TOPIC 4
The practical computation of Fourier (e.g., with Matlab)
PART 1

FIRST of all,
RECALL something

RECALL

For a signal x[n] in discrete time:

PERIODICITY in the transformed domain !

$$X_k = a_k \Longrightarrow \boxed{a_k = a_{k+N}}$$

$$X(\Omega) \Longrightarrow \boxed{X(\Omega) = X(\Omega + 2\pi)}$$

In the next slides, we will recall some important concepts and formula, even with a simple example.

Module of Fourier series – discrete time

If x[n] is real: $a_k = a^*_{-k}$

As a consequence :

even !

$$P_k = |a_k| \rightarrow P_k = P_{-k}$$

$$\left.\begin{aligned} a_k &= a_{k+N} \\ a_k &= a^*_{-k} \end{aligned}\right\}$$

This generates more "symmetries" in P_k… but do not get confused (no important),

Module of Fourier transform – discrete time

If x[n] is real: $X(\Omega) = X^*(-\Omega)$

As a consequence : even !

$$P(\Omega) = |X(\Omega)| \rightarrow P(\Omega) = P(-\Omega)$$

$$X(\Omega) = X(\Omega + 2\pi)$$

$$X(\Omega) = X^*(-\Omega)$$

This generates more "symmetries" in P(Omega)… but do not get confused (no important).

Example (also problem of exam)

Also possible problem of the exams: $x[n] = \begin{cases} 1 & n = 0, \ldots, 4 \\ 0 & n \leq -1 \text{ o } n \geq 5 \end{cases}$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n} = \sum_{\boxed{n=0}}^{\boxed{4}} e^{-j\Omega n} = \frac{1 - e^{-j5\Omega}}{1 - e^{-j\Omega}} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$

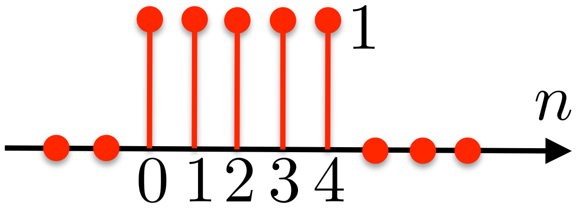
$$\frac{1 - e^{-j5\Omega}}{1 - e^{-j\Omega}} = \frac{e^{-j\frac{5}{2}\Omega}(e^{j\frac{5}{2}\Omega} - e^{-j\frac{5}{2}\Omega})}{e^{-j\frac{1}{2}\Omega}(e^{j\frac{1}{2}\Omega} - e^{-j\frac{1}{2}\Omega})}$$

$$= e^{j\frac{-5+1}{2}\Omega} \frac{e^{j\frac{5}{2}\Omega} - e^{-j\frac{5}{2}\Omega}}{e^{j\frac{1}{2}\Omega} - e^{-j\frac{1}{2}\Omega}} = e^{-j2\Omega} \frac{2j\sin(\frac{5}{2}\Omega)}{2j\sin(\frac{1}{2}\Omega)}$$

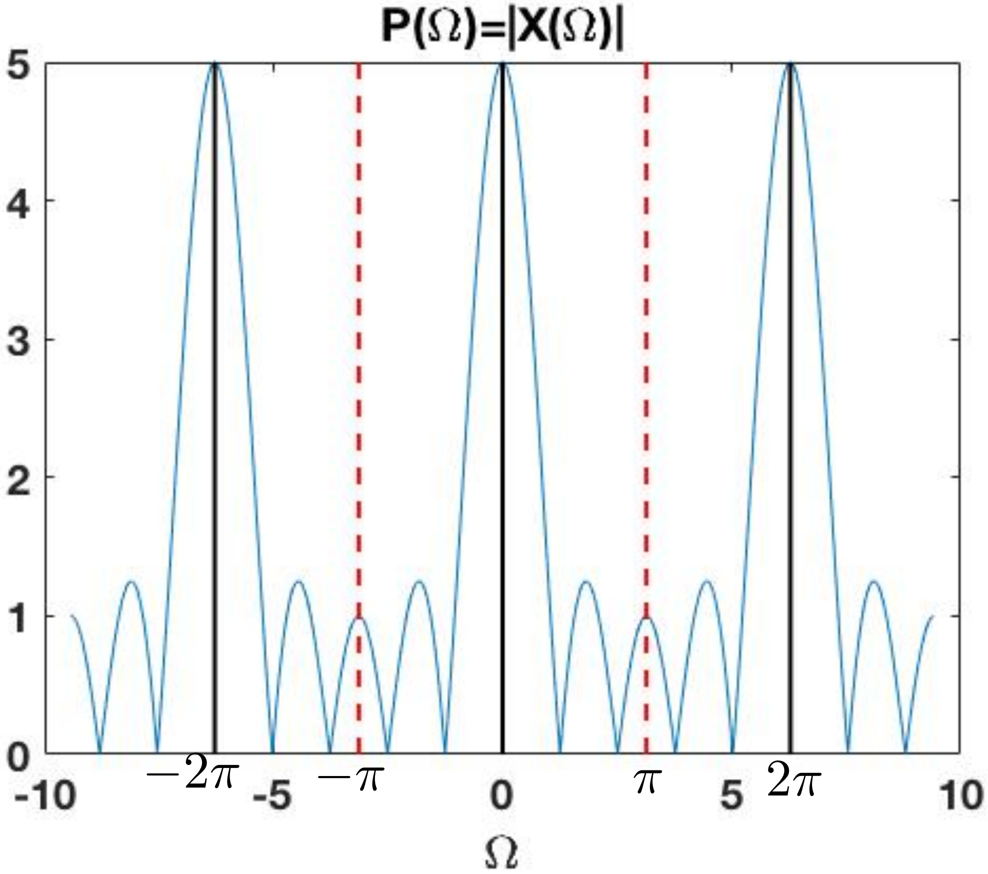
$$= e^{-j2\Omega} \frac{\sin(\frac{5}{2}\Omega)}{\sin(\frac{1}{2}\Omega)}$$

$$x[n] = \begin{cases} 1 & n = 0, ..., 4 \\ 0 & n \leq -1 \ \text{ o } \ n \geq 5 \end{cases}$$

⇒

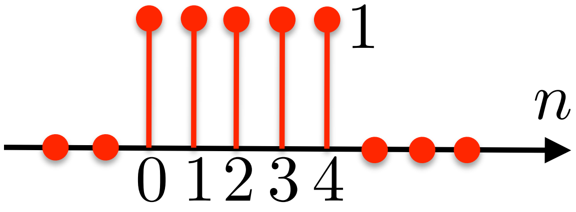


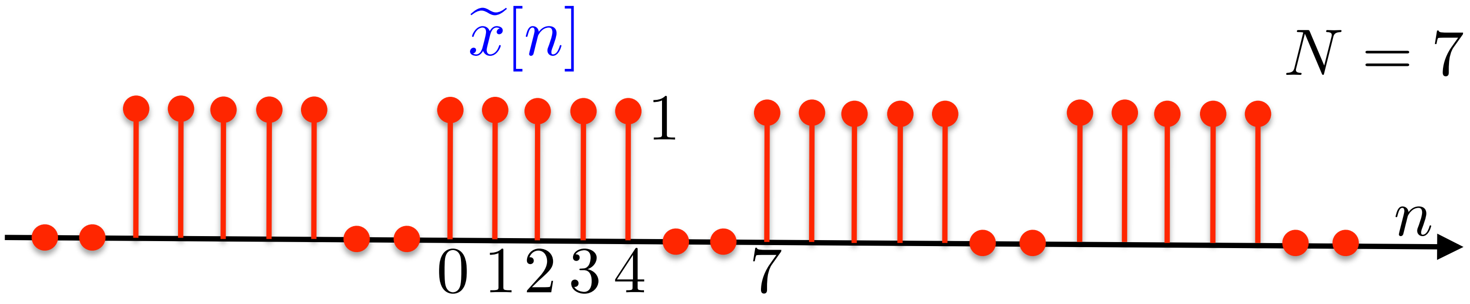
$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n} = \sum_{\boxed{n=0}}^{\boxed{4}} e^{-j\Omega n} = \frac{1 - e^{-j5\Omega}}{1 - e^{-j\Omega}} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$


P(Ω)=|X(Ω)|

Repeating periodically a finite-length signal

$$x[n] = \begin{cases} 1 & n = 0, \ldots, 4 \\ 0 & n \leq -1 \text{ o } n \geq 5 \end{cases}$$





$\widetilde{x}[n] = x[n]$ within a period

$$\widetilde{x}[n] = x[n] \text{ within a period}$$

it is periodic (with period N):
Fourier series !!

the coefficients can be easily computed as:

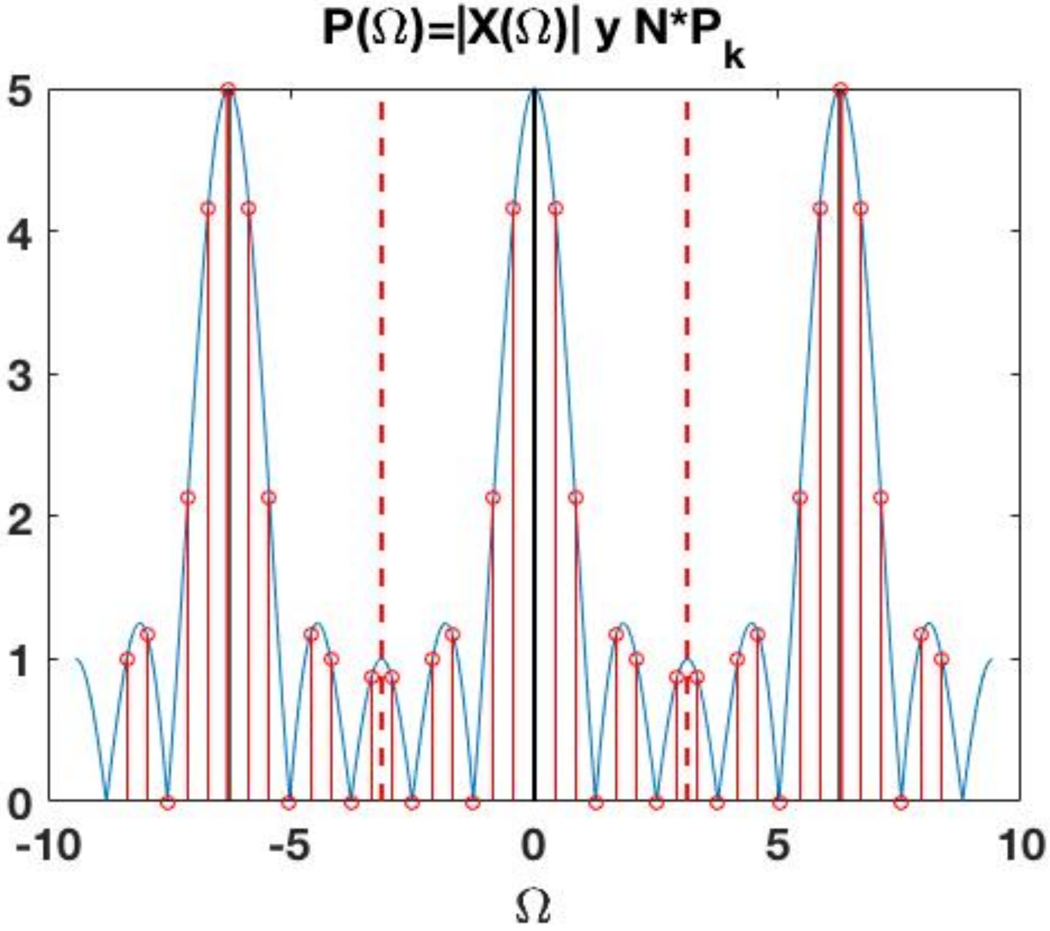
$$\boxed{X_k = a_k = \frac{1}{N} X(k\Omega_0) = \frac{1}{N} X\left(k\frac{2\pi}{N}\right)}$$

$$\boxed{\Omega_0 = \frac{2\pi}{N}}$$

$$X(\Omega) = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$

$$\boxed{Na_k = \frac{\sin(\frac{5}{2}k\Omega_0)}{\sin(\frac{1}{2}k\Omega_0)} e^{-j2k\Omega_0}}$$

$$\boxed{X_k = a_k = \frac{1}{N} X(k\Omega_0) = \frac{1}{N} X\left(k\frac{2\pi}{N}\right)}$$



Example with N=15

RECALL the formulas of the FOURIER SERIES

$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \rightarrow$ **Synthesis Equation**

$a_k = \boxed{\frac{1}{N}} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} \rightarrow$ **Analysis Equation**

Pay attention on this factor !!

- Recall that the sums can be done in any interval of consecutive values of length N: from 0 to N-1, from 1 to N, from 2 to N+1, form -1 to N-2 etc.

Now “real life”
(for instance, with Matlab)

“In Real Life”

- **Data/SIGNAL:** a vector (**finite sequence**) of values, with **finite length L.**

We have to define x_L[n] as:
....0,0,x_L[-1]=0, x_L[0]=x_1,
x_L[1]=x_2,...,x_L[L-1]=x_L,
x_L[0]=0,0,0,0....

Let us define the signal as
x_L[n];
we could compute (by
hand) X_L(Omega) !!!!

$$x_1, x_2, \ldots, x_L$$

$$x[0], x[1], x[2], \ldots, x[L-1]$$

Better notation x_L[n] !!!

Interpretation as a *periodic signal*

The easiest (computationally speaking) is to interpret the sequence of values as one period of a ***periodic signal*** of period N>=L

$$\boxed{N \geq L}$$

(we will consider other possibilities in other slides)

- **We said N>=L since we can always fill with N-L zeros (adding more zeros) our sequence of values:**

$$\overbrace{\underbrace{x_1, x_2, \ldots, x_L}_{L}, \underbrace{0, 0, \ldots, 0}_{N-L}}^{N}$$

DISCRETE FOURIER TRANSFORM (DFT)

THIS COMPUTATIONAL PROCEDURE is called "Discrete Fourier Transform" (DFT): mathematically ALMOST coincides with FOURIER SERIES of a *periodic signal* defined in a discrete domain.

Analysis Equation of DFT

$$\boxed{X_N[k]} = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, ..., N-1$$

Synthesis equation of DFT

$$x[n] = \boxed{\frac{1}{N}} \sum_{k=0}^{N-1} X_N[k] \cdot e^{+j\frac{2\pi}{N}nk}, \qquad n = 0, ..., N-1$$

DISCRETE FOURIER TRANSFORM (DFT)

Analysis Equation of DFT

$$X_N[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, ..., N-1$$

- Output of DFT: a sequence of N values

- They play the role of a_k in the Fourier Series

- Its definition is not unique: it depends on the choice of N !!

DFT versus Fourier series

Analysis Equation of DFT

$$X_N[k] \quad = \quad \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, ..., N-1$$

Synthesis equation of DFT

$$x[n] \quad = \quad \frac{1}{N}\sum_{k=0}^{N-1} X_N[k] \cdot e^{+j\frac{2\pi}{N}nk}, \qquad n = 0, ..., N-1$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \rightarrow$$ **Synthesis Equation**

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\Omega_0 n} \rightarrow$$ **Analysis Equation**

- **Only the position of the factor 1/N, is different !!**

Corresponding frequencies

Exactly as in the Fourier Series:

$X_N[k]$

a_k

$$\boxed{\Omega_k = k\Omega_0 = k\frac{2\pi}{N}}$$

As the Fourier Series and the corresponding coefficients a_k, we could be evaluate X_N[k] with k<0 and/or k>N, and we could assume:

$$X_N[k] = X_N[k+N]$$

Example of working with DFT

Example: evaluating DFT at 0,

$$X_N[0] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}0n} = \sum_{n=0}^{N-1} x[n] \cdot e^{-j0} = \sum_{n=0}^{N-1} x[n]$$

Example: evaluating DFT with N=4 at 0, 1,

$$X_4[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3]$$

$$=$$

$$\begin{aligned} X_4[1] &= \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}1n} = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\frac{\pi}{2}})^n \sum_{n=0}^{3} x[n] \cdot (-j)^n \\ &= x[0](-j)^0 + x[1](-j)^1 + x[2](-j)^2 + x[3](-j)^3 = x[0] - x[1]j - x[2] + x[3]j \end{aligned}$$

Example of working with DFT

Example: evaluating DFT with N=4 at 0, 1,2,3

$$X_4[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3]$$

$$\begin{aligned} X_4[1] &= \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}1n} = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\frac{\pi}{2}})^n \sum_{n=0}^{3} x[n] \cdot (-j)^n \\ &= x[0](-j)^0 + x[1](-j)^1 + x[2](-j)^2 + x[3](-j)^3 = x[0] - x[1]j - x[2] + x[3]j \end{aligned}$$

$$X_4[2] \quad = \quad \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}2n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\pi})^n = x[0] + x[1](-1) + x[2](-1)^2 + x[3](-1)^3$$

$$X_4[3] \quad = \quad \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{2\pi}{4}3n} = \sum_{n=0}^{3} x[n] \cdot (e^{-j\frac{3\pi}{2}})^n = x[0] + x[1]j + x[2]j^2 + x[3]j^3$$

The same as with the Fourier Series (different only for a factor 1/N)

Example of working with DFT

We arrive to:

$$
\begin{aligned}
X_4[0] &= x[0] + x[1] + x[2] + x[3] \\
X_4[1] &= x[0] - jx[1] - x[2] + x[3] \\
X_4[2] &= x[0] - x[1] - x[2] - x[3] \\
X_4[3] &= x[0] + jx[1] - x[2] - jx[3]
\end{aligned}
$$

We can write it as a linear system !

The linear System for DFT

With the previous case (N=4):

$$\begin{bmatrix} X_4[0] \\ X_4[1] \\ X_4[2] \\ X_4[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$\mathbf{F}$

$\mathbf{x}$

$\mathbf{X}_4$

F is a **Vandermonde matrix** ! Each row is "geometric progression" (see next slide)

$$\boxed{\mathbf{X}_4 = \mathbf{F}\mathbf{x}}$$

The linear System for DFT

Generic N:

$$\mathbf{F} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{2\pi}{N}2} & \cdots & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}_{N\times N}$$

$$\boxed{\mathbf{X}_N = \mathbf{F}\mathbf{x}}$$

COMPUTING DFT: easy!

SIGNAL

DFT

$$\boxed{\mathbf{X}_N = \mathbf{F}\mathbf{x}}$$

EASY: JUST MULTIPLICATIONS and SUMS !

$$\mathbf{X}_N = [X_N[0], X_N[1], ..., X_N[N-1]]^\top$$

Computing DFT analitically

In simple artificial cases, we can compute the DFT analytically: (clearly, we always consider finite-length signals, as in the "real world" case)

Example 1

$$x[n] = 3\delta[n-2]$$

$$X_N[k] \quad = \quad \sum_{n=0}^{N-1} 3\delta[n-2] \cdot e^{-j\frac{2\pi}{N}kn} = 3\sum_{n=0}^{N-1} \delta[n-2] \cdot e^{-j\frac{2\pi}{N}k2} = 3e^{-j\frac{4\pi}{N}k}$$

Example 2

$$x[n] = (-1)^n, \qquad n = 0, ..., N-1$$

$$\begin{aligned} X_N[k] \quad &= \quad \sum_{n=0}^{N-1}(-1)^n \cdot e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}(e^{-j\pi})^n \cdot (e^{-j\frac{2\pi}{N}k})^n = \sum_{n=0}^{N-1}(e^{-j(\frac{2\pi}{N}k+\pi)})^n \\ &= \quad \frac{1-(e^{-j(\frac{2\pi}{N}k+\pi)})^N}{1-(e^{-j(\frac{2\pi}{N}k+\pi)})} = \frac{1-e^{-jN\pi}}{1+e^{-j\frac{2\pi}{N}k}} \end{aligned}$$

VERY IMPORTANT EXAMPLE

Example 3

$$x[n] = \begin{cases} 1 & \text{if } n = 0, ..., L-1 \\ 0 & \text{if } n = L, ..., N \end{cases}$$

This signal has length N but N-L samples are zeros (the effective length is L).

However, we would like to consider it of length N and we compute X_N[k]:

here is N-1

$$X_N[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} = \sum_{n=0}^{L-1} e^{-jk\frac{2\pi}{N}n} = \frac{1-e^{-jk\frac{2\pi}{N}L}}{1-e^{-jk\frac{2\pi}{N}}} = \frac{e^{-jk\frac{\pi}{N}L}\left(e^{jk\frac{\pi}{N}L} - e^{-jk\frac{\pi}{N}L}\right)}{e^{-jk\frac{\pi}{N}}\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}\right)}$$

$$\boxed{X_N[k] = e^{-j(L-1)\frac{\pi}{N}k} \frac{\sin(\frac{L\pi}{N}k)}{\sin(\frac{\pi}{N}k)}}$$

We will use this formula later.

DFT versus Fourier Transform

We have already compared DFT and the Fourier Series (a_k).
Now we compare with the Fourier Transform of x[n].

$$X_N[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, \ldots, N-1$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n}, \qquad -\infty < \Omega < \infty$$

But, in the real world, we have a finite-length sequence ➔➔

DFT versus Fourier Transform

…considering that the signal as a periodic signal (and we know one period), then we can do as a “Fourier Series”:

$$X_N[k] \quad = \quad \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, ..., N-1$$

…considering that the signal is an infinite signal with non-zero values only the values that we have, then we can do FT:

$$X(\Omega) = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n} \qquad -\infty < \Omega < \infty$$

… we have a finite-length sequence

DFT and Fourier Transform

THEN, WE CAN WRITE:

$$X_N[k] = X(\Omega)\Big|_{k\frac{2\pi}{N}} = X\left(k\frac{2\pi}{N}\right)$$

$$X_N[k] = X(\Omega)\Big|_{k\Omega_0} = X\left(k\Omega_0\right)$$

$$\Omega_0 = \frac{2\pi}{N}$$

Recall that:

$$a_k = \frac{1}{N}X(\Omega)\Big|_{k\Omega_0} = \frac{1}{N}X\left(k\Omega_0\right)$$

It is exactly the same but with the factor 1/N…

DFT and Fourier Transform

$$X_N[k] = X(\Omega)\Big|_{k\frac{2\pi}{N}} = X\left(k\frac{2\pi}{N}\right)$$

$$X_N[k] = X(\Omega)\Big|_{k\Omega_0} = X\left(k\Omega_0\right)$$

$$\Omega_0 = \frac{2\pi}{N}$$

THEN, WE ARE "SAMPLING" the FOURIER TRANSFORM in frequency domain.

Come back to the examples:

$$x[n] = \begin{cases} 1 & n = 0, \ldots, 4 \\ 0 & n \leq -1 \text{ o } n \geq 5 \end{cases}$$

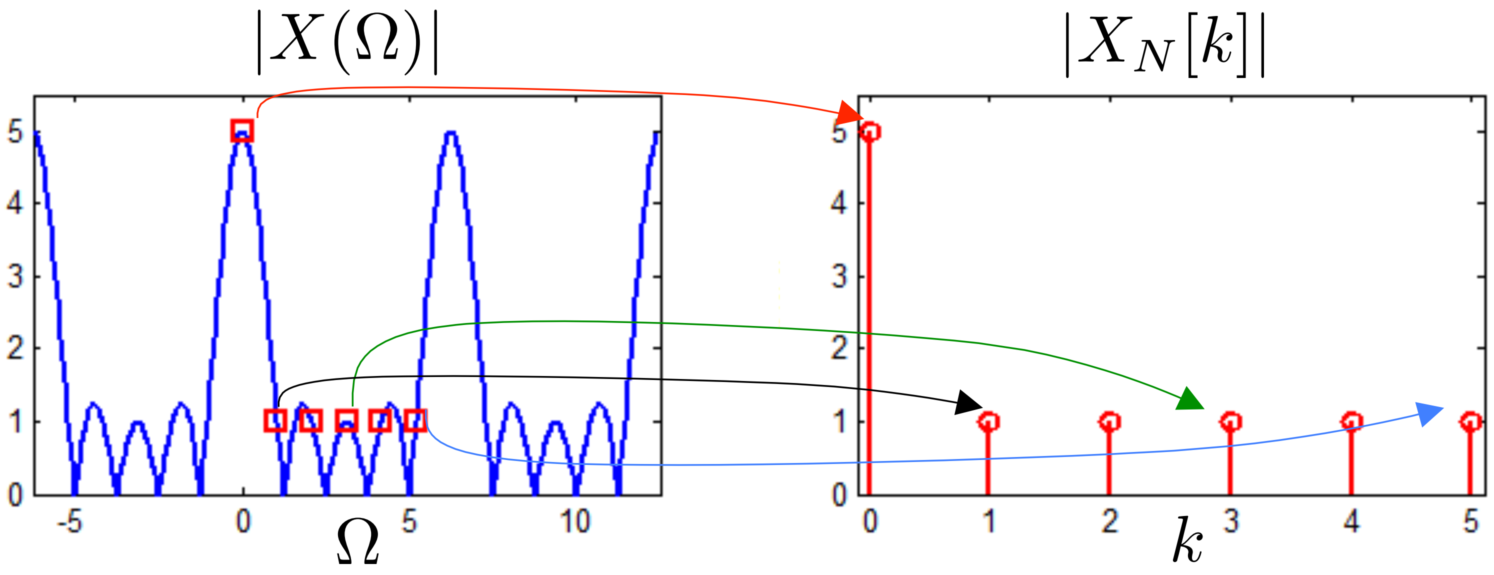
We have L=5 BUT we assume:

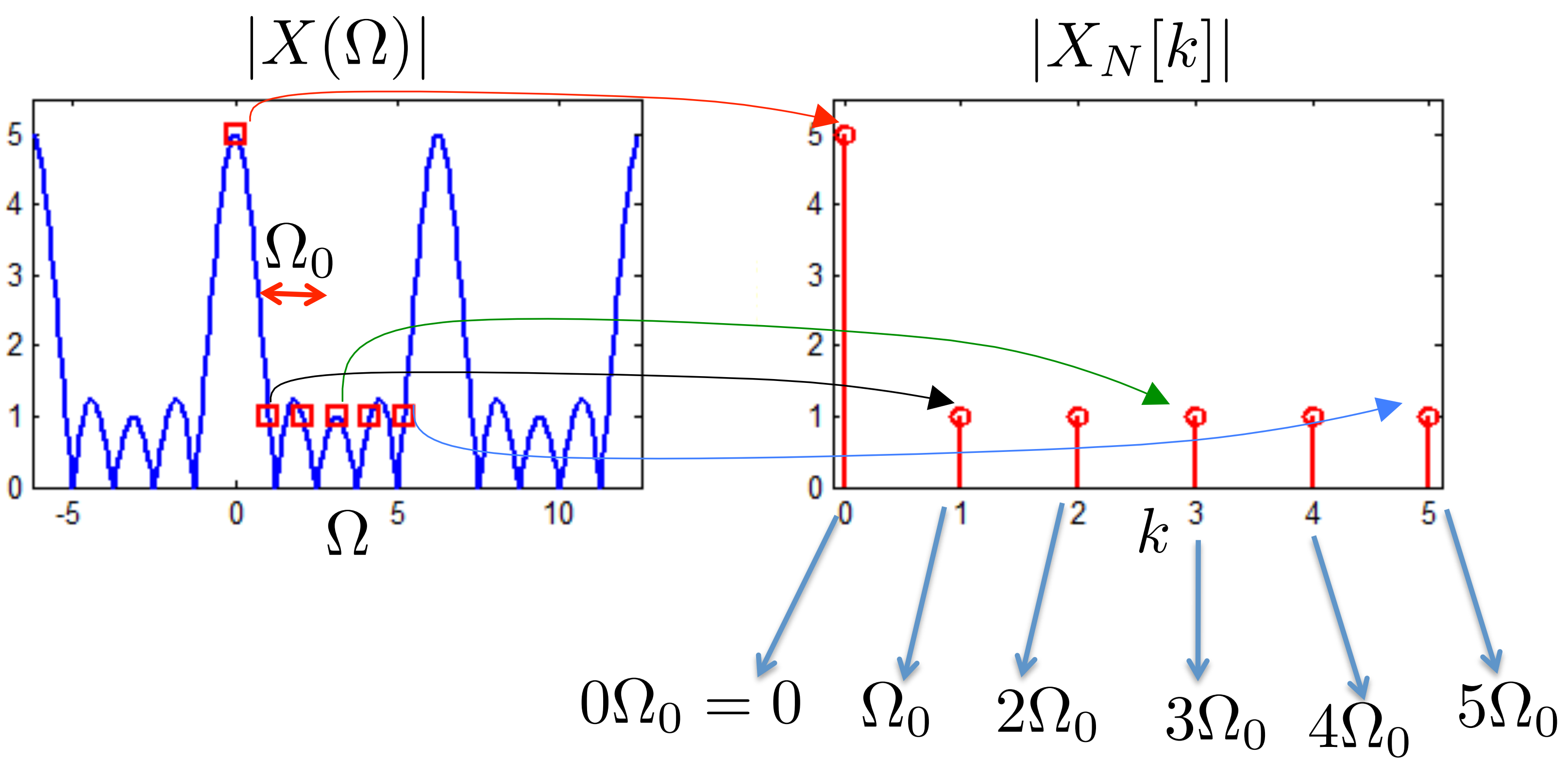
$$N = 6$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=0}^{4} e^{-j\Omega n} = \frac{1-e^{-j5\Omega}}{1-e^{-j\Omega}} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$

$$N-1 \leftarrow$$

$$X_6[k] = \sum_{n=0}^{5} x[n] \cdot e^{-j\frac{2\pi}{6}kn} = \frac{\sin\left(\frac{5\pi}{6}k\right)}{\sin\left(\frac{\pi}{6}k\right)} e^{-j\frac{4\pi}{6}k}, \qquad k = 0, \ldots, 5$$





$$k\Omega_0 = k\frac{2\pi}{N} = k\frac{2\pi}{6} = k\frac{\pi}{3}$$

$X_N[k]$

a_k

$$\boxed{\Omega_k = k\Omega_0 = k\frac{2\pi}{N}}$$

Again the example:

$$x[n] = \begin{cases} 1 & n = 0, ..., 4 \\ 0 & n \leq -1 \text{ o } n \geq 5 \end{cases}$$

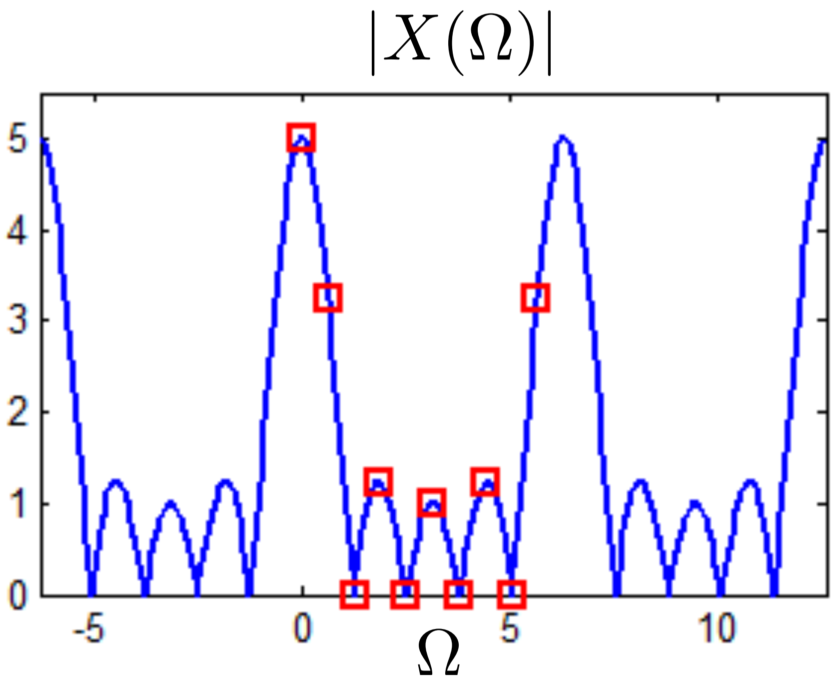
We have L=5 BUT NOW we assume:

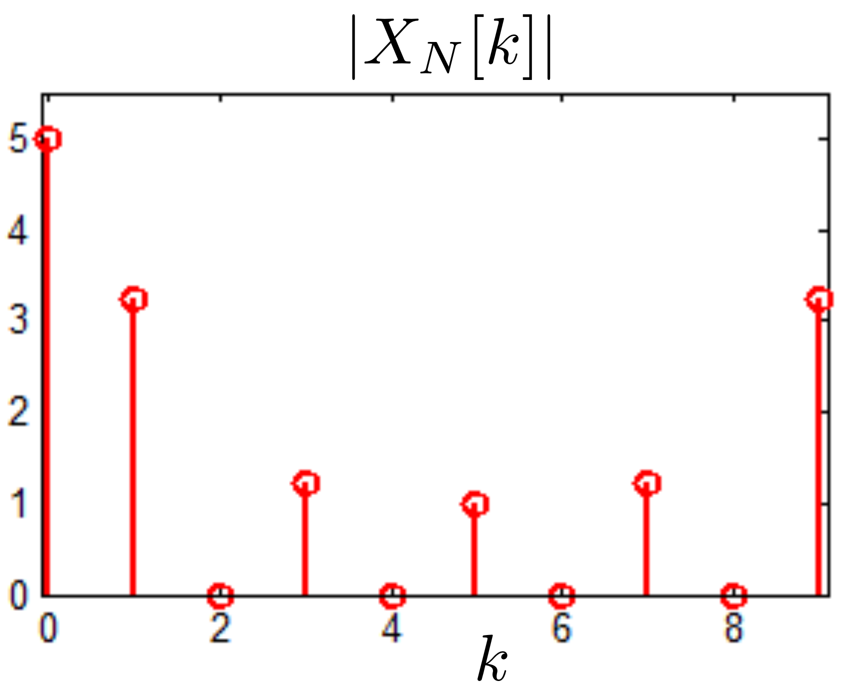
$$N = 10$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=0}^{4} e^{-j\Omega n} = \frac{1 - e^{-j5\Omega}}{1 - e^{-j\Omega}} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$

$$N - 1$$

$$X_{10}[k] = \sum_{n=0}^{9} x[n] \cdot e^{-j\frac{2\pi}{10}kn} = \frac{\sin\left(\frac{5\pi}{10}k\right)}{\sin\left(\frac{\pi}{10}k\right)} e^{-j\frac{4\pi}{10}k}, \qquad k = 0, ..., 9$$





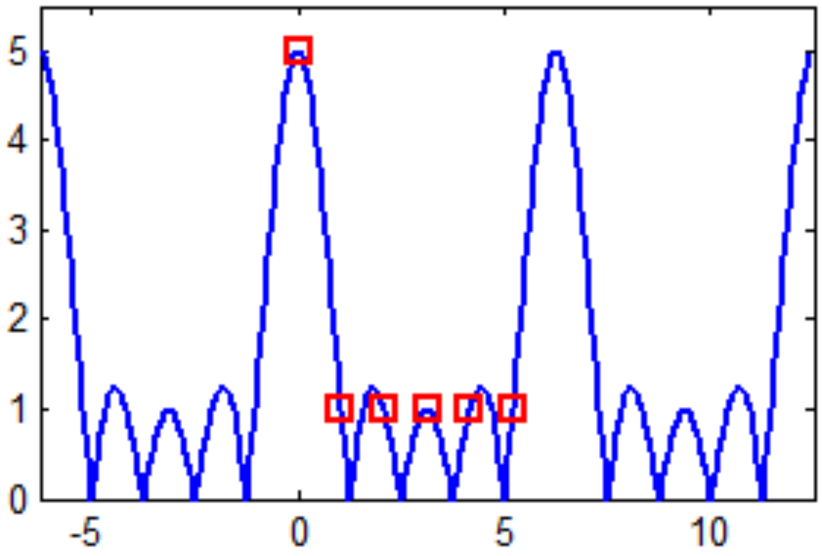
$$\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{5}$$

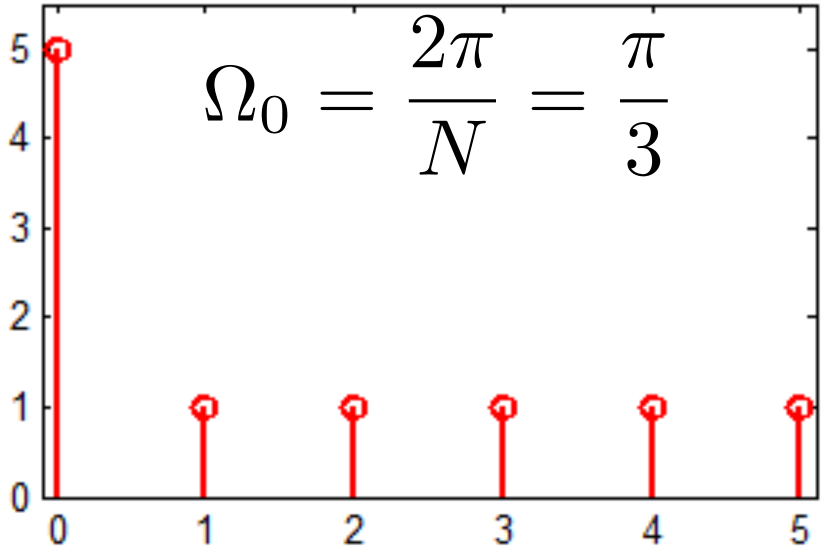
HERE Omega_0 is smaller since N is bigger !!!

Then, we get more points from the FT !!

COMPARE:

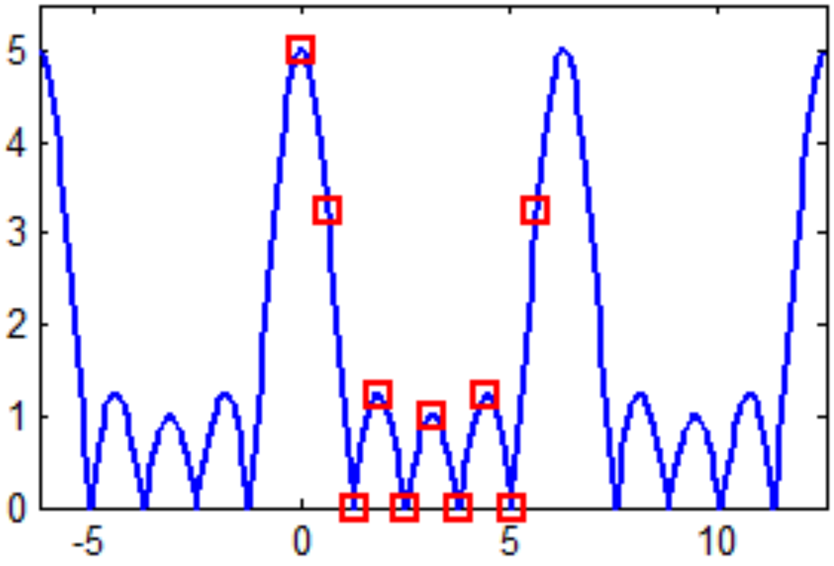
$N = 6$

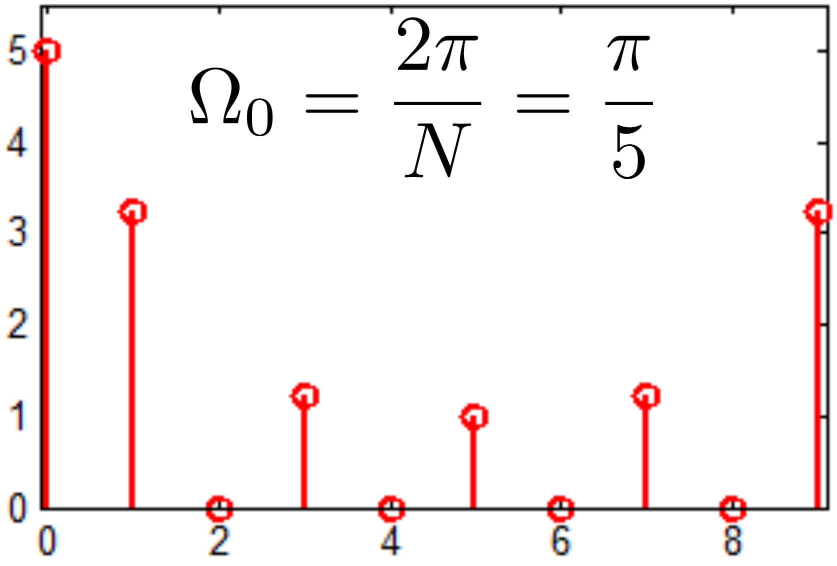




$$\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{3}$$

$N = 10$





$$\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{5}$$

Again the example:

$$x[n] = \begin{cases} 1 & n = 0, ..., 4 \\ 0 & n \leq -1 \text{ o } n \geq 5 \end{cases}$$

We have L=5 BUT NOW we assume:

$$N = 3$$

$$L > N!!!$$

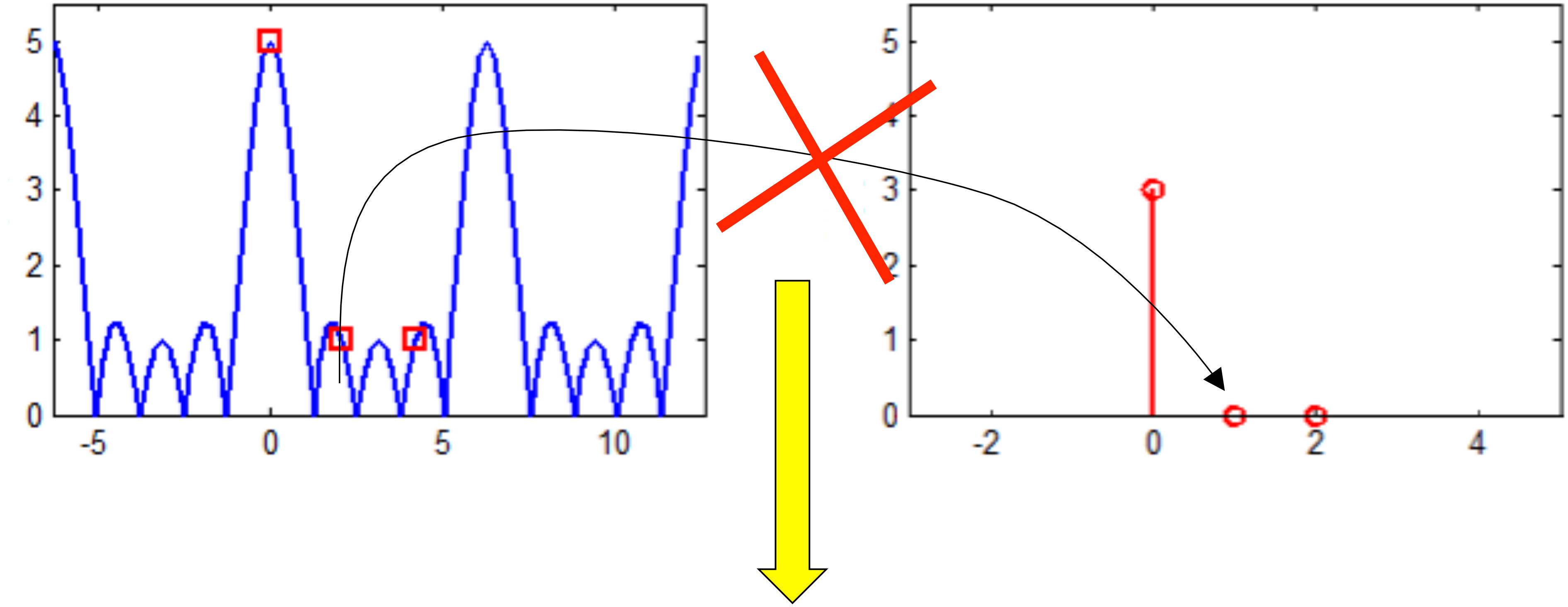
We need: $\boxed{L \leq N}$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n} = \sum_{n=0}^{4} e^{-j\Omega n} = \frac{1 - e^{-j5\Omega}}{1 - e^{-j\Omega}} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$$

$$N - 1$$

$$X_3[k] = \sum_{n=0}^{\textcircled{2}} x[n] \cdot e^{-j\frac{2\pi}{3}kn} = \frac{\sin\left(\frac{3\pi}{3}k\right)}{\sin\left(\frac{\pi}{3}k\right)} e^{-j\frac{2\pi}{3}k}, \qquad k = 0, ..., 2$$

graphically:



NO CORRESPONDENCE !

Since L>N !!!

We need: $L \leq N$

Other way to see the DFT

One can consider the DFT as a completely different mathematical operator (with respect to the Fourier series) that is function of x[n] and N:

$$X_N[k] \quad = \quad \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, \ldots, N-1$$

With the condition that: $\boxed{N \geq L}$

Where L is the length of x[n]