstruction (we use x[n]).

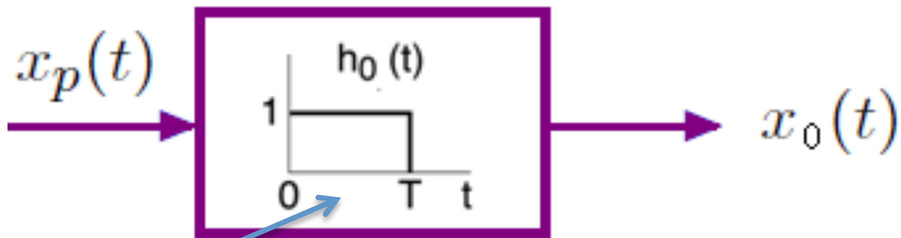
Suboptimal interpolation

Suboptimal interpolation



We change this filter:
Using a non-ideal one

Zero-order interpolator (piecewise constant)



finite temporal length!

$$x_0(t) = x_p(t) * h_0(t)$$

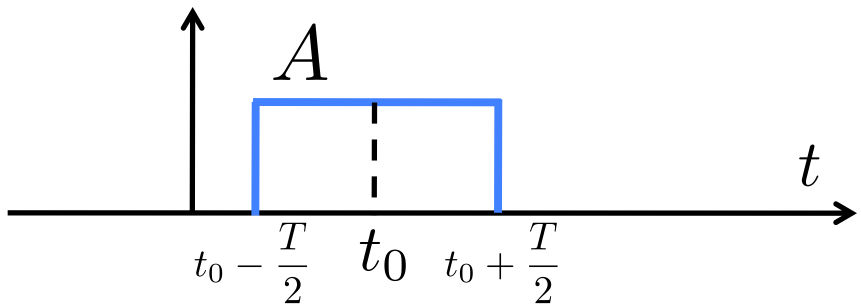
A rectangle: but in TIME !

A sinc function in FREQUENCY !!

but an infinite length response in frequency !

RECALL:
Fourier Trasform of a rectangle

$$x(t) = A \text{ rect}\left(\frac{t - t_0}{T}\right) = \begin{cases} A & \text{for } |t - t_0| < \dfrac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



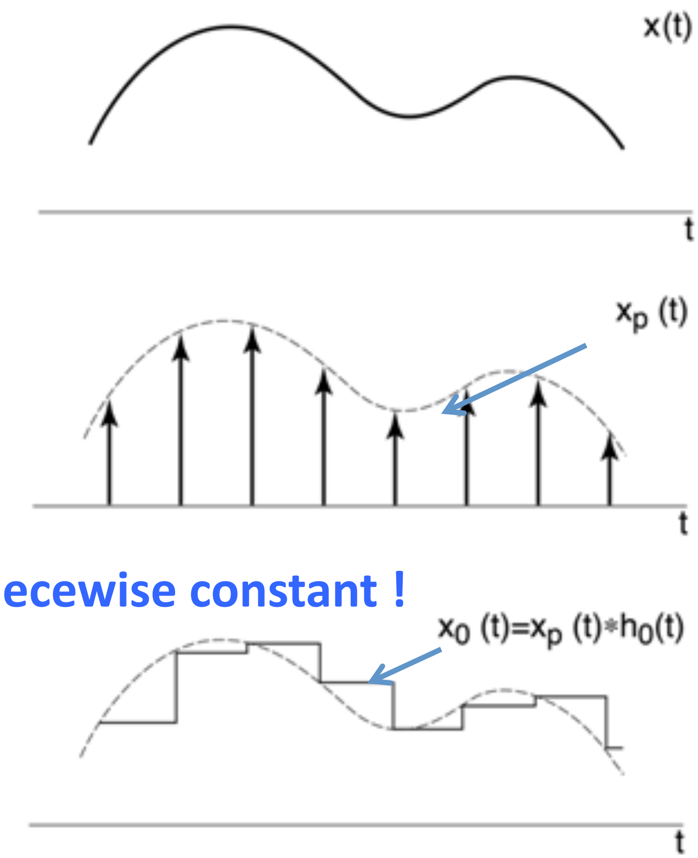
In this figure
$t_0 > 0$

$$X(\omega) = AT \text{ sinc}(\omega T/2) e^{-j\omega t_0}$$

$$= AT \frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega t_0} = 2A \frac{\sin(\omega \frac{T}{2})}{\omega} e^{-j\omega t_0}$$

Zero-order interpolator

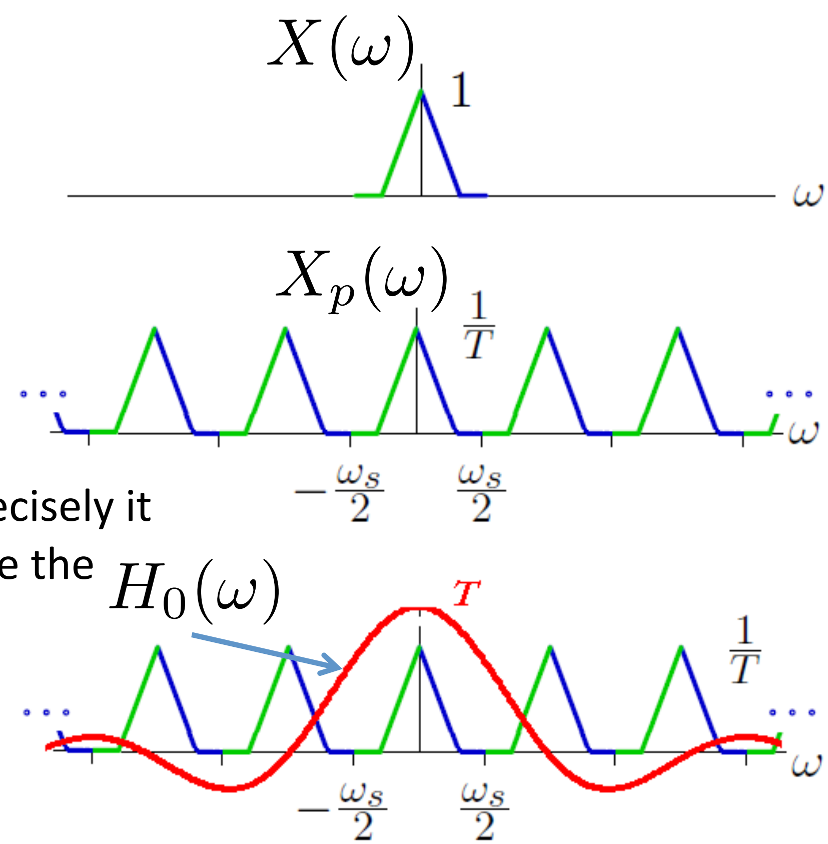
TIME DOMAIN



Piecewise constant !

$$x_0(t) = x_p(t) * h_0(t)$$

FREQUENCY DOMAIN



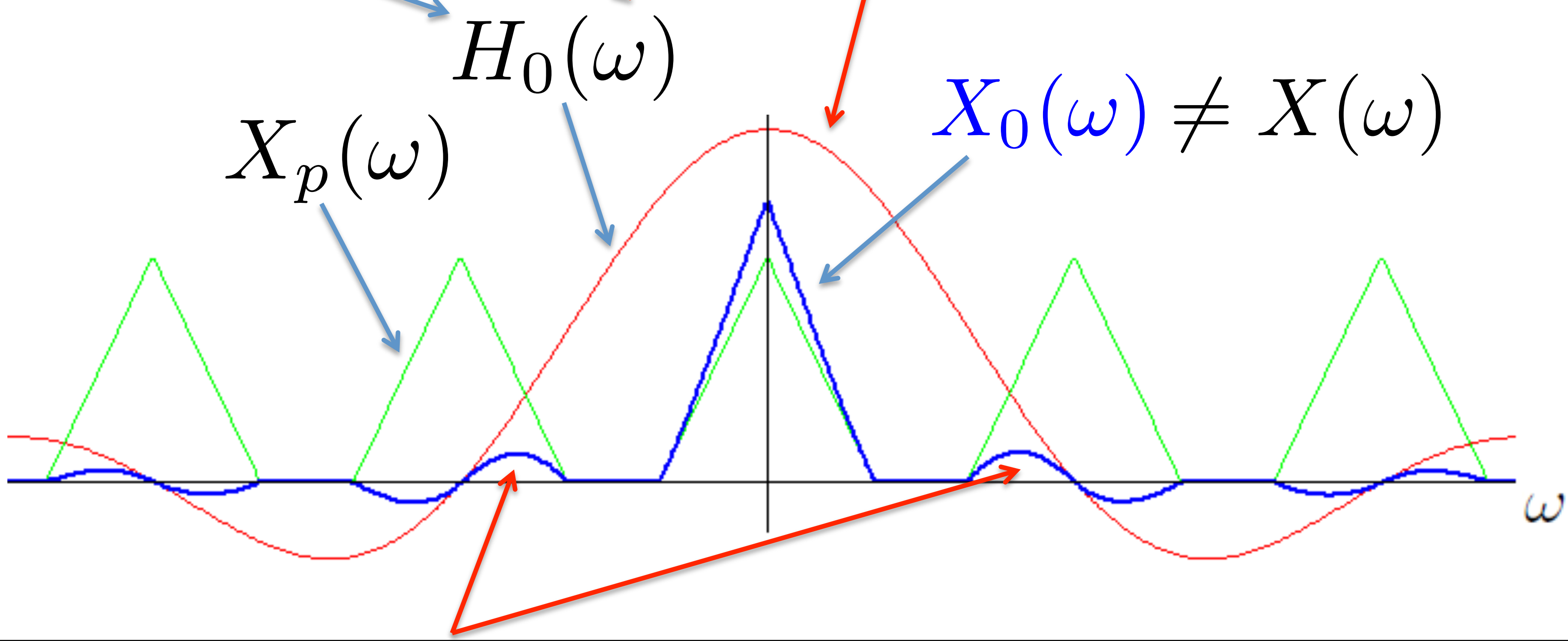
more precisely it should be the module

$$X_0(\omega) = X_p(\omega) H_0(\omega)$$

Zero-order interpolator

Recall that, in general, the FT takes complex values (then, e.g., in that case, we represent the module)

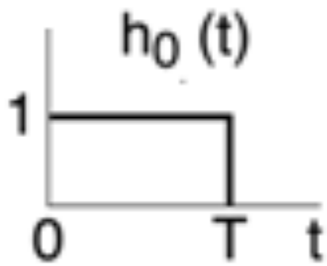
infinite length response in frequency !



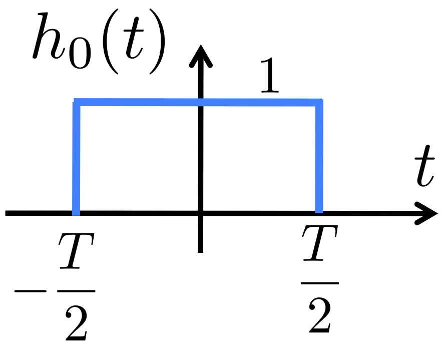
<u>NO perfect reconstruction</u>: some high-frequencies are included ("distortion"), that are not contained in the original signal x(t)

Zero-order
Nearest-Neighbor interpolator

If instead of

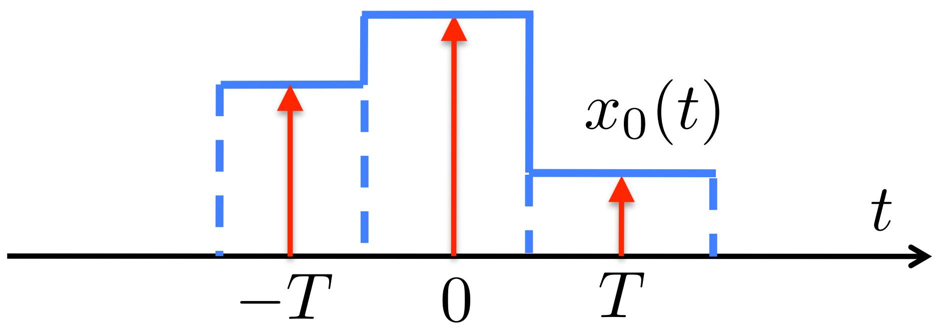


We use



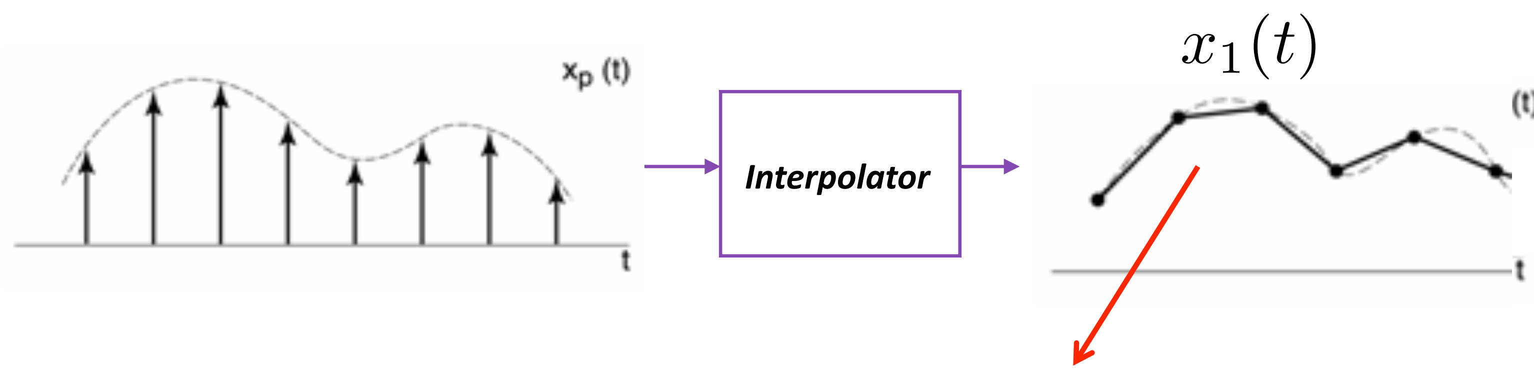
1) $H_0(\omega)$ is real (no complex); is directly the sinc function (without multiplying for complex exponentials)

2) We obtain a **Nearest-Neighbor interpolator !**



First-order interpolator

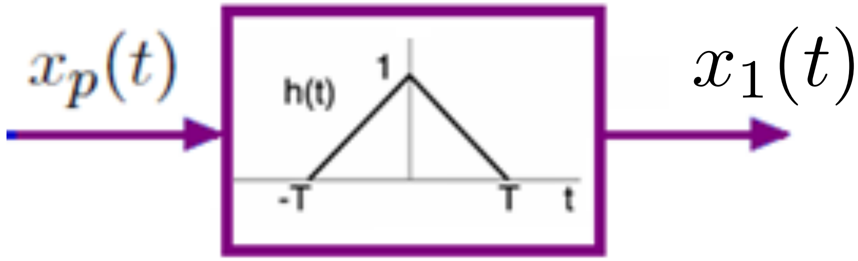
Time Domain



Piecewise linear interpolator !

Matlab uses this interpolator in its plots (for default).

First-order interpolator



Triangular filter in time - h(t)

A squared sinc (squared "octopus") in frequency !

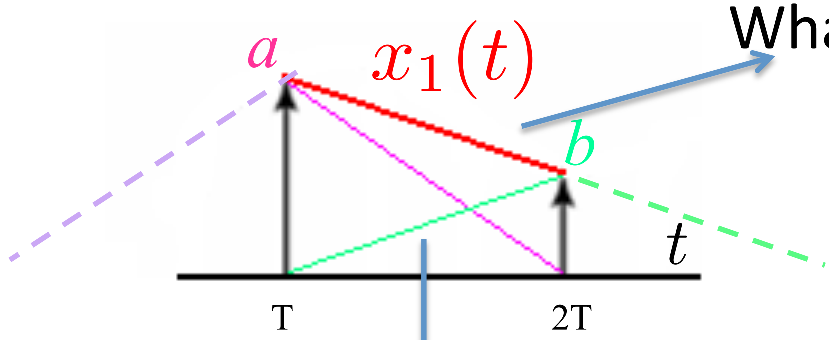
$$x_1(t) = x_p(t) * h(t)$$

Convolution with a triangular function of length 2T

First-order interpolator

$$x_1(t) = x_p(t) * h(t)$$

Convolution with a triangular function of length 2T



What is the value of the signal at t=1.4T ?

$x_1(1.4T) = 0.6 \cdot a + 0.4 \cdot b$

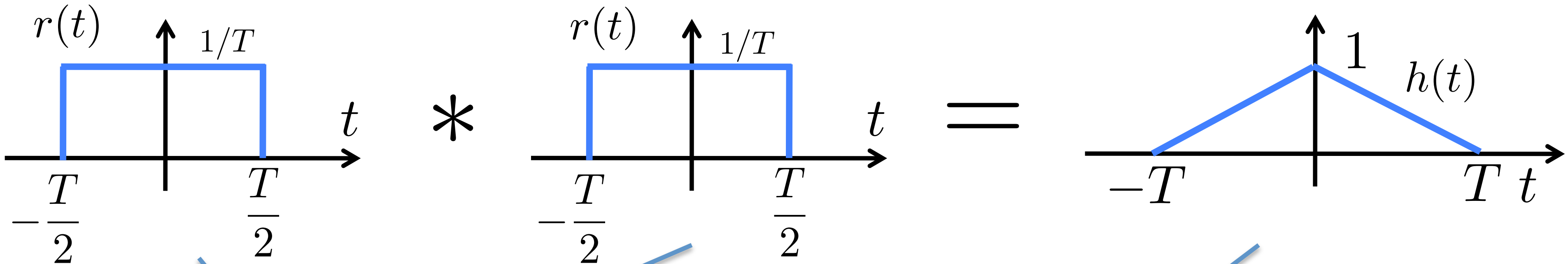
Linear combination of two pieces of two triangles during the convolution

And the value of the signal at t=rT with r>=1 and r<=2?

$$x_1(t) = (2 - r)a + (r - 1)b$$

Triangle as convolution of two rectangles

RECALL:

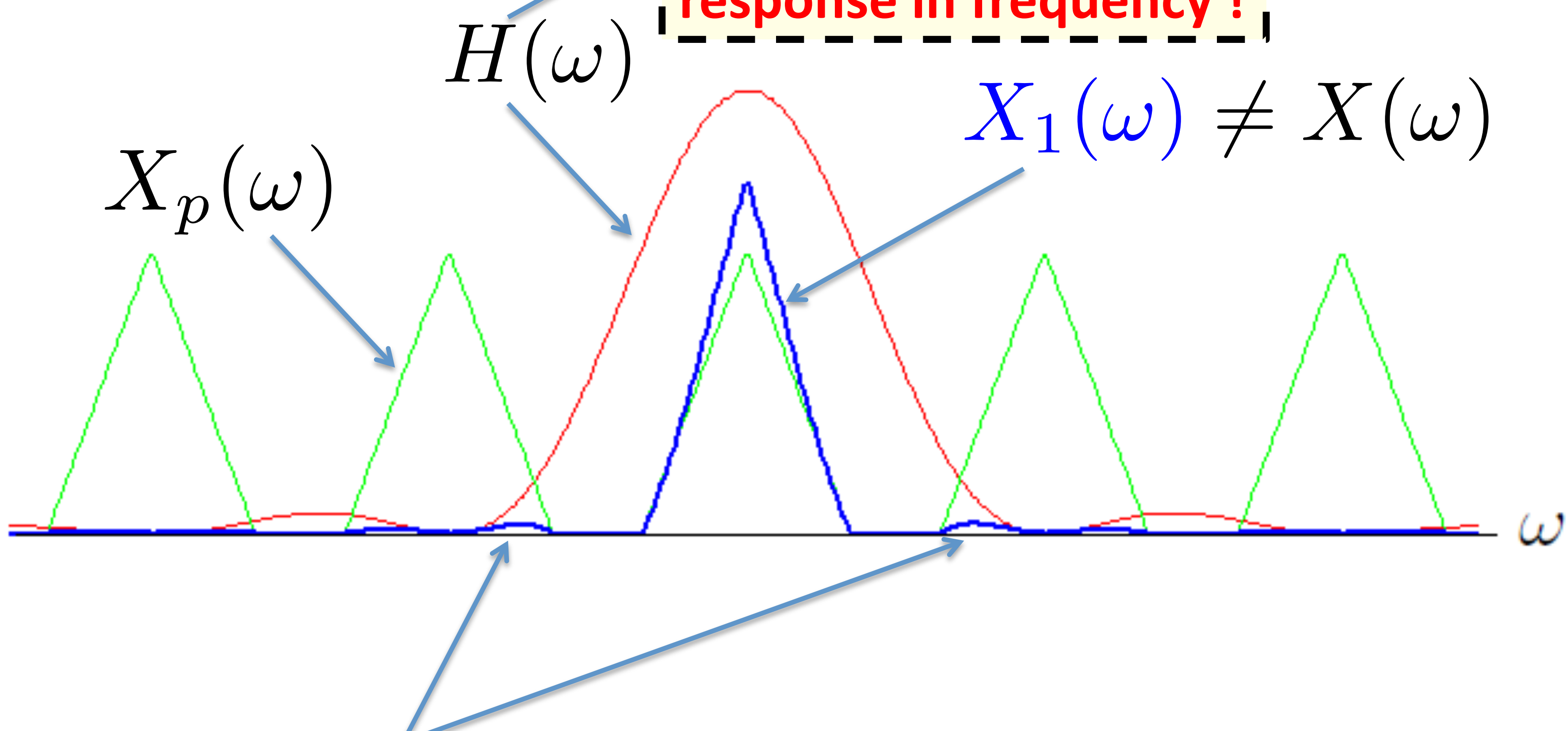


$$R(\omega) = \frac{2}{T}\frac{\sin(\omega\frac{T}{2})}{\omega}$$

$$H(\omega) = R(\omega)^2 = \left(\frac{2}{T}\frac{\sin(\omega\frac{T}{2})}{\omega}\right)^2 = \frac{4\sin(\omega\frac{T}{2})^2}{\omega^2 T^2}$$

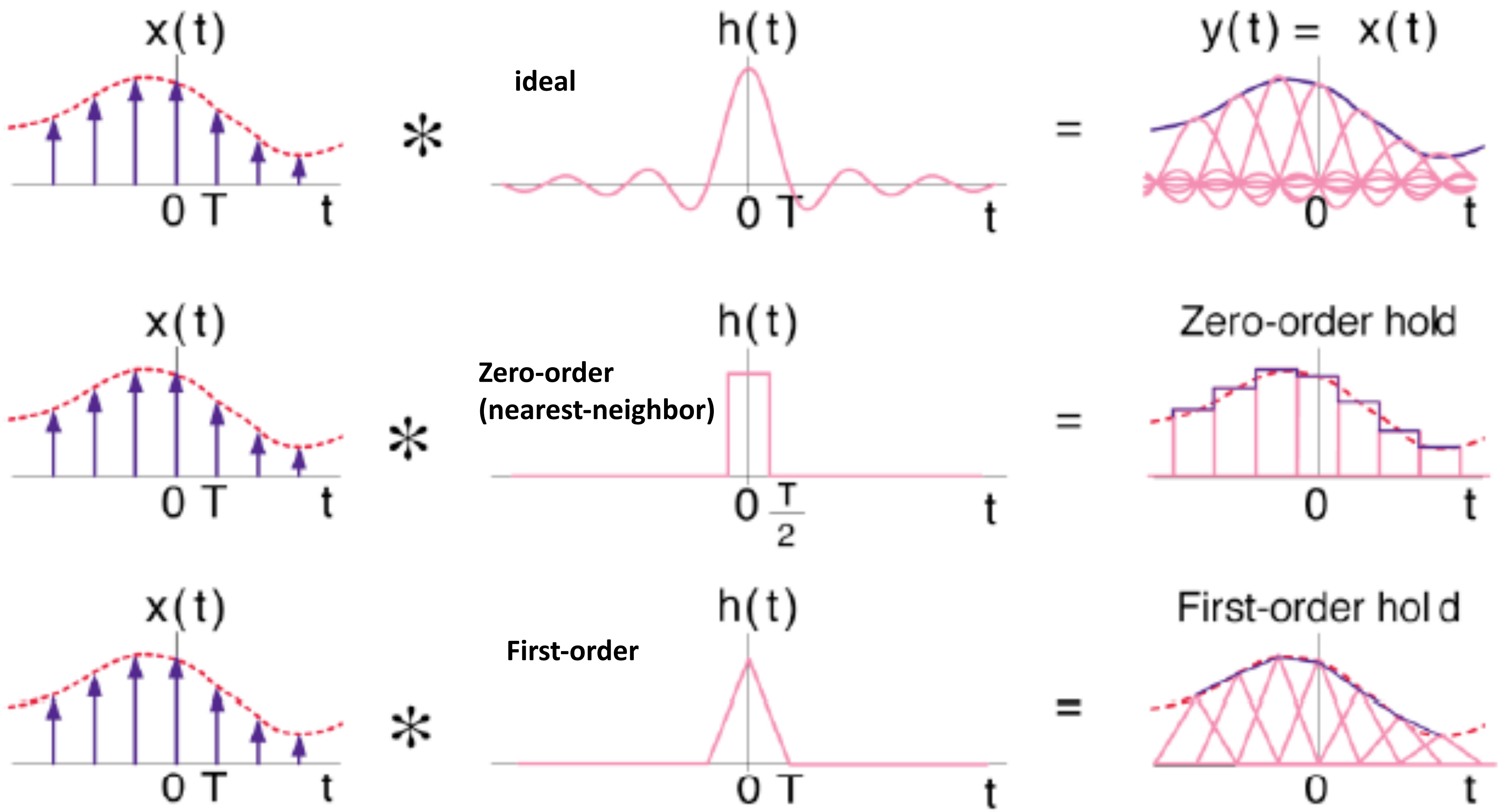
First-order interpolator in frequency

infinite length response in frequency !



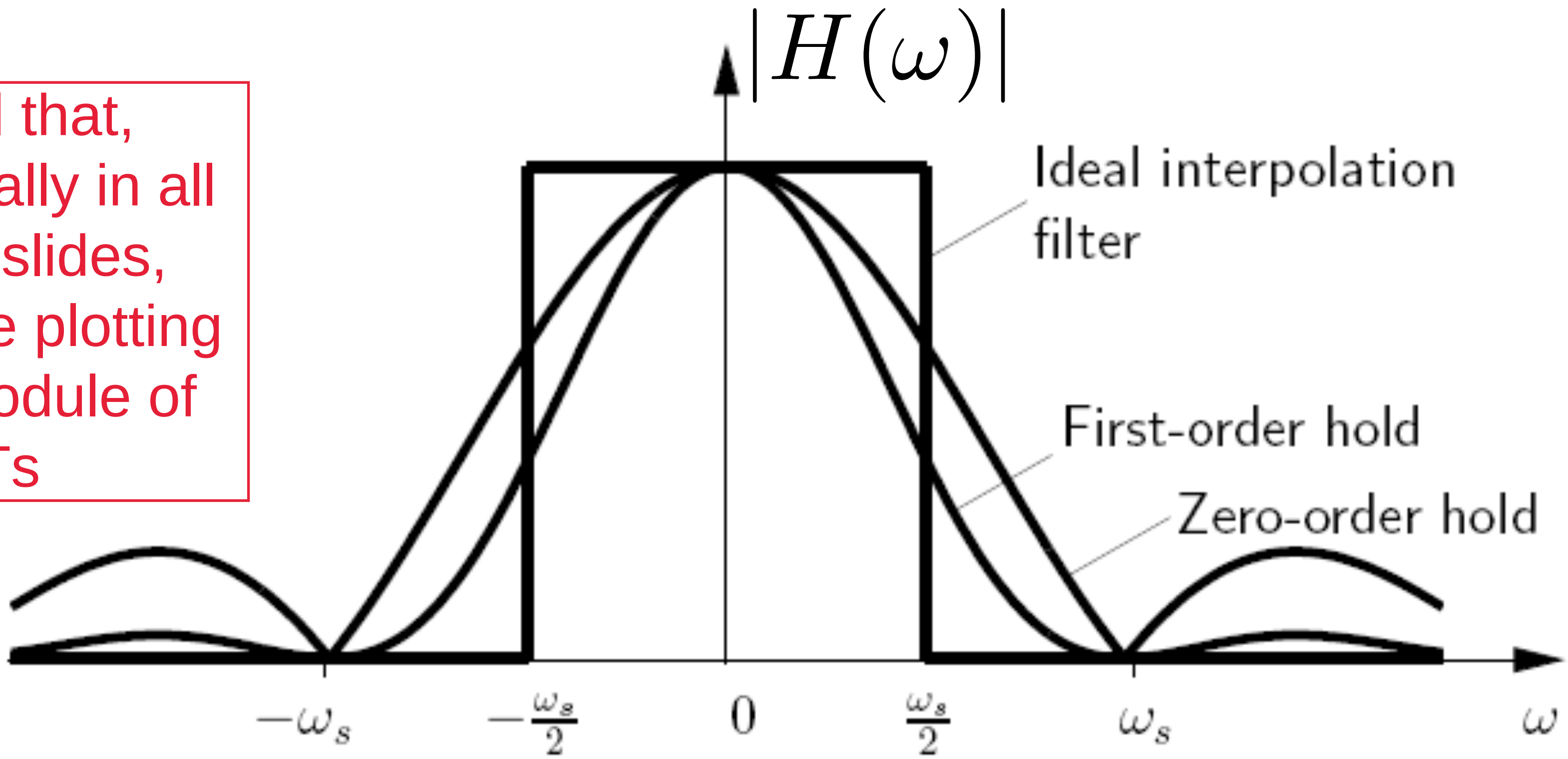
Again <u>NO perfect reconstruction</u>: some high-frequencies are included (but less than before with the zero-order)

Summary (time domain)



Summary (frequency domain)

Recall that, generally in all these slides, we are plotting the module of the FTs



Note that the increasing the order provides a better approximation of the ideal case