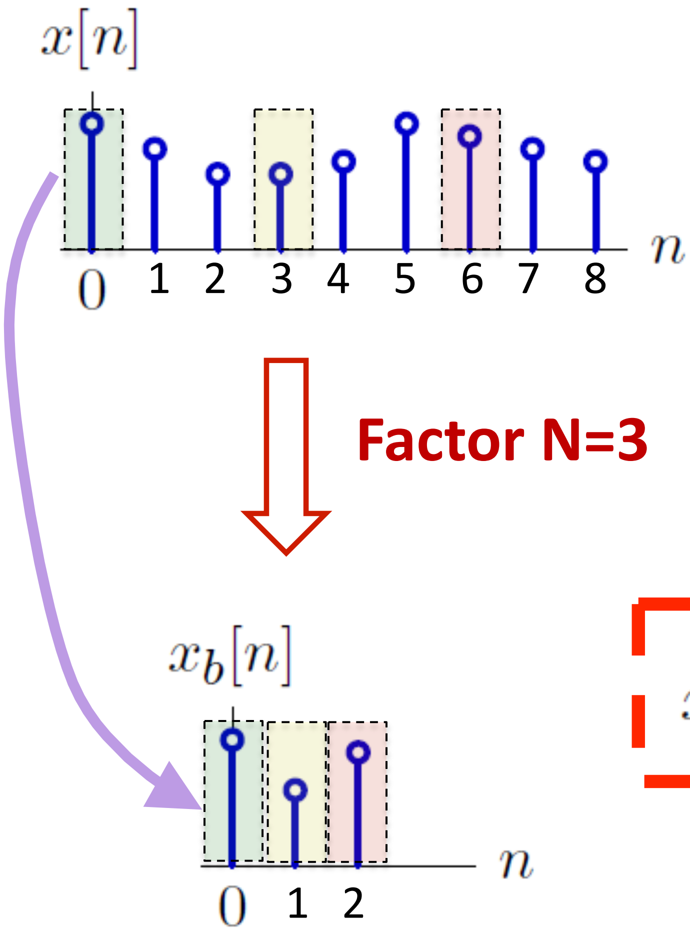
Sampling (in discrete time)

FOURTH PART

(down-sampling, “diezmado”;
and up-sampling, “interpolation in discrete time”)

Down-sampling ("Diezmado")

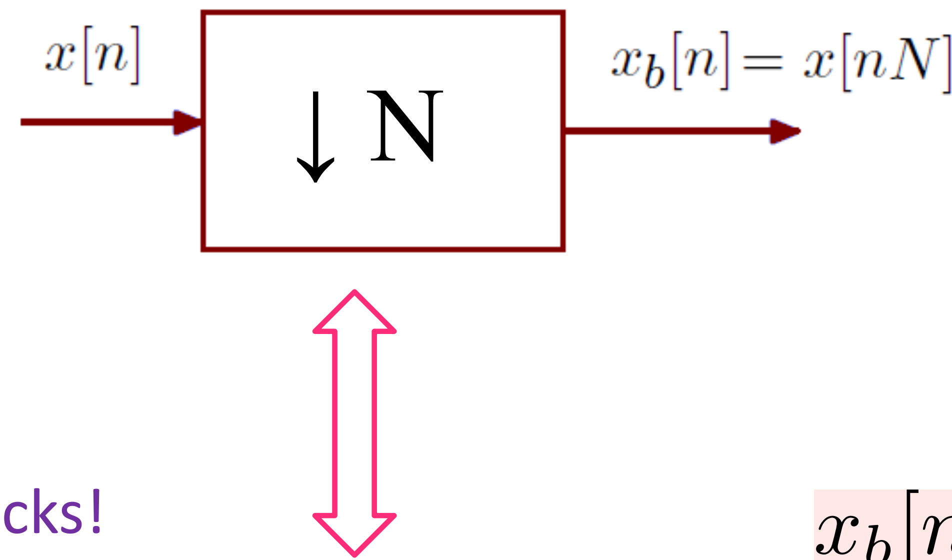


With one block: directly (see next slides)

$$x_b[n] = x[nN]$$

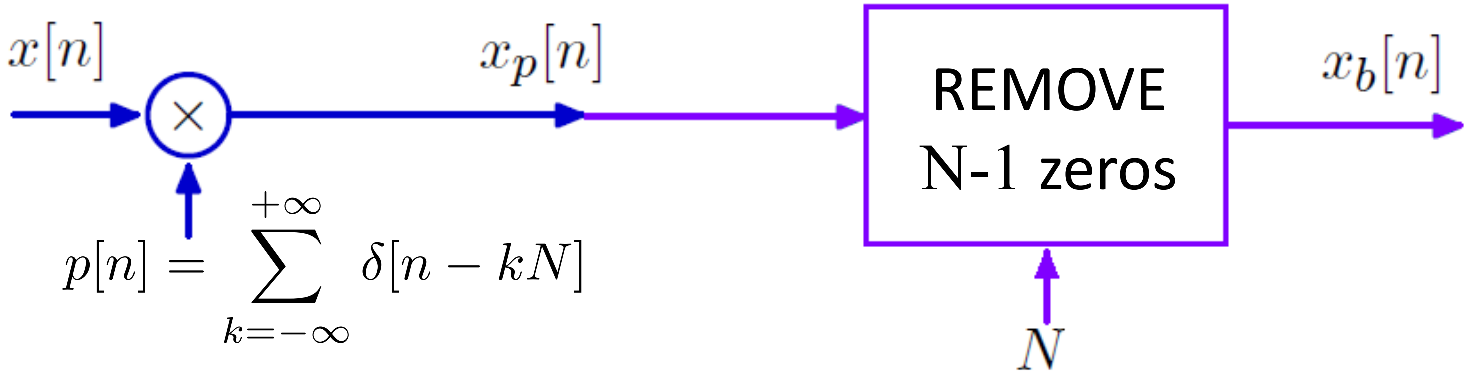
$x_b[-1] = x[-3]$
$x_b[0] = x[0]$
$x_b[1] = x[3]$
$x_b[2] = x[6]$
…

Down-sampling ("Diezmado")



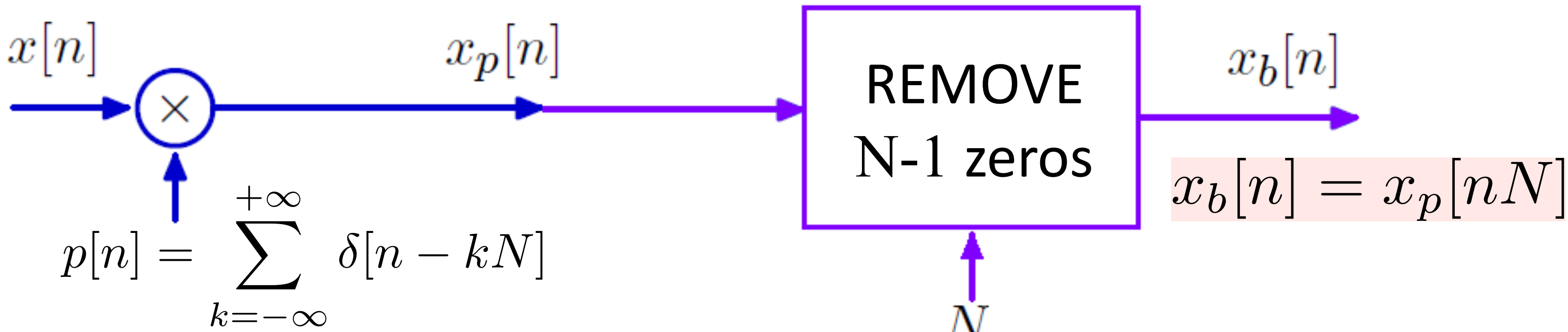
With two blocks!

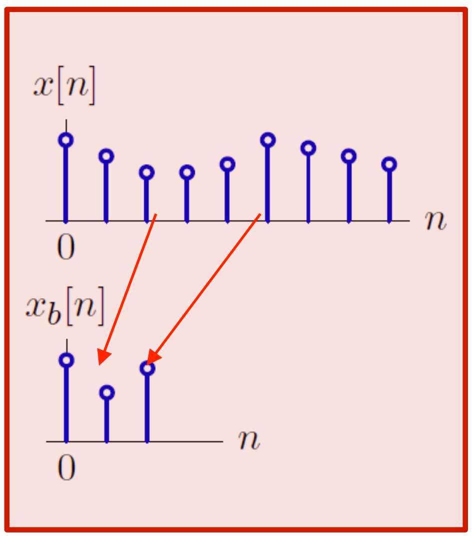
$$x_b[n] = x_p[nN]$$



Multiplication with an infinite train of delta: the results is that some values go to zero and the rest of samples keep the same values.

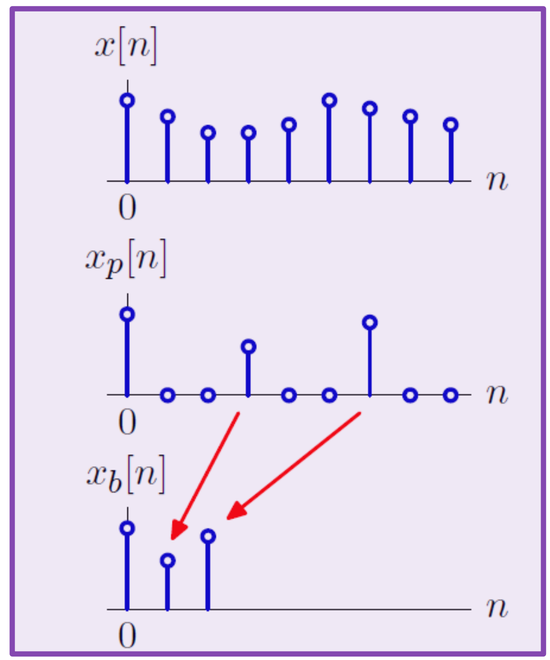
Down-sampling ("Diezmado")





With one block

THE SAME RESULTS !!



With two blocks

Down-sampling in frequency: first step

- Let us consider that:

(p[n] can be considered periodic of period N !)

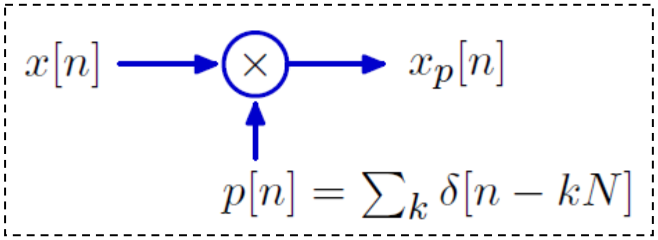
Generalized Fourier Transform

$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \Longleftrightarrow P_G(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N}\right)$$

$$x_p[n] = x[n]p[n] \Longleftrightarrow X_p(\Omega) = \frac{1}{2\pi} \int_0^{2\pi} X(\theta) P_G(\Omega - \theta) d\theta$$

$$X_P(\Omega) = \frac{1}{N} \sum_{k=-\infty}^{+\infty} X\left(\Omega - k\frac{2\pi}{N}\right)$$

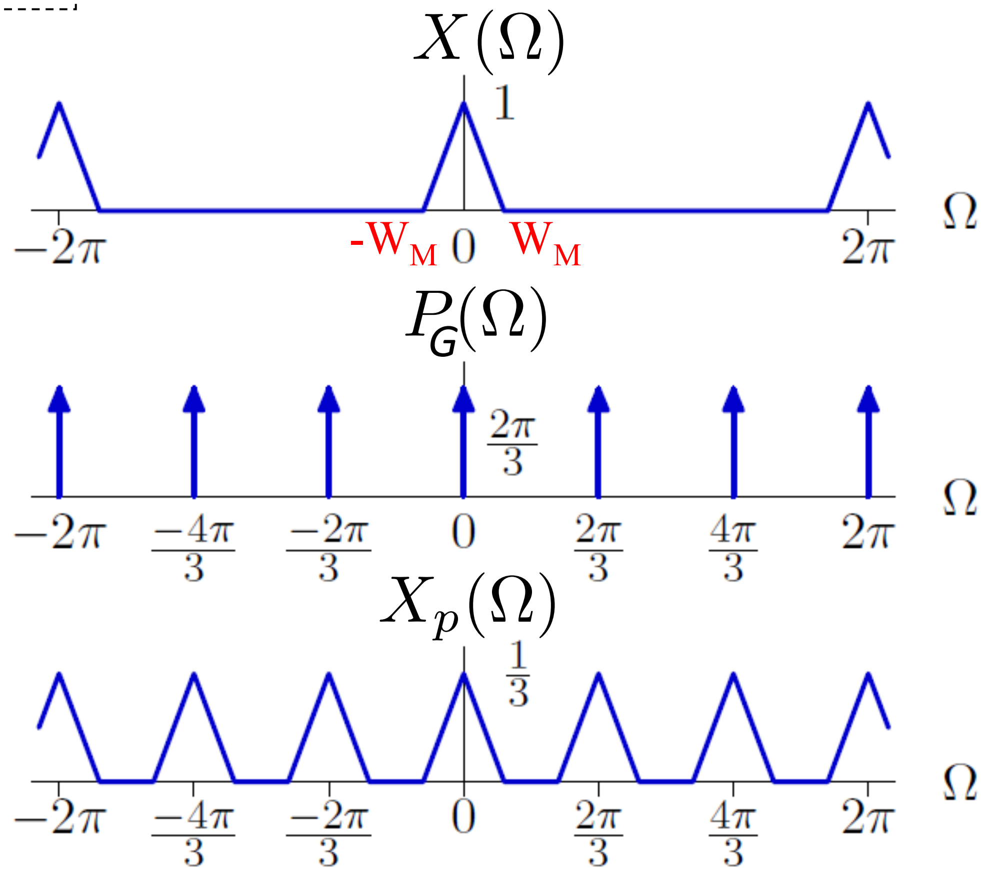
Down-sampling in frequency: first step



Recall that we are plotting the module of the FTs

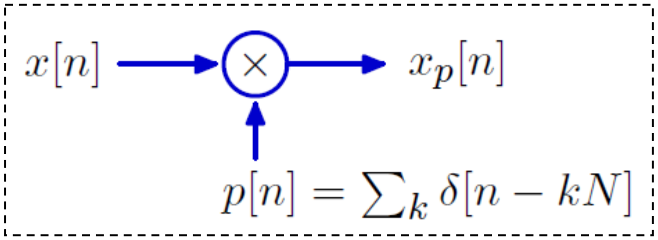
It is **already** periodic, since it is the FT of x[n] (discrete time)

$N = 3$



In this example, we are lucky: no overlapping !!!

first step: avoiding overlapping



No overlapping if:

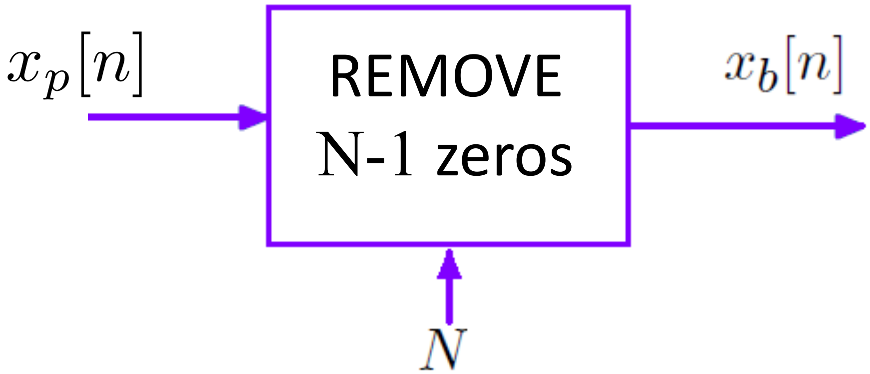
$$2W_M < \frac{2\pi}{N}$$

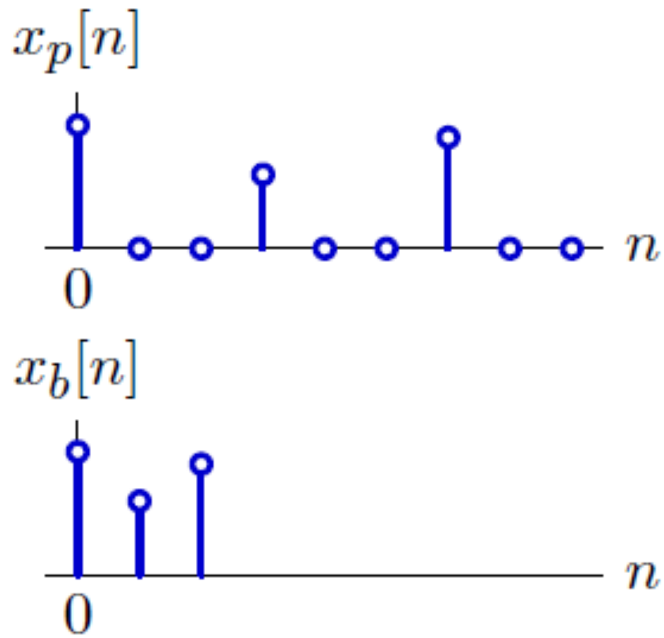
$$W_M < \frac{\pi}{N}$$

$$N < \frac{\pi}{W_M}$$

Equivalent to the Nyquist condition

Down-sampling: second step





$$x_b[n] = x_p[nN]$$

$$X_b(\Omega) = \sum_{n=-\infty}^{+\infty} x_b[n] e^{-j\Omega n}$$

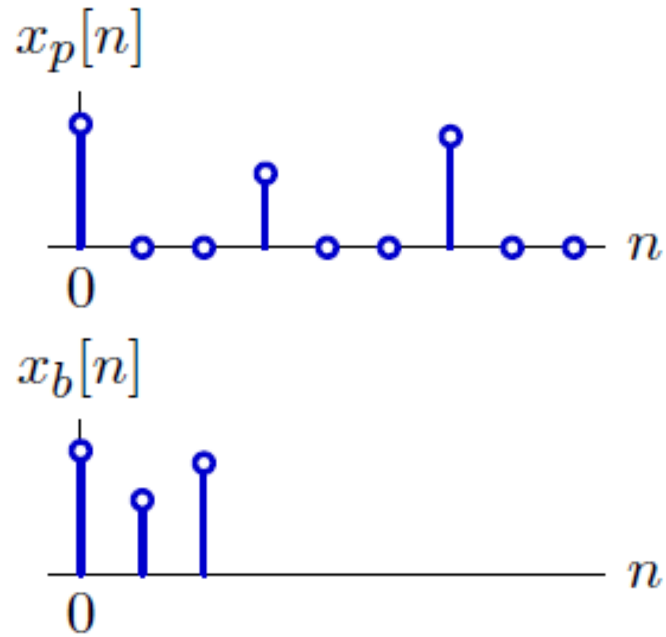
Change of variable n'=nN

$$= \sum_{n=-\infty}^{+\infty} x_p[nN] e^{-j\Omega n} = \sum_{n'=-\infty}^{+\infty} x_p[n'] e^{-j\frac{\Omega}{N}n'} = X_p\left(\frac{\Omega}{N}\right)$$

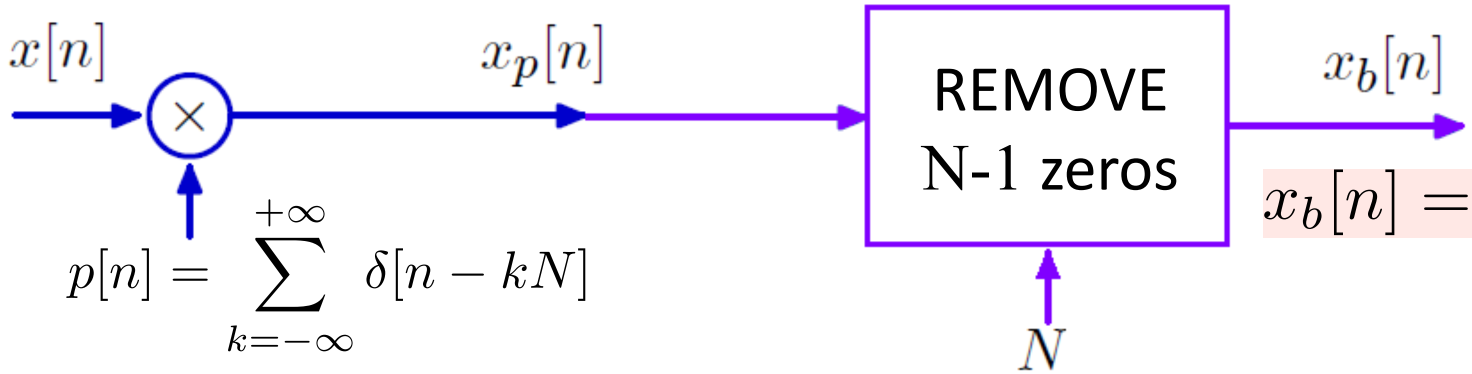
Down-sampling: second step

$$x_b[n] = x_p[nN]$$

$$X_b(\Omega) = X_p\left(\frac{\Omega}{N}\right)$$



Down-sampling: summary

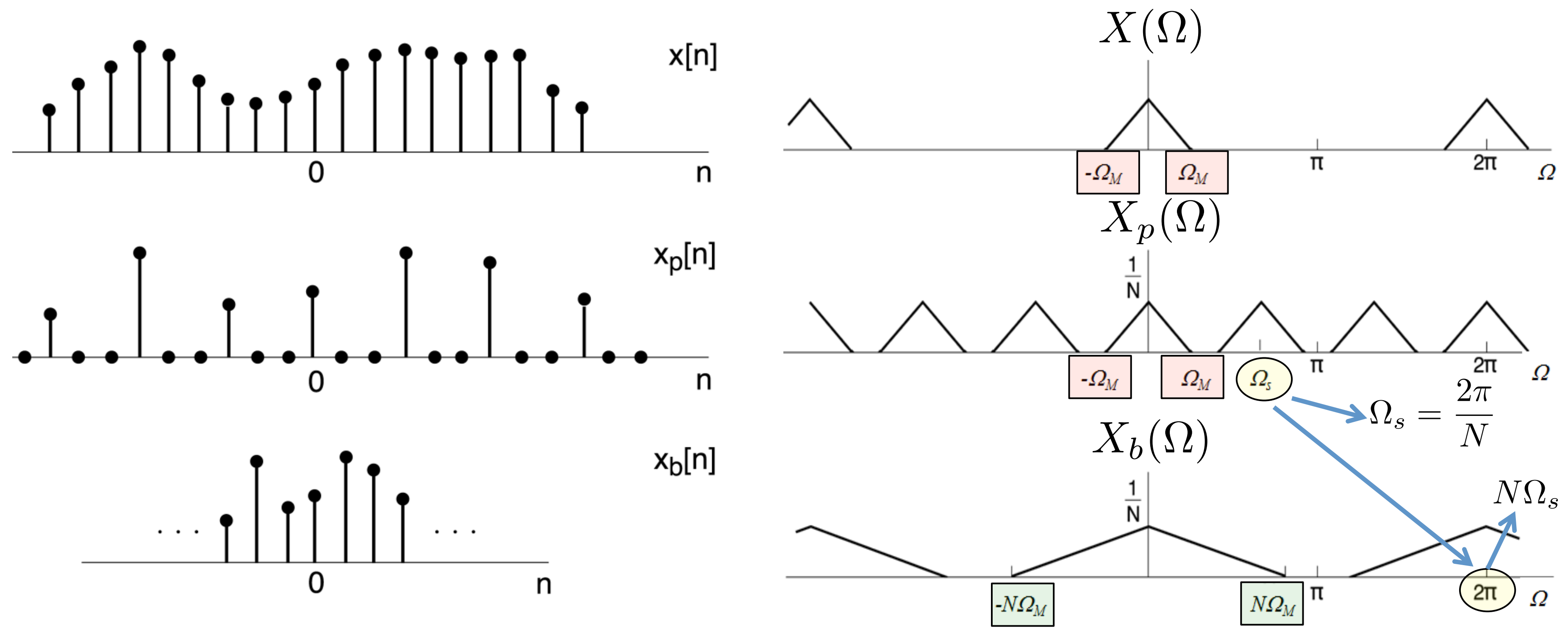


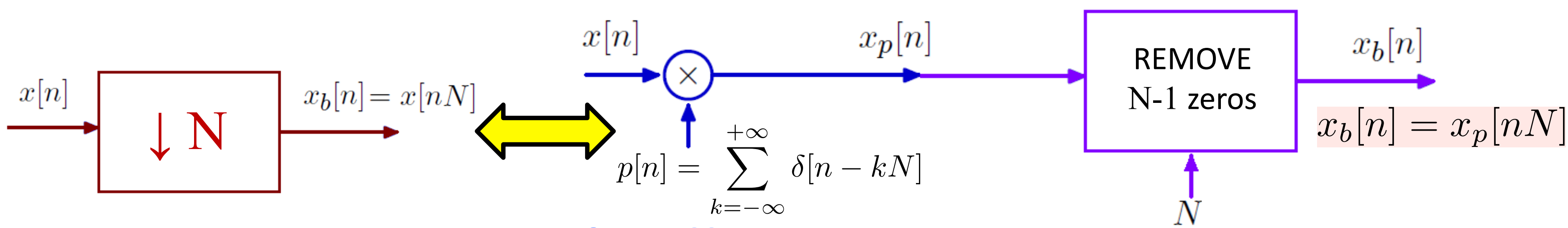
$$x_b[n] = x_p[nN]$$

$$X_b(\Omega) = X_p\left(\frac{\Omega}{N}\right) = \frac{1}{N}\sum_{k=-\infty}^{+\infty} X\left(\frac{\Omega}{N} - k\frac{2\pi}{N}\right)$$

$$X_b(\Omega) = \frac{1}{N}\sum_{k=-\infty}^{+\infty} X\left(\frac{\Omega - k2\pi}{N}\right)$$

Down-sampling: summary





Equivalent !!

One block= two blocks

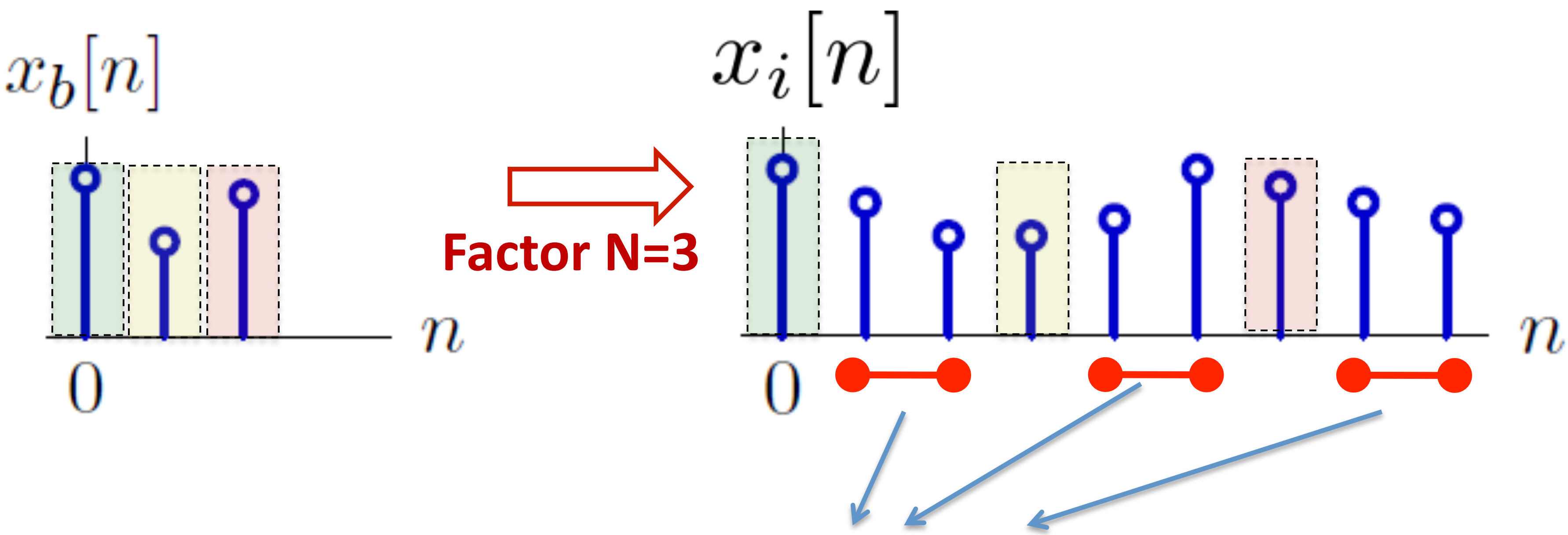
Recall that we are plotting the module of the FTs

Up-sampling: "adding samples"

Now, we consider the "opposite" problem, we come back.... We consider the problem of "adding samples" is a suitable/proper way.

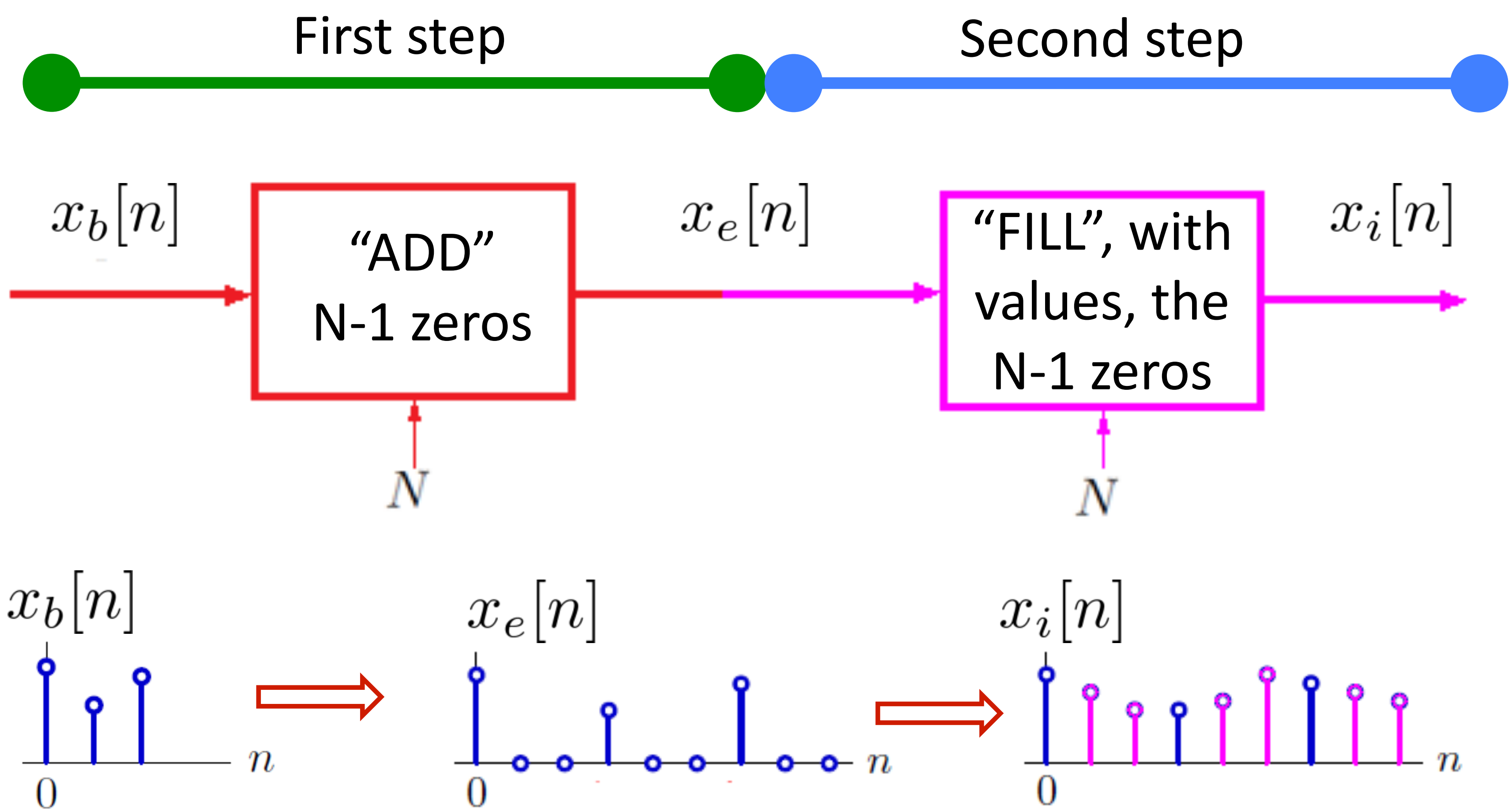
This is related to the interpolation problem:
Interpolating a signal in discrete time and obtaining other signal in discrete time (instead of obtaining a signal in continuous time as we have previously seen).

Up-sampling: try to recover the down-sampled signal



We "decide" these values…
Can we reconstruct/recover perfectly the previously down-sampled signal?

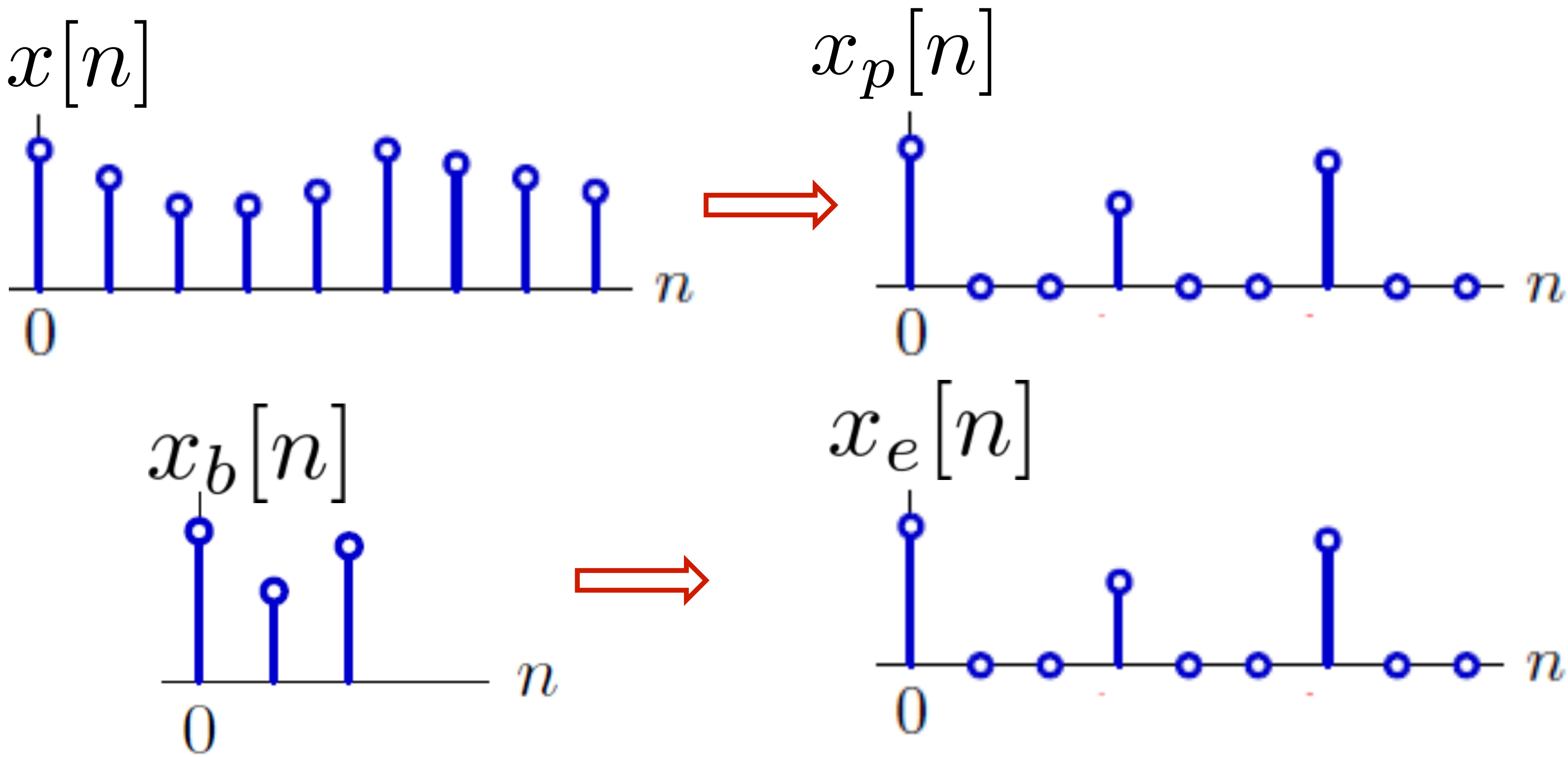
Up-sampling: two steps (again)



We would like: $x_i[n] = x[n]$

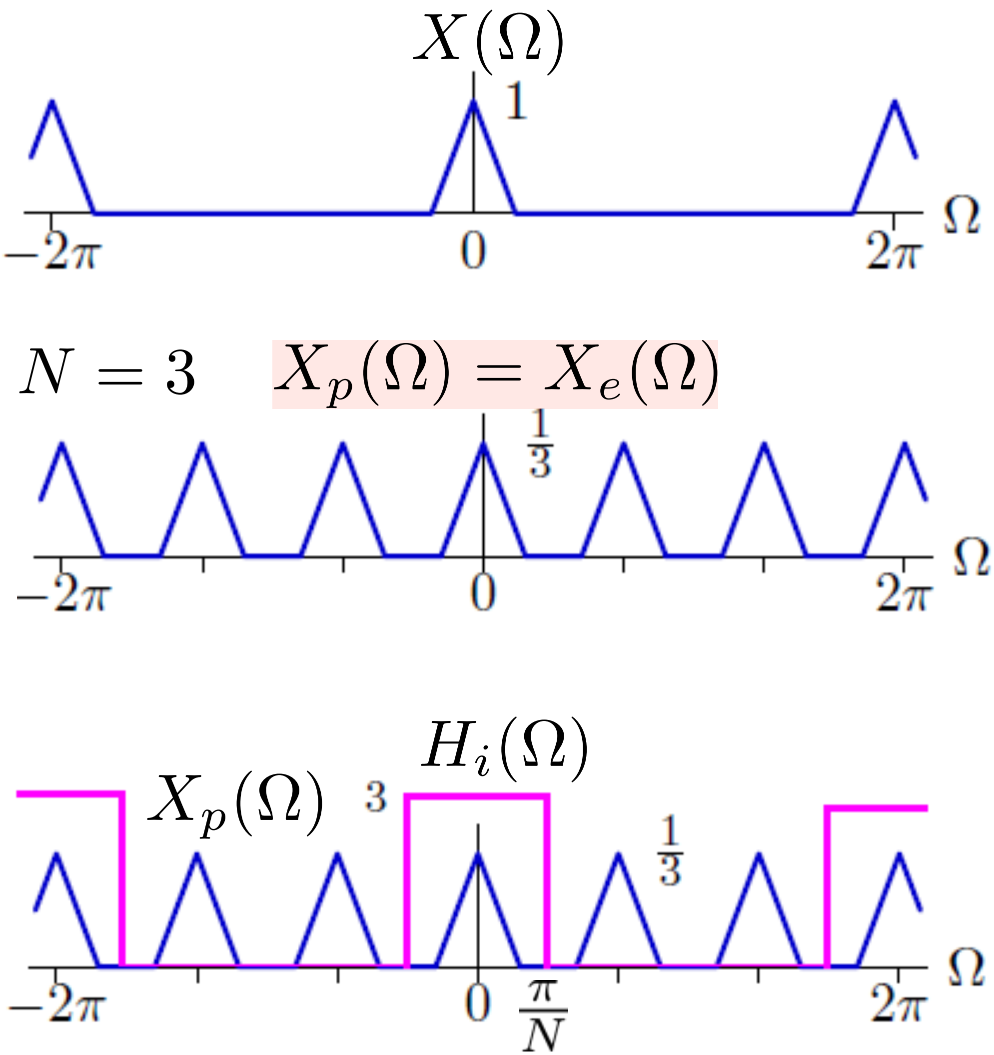
Up-sampling: first step

Note that x_p[n] is equal to x_e[n] !!!



$$x_e[n] = x_p[n]$$

Up-sampling: recover the down-sampled signal (second step)



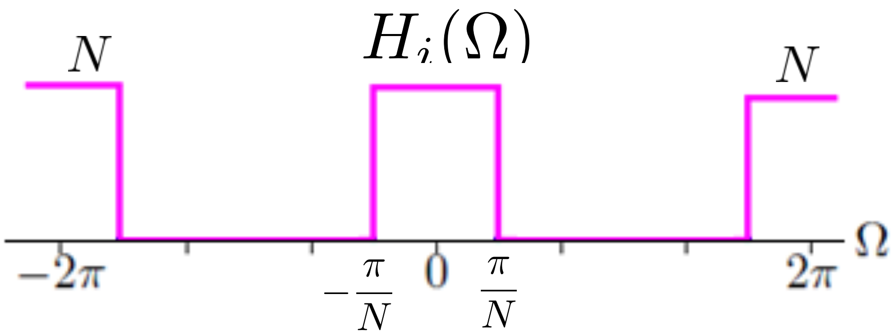
$$X_i(\Omega) = H_i(\Omega)X_p(\Omega)$$

In this example, we obtain:

$$X_i(\Omega) = X(\Omega)$$

Recall that we are plotting the module of the FTs

Ideal interpolator for up-sampling



$$H_i(\Omega) \begin{cases} N, & |\Omega| \leq \pi/N \\ 0, & \pi/N \leq |\Omega| \leq \pi \end{cases}$$

periodic 2π

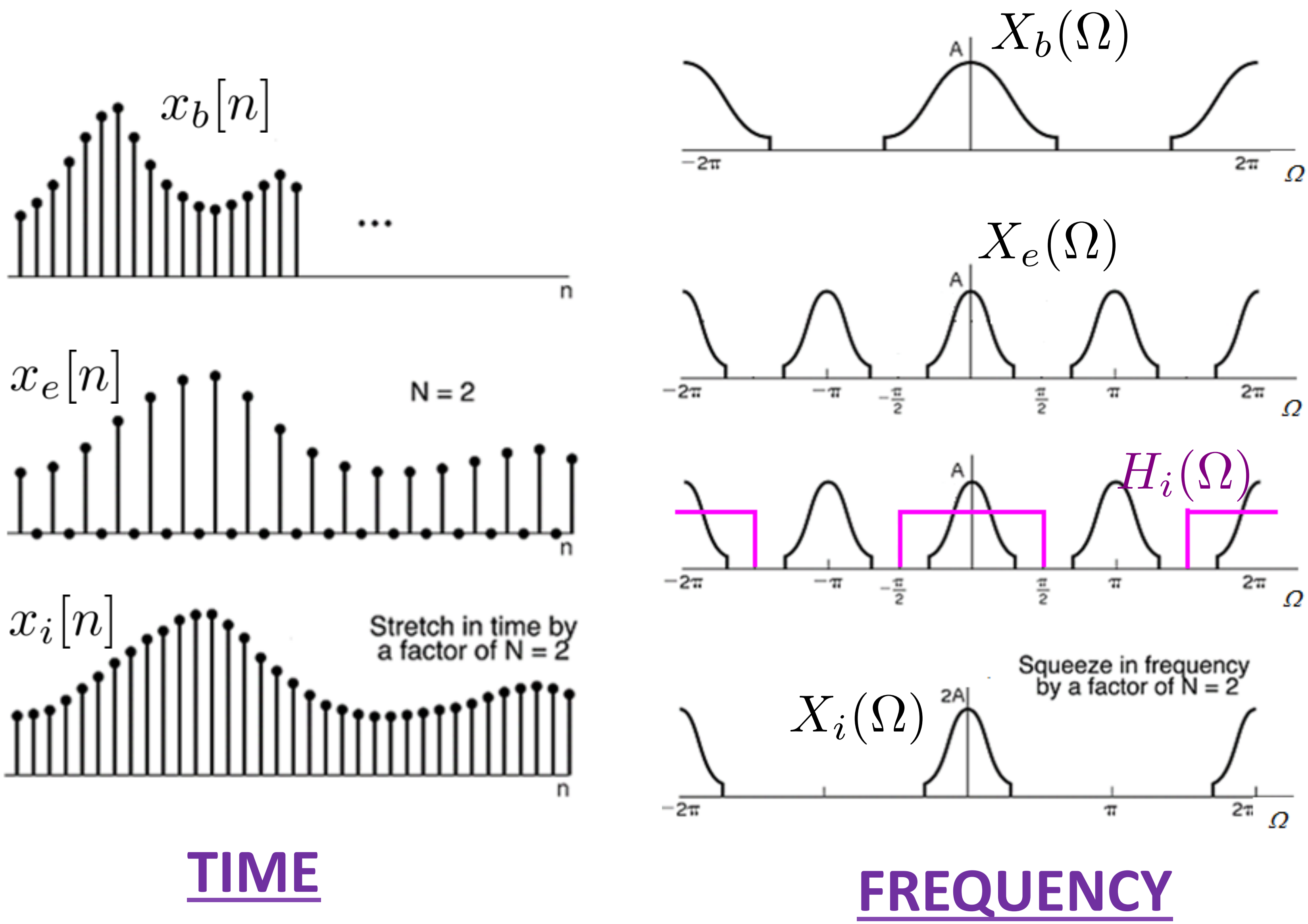
In time, it will a convolution with an "octopus" :

$$h_i[n] = \frac{1}{2\pi} \int_{-\pi/N}^{\pi/N} N \cdot e^{j\Omega n} d\Omega = \frac{N}{2\pi} \frac{2j sin(\pi n/N)}{jn} = \frac{sin(\pi n/N)}{\pi n/N} = sinc(n/N)$$

It is zero at each multiple of N (in this example, N=3)

Sinc function in discrete time ("octopus" in discrete time; "el pulpo"!!)

Ideal interpolator for up-sampling



Recall that we are plotting the module of the FTs