TOPIC 4
The practical computation of Fourier (e.g., with Matlab)
PART 2

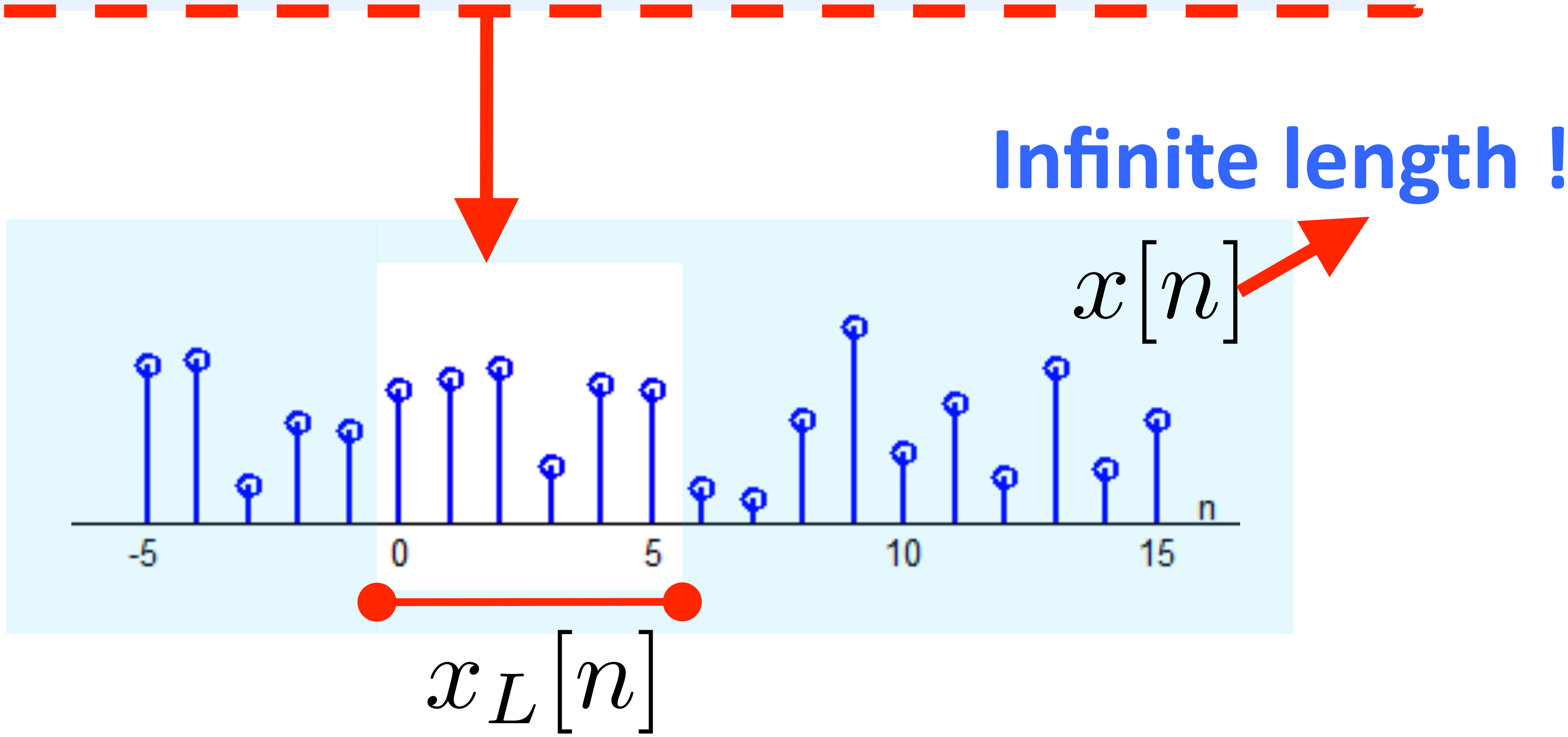
So far…

- We have interpreted the data (the finite sequence of values) as a periodic signal.

- We can consider another approach: the finite sequence of values (that we observe) is a piece of infinite signal.

Just a “piece” of a signal…

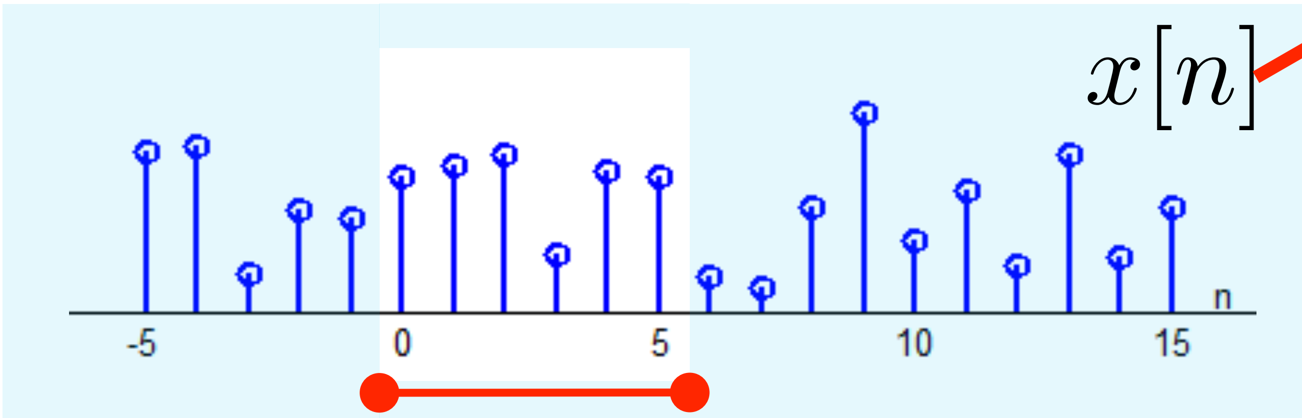
- We can consider another approach: the finite sequence of values (that we observe) is a piece of infinite signal.



Just a "piece" of a signal…

Infinite length !

$x[n]$



$$x_L[n] = x[n]w_L[n]$$

windowed signal

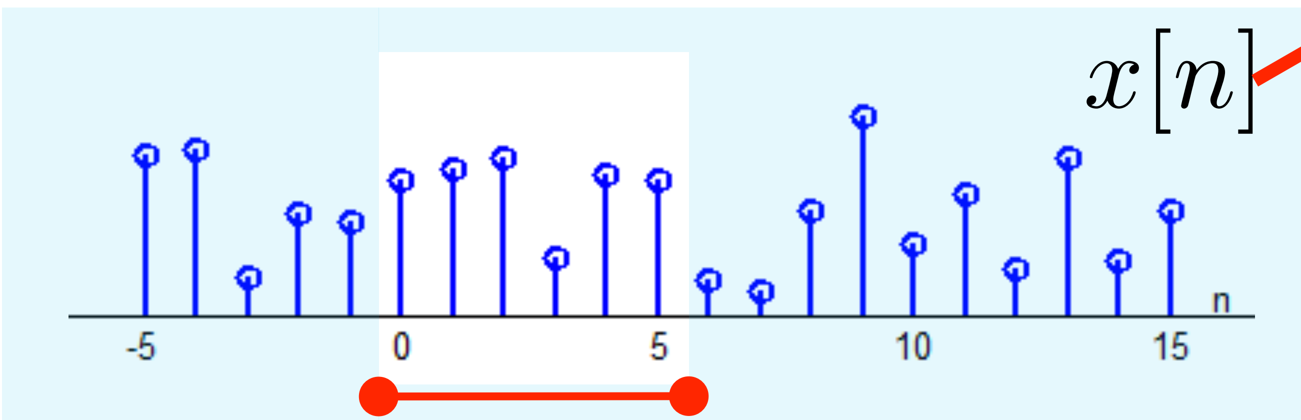
"window"

$$w_L[n] = \begin{cases} 1 & n = 0, \ldots, L-1 \\ 0 & \text{Otherwise} \end{cases}$$

Just a "piece" of a signal...

SUMMARY: Now, we interpret that we "see" a windowed signal.

Infinite length !



$$x_L[n] = x[n]w_L[n]$$

windowed signal

In frequency

$$x_L[n] = x[n]w_L[n]$$

$$X_L(\Omega) = X(\Omega) * W_L(\Omega)$$

$$X_L(\Omega) = \frac{1}{2\pi}\int_{2\pi} X(\theta)W_L(\Omega-\theta)d\theta$$

In frequency

$$w_L[n] = \begin{cases} 1 & n = 0, ..., L-1 \\ 0 & \text{Otherwise} \end{cases}$$

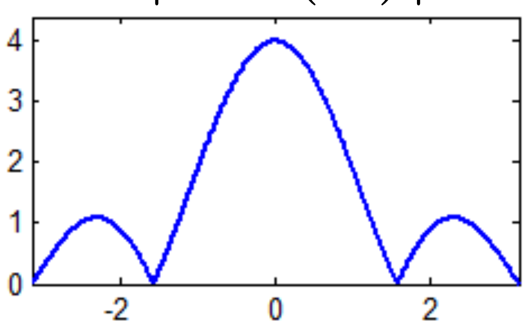
$$W_L(\Omega) = \frac{\sin(\frac{L\Omega}{2})}{\sin(\frac{\Omega}{2})} e^{-j\frac{(L-1)\Omega}{2}}$$

RECALL that is PERIODIC of period 2π

In frequency

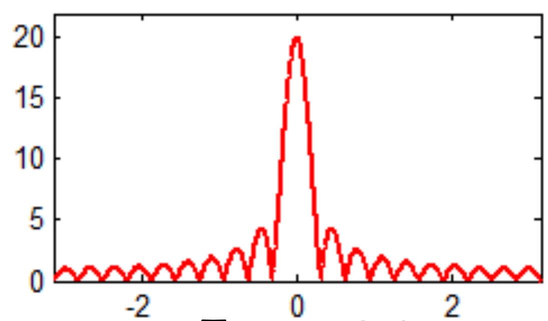
$$W_L(\Omega) = \frac{\sin(\frac{L\Omega}{2})}{\sin(\frac{\Omega}{2})} e^{-j\frac{(L-1)\Omega}{2}}$$

$|W_L(\Omega)|$



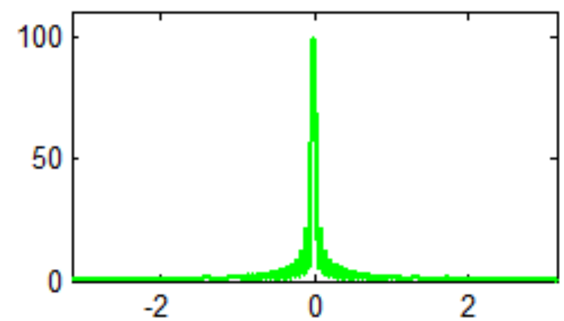
$L = 4$

$|W_L(\Omega)|$



$L = 20$

$|W_L(\Omega)|$



$L = 100$

As L grows, the FT of "the window" becomes more similar to delta in frequency.

Summary (so far)

TIME:

$$w_L[n] \xrightarrow{L \to \infty} 1$$

$$x_L[n] = x[n]w_L[n] \xrightarrow{L \to \infty} x[n]$$

Frequency:

$$W_L(\Omega) \xrightarrow{L \to \infty} 2\pi \sum_{k=-\infty}^{+\infty} \delta(\Omega - 2\pi k)$$

Generalized FT

$$X_L(\Omega) \xrightarrow{L \to \infty} X(\Omega)$$

also

$$X_L(\Omega) = X(\Omega) * W_L(\Omega) = \frac{1}{2\pi} \int_{2\pi} X(\theta)W_L(\Omega - \theta)d\theta$$

RECALL

We could always consider a larger sequence:

$$N \geq L$$

Adding zeros to the windowed signal, $x_L[n]$

(or considering the alternative mathematical view of DFT… see last slide of TOPIC 4- part 1)

Relationship with DFT

$$X_N[k] = \sum_{n=0}^{N-1} x_L[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{L-1} x_L[n] e^{-jk\frac{2\pi}{N}n}$$

$$x_L[n] = x[n]w_L[n]$$

$$X_L(\Omega) = X(\Omega) * W_L(\Omega)$$

$$X_L(\Omega) = \sum_{n=-\infty}^{+\infty} x_L[n] e^{-jk\Omega n} = \sum_{n=0}^{L-1} x_L[n] e^{-j\Omega n}$$

Relationship with DFT

COMPARING:

(Matlab gives you N numbers)

$$X_N[k] = \sum_{n=0}^{L-1} x_L[n] e^{-jk\frac{2\pi}{N}n} \qquad 0 \leq k \leq N-1$$

But actually we know that we have $X_N[k] = X_N[k+N]$

$$X_L(\Omega) = \sum_{n=0}^{L-1} x_L[n] e^{-j\Omega n} \qquad -\infty < \Omega < \infty$$

Periodic of period 2π

THEN AGAIN:

$$X_N[k] = X_L(\Omega)\Big|_{k\frac{2\pi}{N}} = X_L\left(k\frac{2\pi}{N}\right)$$

Relationship with DFT

$$X_N[k] = X_L(\Omega)\Big|_{k\frac{2\pi}{N}} = X_L\left(k\frac{2\pi}{N}\right)$$

$$X_N[k] = X_L(\Omega)\Big|_{k\Omega_0} = X_L(k\Omega_0)$$

$$\Omega_0 = \frac{2\pi}{N}$$

THEN, WE ARE "SAMPLING" the FOURIER TRANSFORM X_L(Omega) in frequency domain.

Exactly the same conclusions…

Relationship with DFT

$X_N[k]$

a_k

$$\Omega_k = k\Omega_0 = k\frac{2\pi}{N}$$

Exactly the same conclusions of PART 1…

VERY IMPORTANT SUMMARY

(1) We have: x[0], x[1], x[2],…,x[L-1]

(2) We compute DFT (e.g., by Matlab) with

$$N \geq L$$

$$X_N[k] = \sum_{n=0}^{L-1} x_L[n] e^{-jk\frac{2\pi}{N}n}$$

(3) Matlab returns N complex numbers:

$$X_N[0], X_N[1], X_N[2] \ldots, X_N[N-1]$$

$$\downarrow \quad\quad \downarrow \quad\quad \downarrow \quad\quad \downarrow$$

$$0 \quad\quad \Omega_0 \quad\quad 2\Omega_0 \quad\quad (N-1)\Omega_0$$

VERY IMPORTANT SUMMARY

(3) Matlab returns N complex numbers:

$$X_N[0], X_N[1], X_N[2] \ldots, X_N[N-1]$$

$$0 \qquad \Omega_0 \qquad 2\Omega_0 \qquad (N-1)\Omega_0$$

$$\Omega_0 = \frac{2\pi}{N}$$

$$X_N[k] = X_L(\Omega)\Big|_{k\Omega_0} = X_L(k\Omega_0)$$

VERY IMPORTANT SUMMARY

(4) Moreover the DFT values, that we get, are "samples" of the Fourier Transform of the windowed signal: we are assuming that we are observing just a "piece" of an infinite (non-periodic) signal…

$$X_N[k] = X_L(\Omega)\Big|_{k\Omega_0} = X_L(k\Omega_0)$$

$$x_L[n] = x[n]w_L[n]$$

$$X_L(\Omega) = X(\Omega) * W_L(\Omega)$$