TOPIC 5
ZETA TRANSFORM
PART 1

ZETA TRANSFORM

Region Of Convergence (ROC)

$z \in \mathrm{ROC}$

DEFINITION:

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \mathcal{Z}\{x[n]\}$$

This is the so-called "Analysis Equation":
in the sense that we go from time domain- x[n] - to the transform domain - X(z).

$z \in \mathbb{C}$ $\qquad$ $X(z) \in \mathbb{C}$

It is a generalization/extension of the Fourier Transform for a signal in discrete time (DTFT).

ZETA TRANSFORM

Region of Convergence

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \quad z \in \mathrm{ROC}$$

$$z = re^{j\Omega}$$

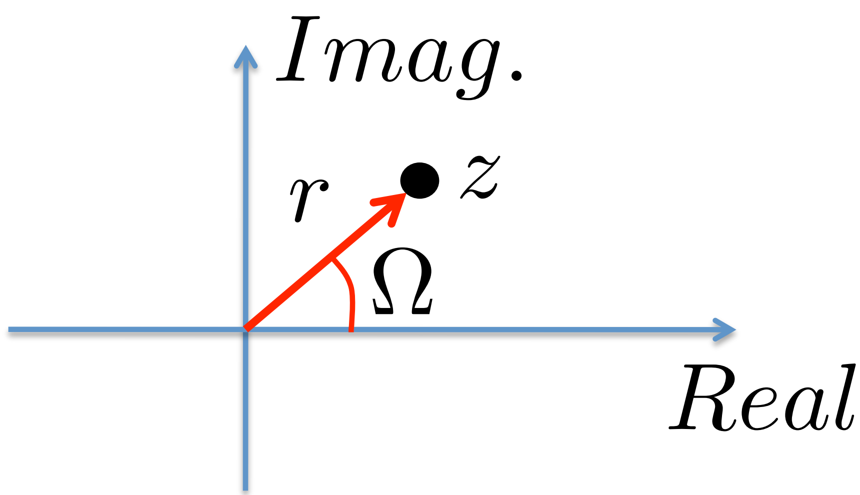
FREQUENCY

We will use the polar representation
for the variable z.

POLAR REPRESENTATION

$$z = re^{j\Omega}$$

- **For each r and Omega we have a complex number z (a point in the complex plane).**



RELATIONSHIP with other topics/fields

- Zeta transform coincides with the so-called Laurent Series - Laurent expansion (in complex analysis).
- The inverse Zeta transform is related to the so-called Cauchy integral formula and residue theorem (in complex analysis).

RELATIONSHIP with other topics/fields

- **Since the Laurent Series generalizes the Taylor series, the Zeta trasform is a generalization of the Taylor series;**
Indeed, consider only the negative values of n, we have

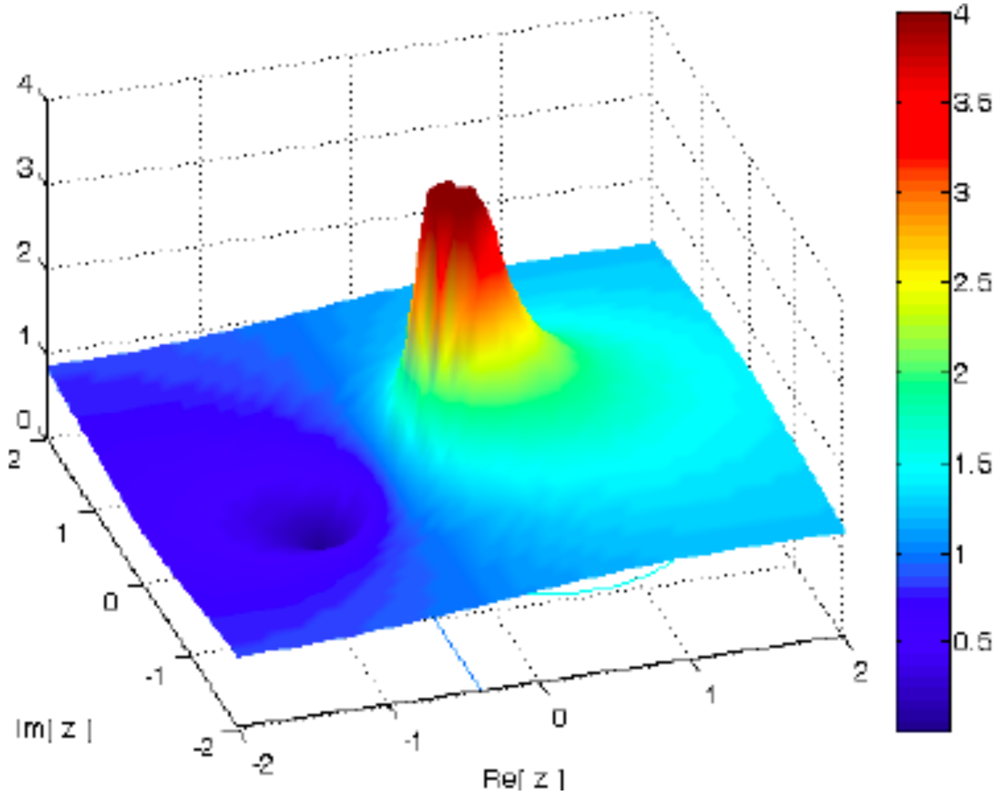
$$X(z) = \sum_{n=-\infty}^{0} x[n]z^{-n} \underset{\substack{\uparrow \\ k=-n}}{=} \sum_{k=0}^{\infty} x[k]z^{k}$$

RELATIONSHIP with other topics/fields

- Zeta transform is related to the solution of the difference equations.
- as a consequence, it is related also to digital filtering theory (ARMA, AR, MA, IIR, FIR etc.).

Graphical example of |X(z)|

- **The Zeta transform is defined in the complex domain (except some points or regions where there is not convergence; later we will talk about that); it takes complex values then we plot the module (for instance).**



Inverse Zeta Transform

- Almost never used in practice.
- Synthesis equation.

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

LINE INTEGRAL – CURVE INTEGRAL (in a "close" "circular" curve)

Recovering the FOURIER TRANSFORM

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n](re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]r^{-n}e^{-j\Omega n}$$

Recovering the FOURIER TRANSFORM

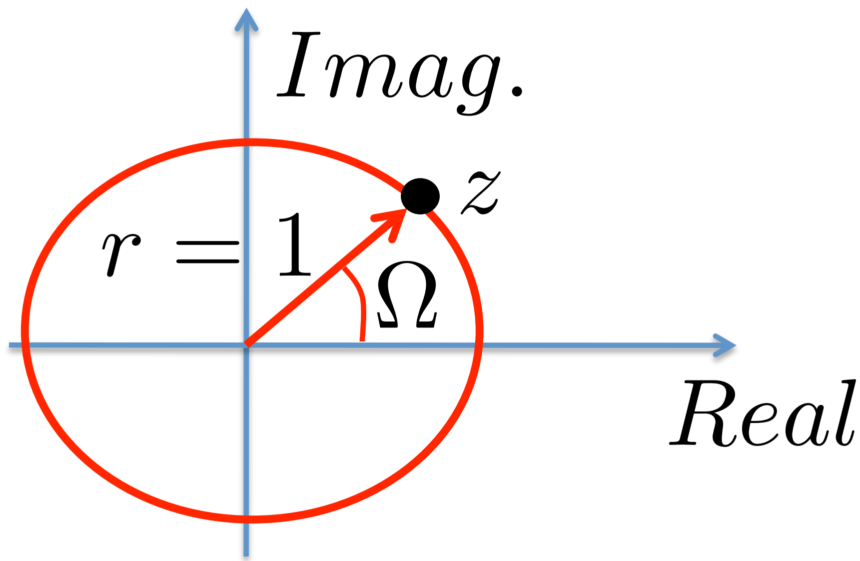
$$X(re^{j\Omega}) = X(r,\Omega) = \sum_{n=-\infty}^{+\infty} x[n]r^{-n}e^{-j\Omega n}$$

$$r = 1$$

$$X(e^{j\Omega}) = X(1,\Omega) = X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

Recovering the FOURIER TRANSFORM

$$X(e^{j\Omega}) = X(1,\Omega) = X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$



- The Fourier Transform is defined in this "circle"
- The ROC must contains this circle in order to have the FT !

Why ZETA TRANSFORM?

$$z = re^{j\Omega}$$

- The variable r can help the convergence of the series (for some signals x[n]).

- RECALL For each r and Omega we have a complex number z (a point in the complex plane).

EXAMPLE

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

ROC !!!!!
Region of convergence

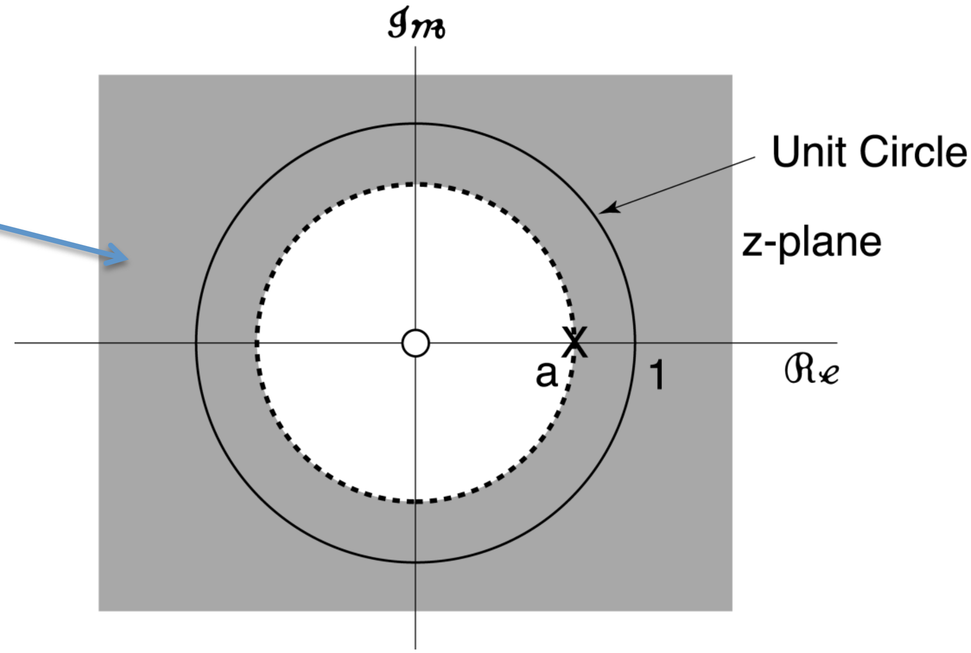
Region of convergence (ROC)

$$x[n] = a^n u[n] \longrightarrow X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

ROC !!!!!
Region of convergence

The ROC is depicted in "grey"

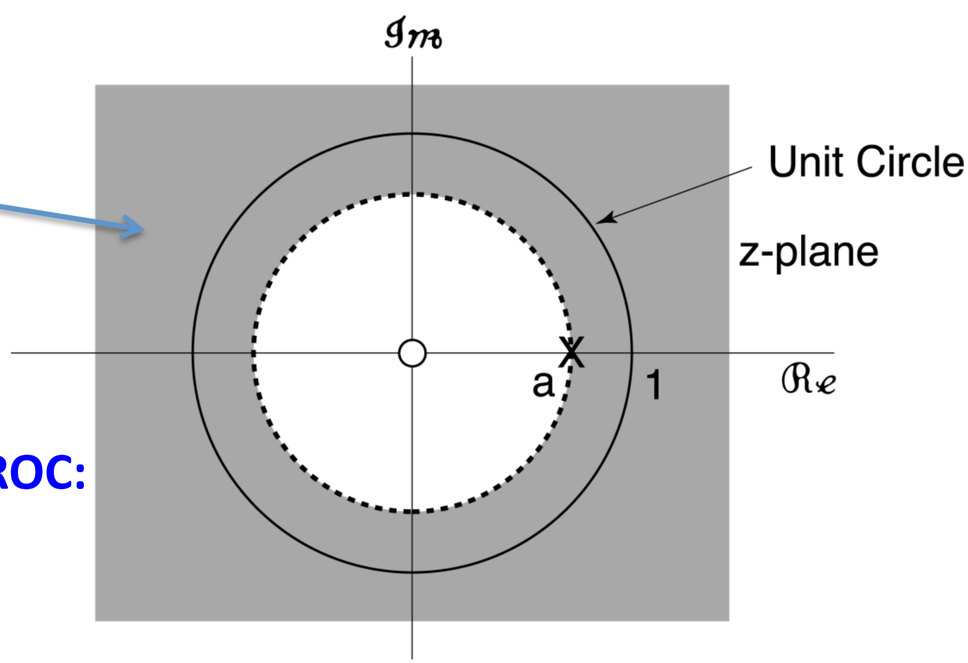


Region of convergence (ROC)

$$\text{ROC} = \{\text{all } z \in \mathbb{C} \text{ such that } X(z) \text{ exists (finite)}\}$$

$$\text{ROC} = \left\{\text{all } z \in \mathbb{C} \text{ such that } \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \text{ converges}\right\}$$

The ROC is depicted in "grey"



Does the FT exist? In this case "yes" !!!
Since the unit circle is contained in the ROC:
if a<1, then the FT exists;
if a>1, then the FT does not exist !

Region of convergence (ROC)

$$x[n] = a^n u[n] \longrightarrow X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

We have always
to obtain/provide:
X(z) jointly with the ROC.

$$\text{ROC} = \{ |z| > |a| \}$$

ROC: Region Of Convergence

Region of convergence (ROC)

The ROC is an essential part of the information.

Other example

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{a^{-1}z}{a^{-1}z-1}$$

$$= \frac{z}{z-a},$$

IT IS THE SAME X(z) of the other example!!

ROC: Region Of Convergence

If $|a^{-1}z| < 1$, $i.e.$, $|z| < |a|$

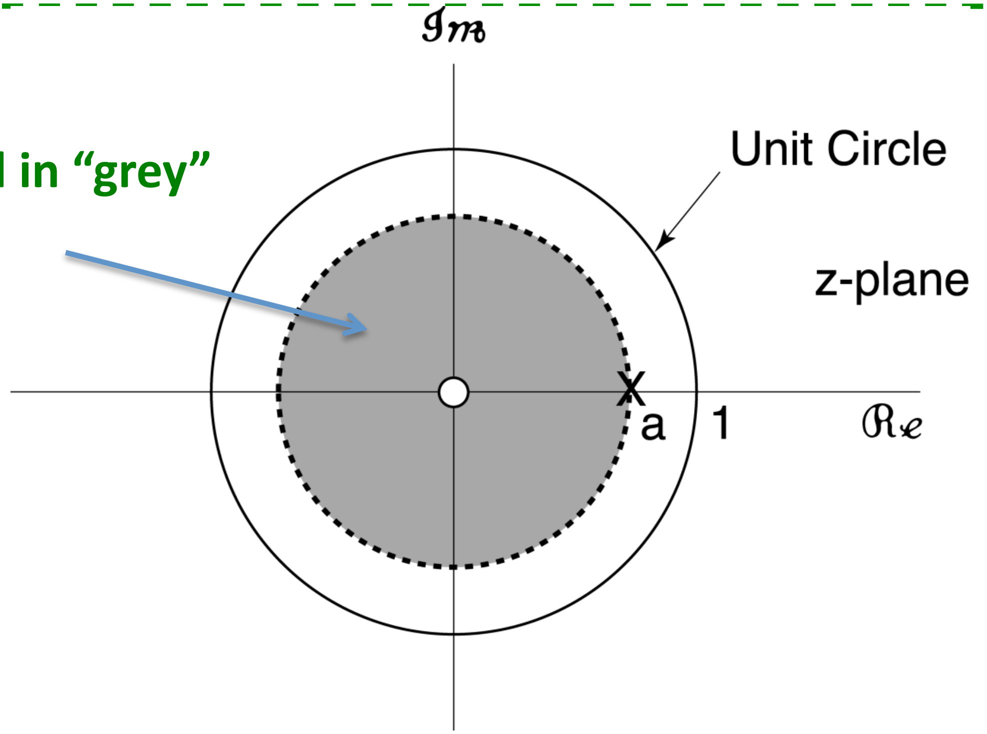
Other example

$$x[n] = -a^n u[-n-1] \longrightarrow X(z) = \frac{z}{z-a}$$

ROC

$$|z| < |a|$$

The ROC is depicted in "grey"



If a<1 then the FT does not exist....
If a>1 then the FT will exist....
(In the figure above a<1, the FT does not exist)

About ROCs of the Zeta transform

The ROCs are always "circular pieces/ portions" of the complex plane, possibly infinite pieces (as we saw in the previous slides; it can be also like a "donut", we will see this case).

SUMMARY of the examples

$$x[n] = a^n u[n] \rightarrow X(z) = \frac{z}{z-a}$$

ROC

$|z| > |a|$

$$x[n] = -a^n u[-n-1] \rightarrow X(z) = \frac{z}{z-a}$$

ROC

$|z| < |a|$

THE SAME ZETA TRASFORM BUT DIFFERENT ROCs !!!!

...Then, the ZETA TRANSFORM IS:

$$X(z) + \mathrm{ROC}$$

Given a signal in time x[n]…

$$x[n] \longrightarrow X(z) + \mathrm{ROC}$$

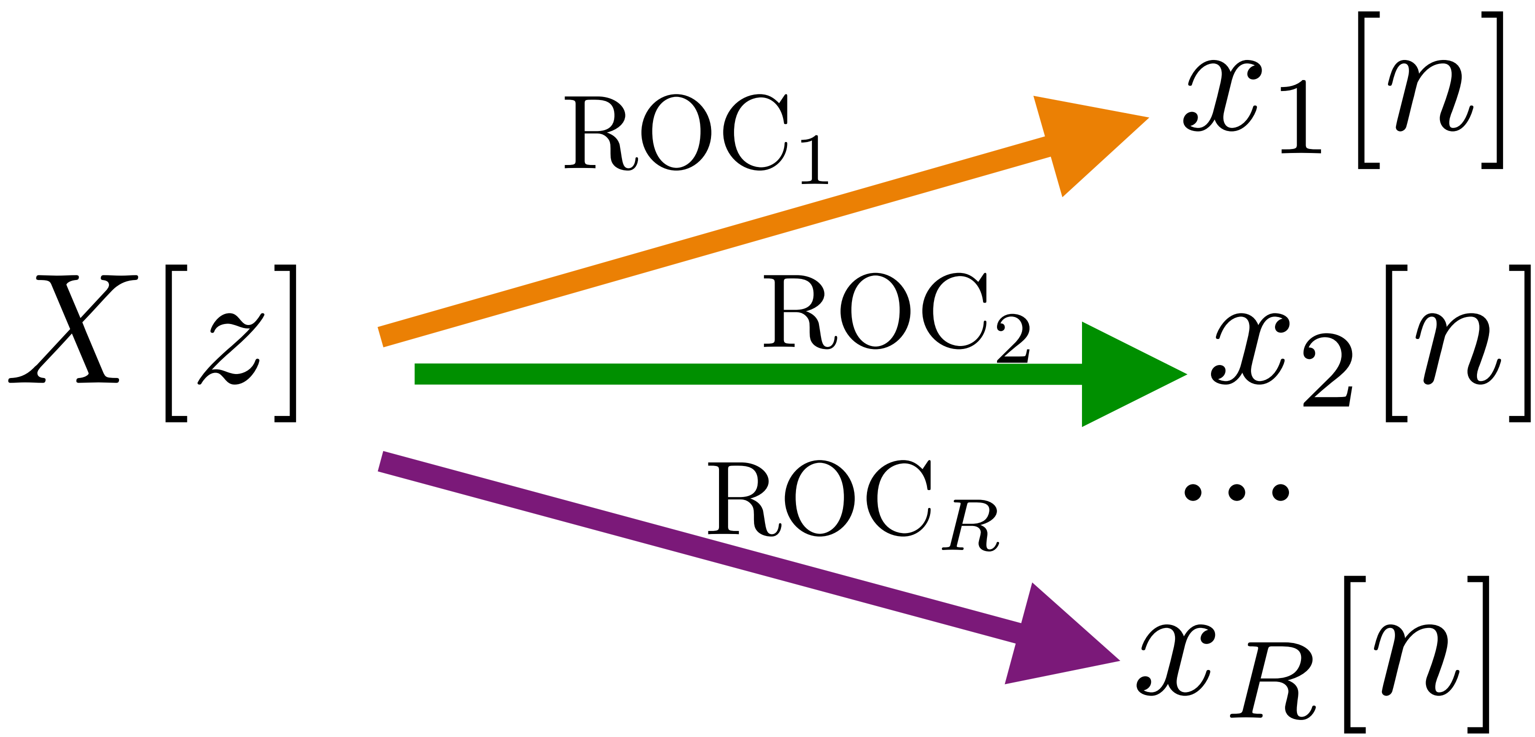
To a signal x[n] in time corresponds one X(z) and one ROC associated. The pair X[z]+ROC is associated to x[n].

Given X[z]+ROC …

$$X(z) + \text{ROC} \longrightarrow x[n]$$

Given a pair X[z]+ROC, in time we a UNIQUE (only one) signal x[n], corresponding to this pair (X[z]+ROC).

Given just X[z] ...



Given only X[z], we different signals in time (which provide the same X[z]). Each signal corresponds to a specific ROC.

Given just X[z]: example

$$X(z) = \frac{z}{z-a}$$

ROC-1

$|z| > a$

$$x[n] = a^n u[n]$$

ROC-2

$|z| < a$

$$x[n] = -a^n u[-n-1]$$

Poles and zeros

POLES: all values of z such that $X(z) \to \infty$

ZEROS: all values of z such that $X(z) = 0$

Poles and zeros: example

$$X(z) = \frac{z}{z-a}$$

One zero at z=0

One pole at z=a

Poles and zeros: other example

$$X(z) = z^2 - 4$$

Two zeros at z=-2
And z=2

ONE pole at
z=Infinity
("almost" no
poles…)

Poles and zeros: other example

$$X(z) = z^2 + 4$$

→ **Two zeros at z=-2j and z=2j**

→ **ONE pole at z=Infinity ("almost" no poles…)**

Poles and zeros: other example

$$X(z) = \frac{z-1}{z^2+4}$$

One zero at z=1
And one zero at z=Infinity

Two poles at z=-2j and z=2j

ROCs and poles

VERY IMPORTANT SLIDE !!

A ROC does not contain poles (never!)

Generally, the poles are in the boundaries of the ROCs. If X(z) is rational (a fraction of polynomials) this is always the case.

ROCs and poles

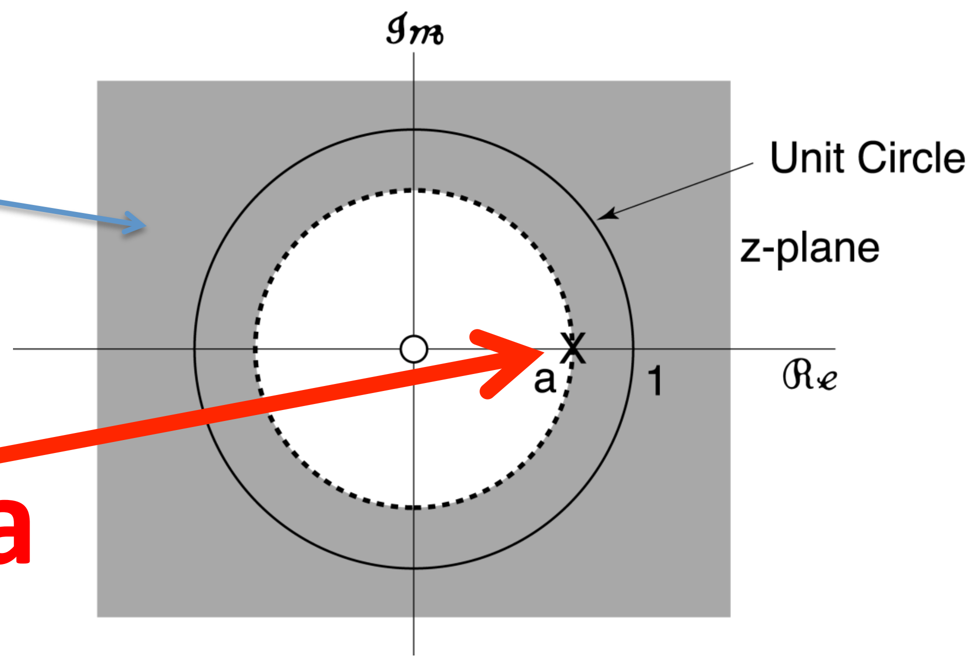
Namely, in the case of X(z) is rational (a fraction of polynomials) the poles "define/determine" the ROC.

ROCs and poles: examples

$$x[n] = a^n u[n] \longrightarrow X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

The ROC is depicted in "grey"



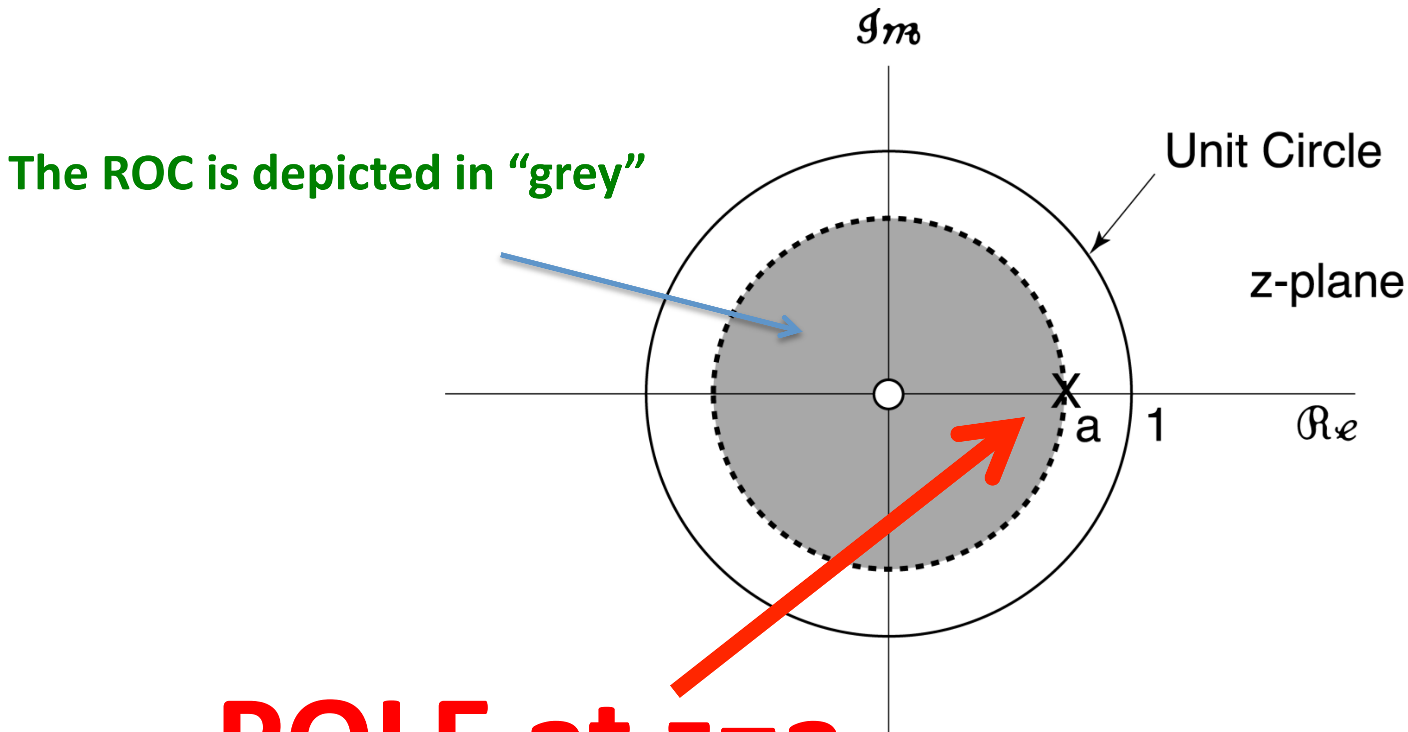
POLE at z=a

ROCs and poles: examples

$$x[n] = -a^n u[-n-1] \longrightarrow X(z) = \frac{z}{z-a}$$

ROC

$|z| < |a|$



POLE at z=a