TOPIC 5
ZETA TRANSFORM
PART 2

RECALL: ZETA TRANSFORM

Region of Convergence

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$z \in \mathrm{ROC}$

$$z = re^{j\Omega}$$

We will use the polar representation
for the variable z.

FREQUENCY

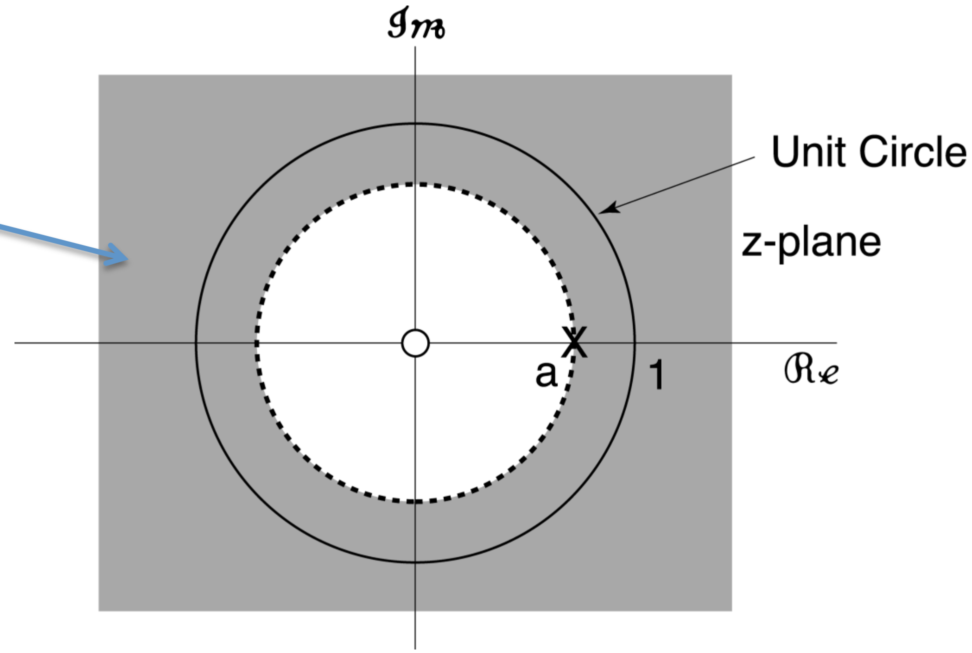
RECALL: Example 1

$$x[n] = a^n u[n] \longrightarrow X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

ROC !!!!!
Region of convergence

The ROC is depicted in "grey"

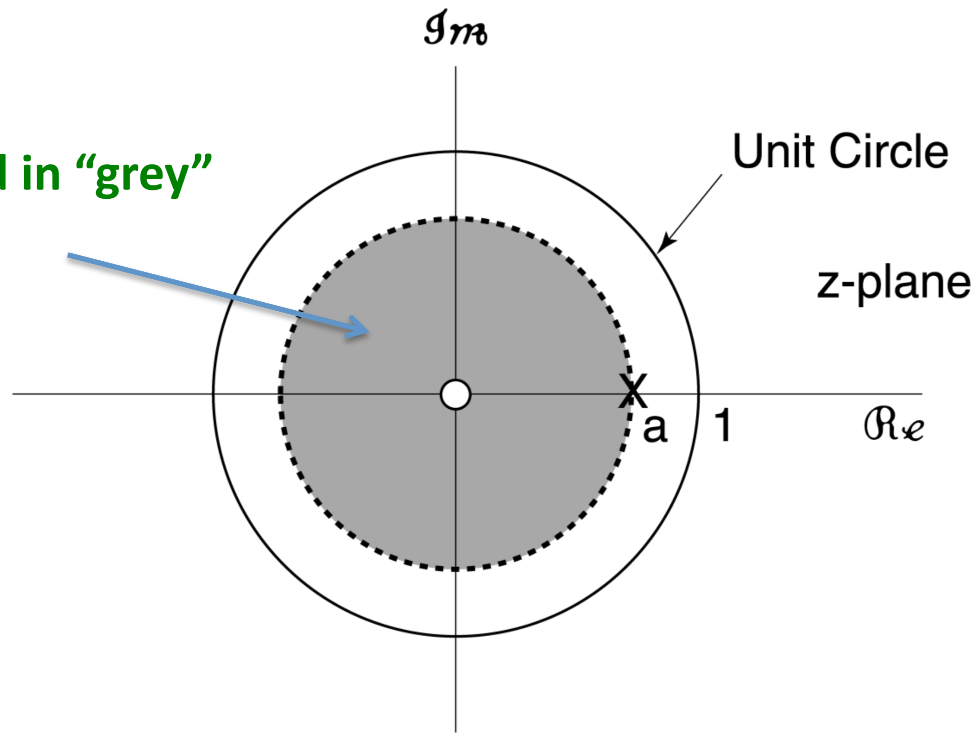


RECALL: Example 2

ROC

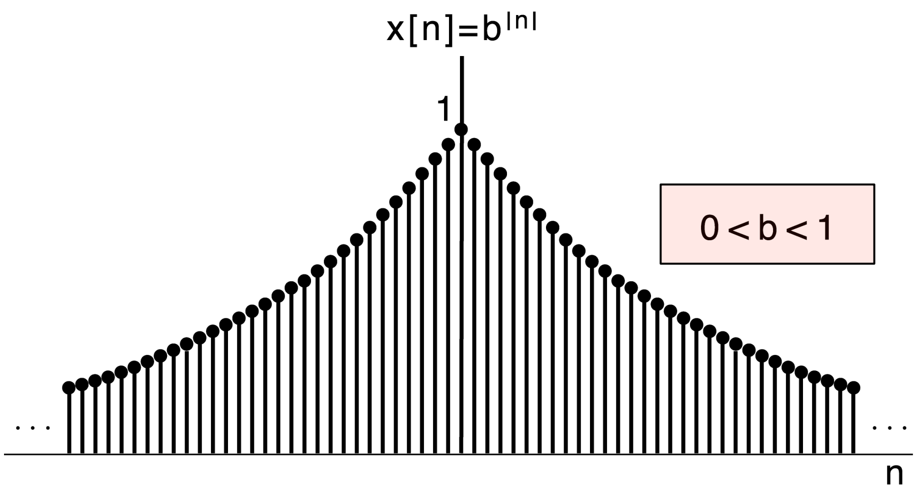
$$x[n] = -a^n u[-n-1] \longrightarrow X(z) = \frac{z}{z-a} \qquad |z| < |a|$$

The ROC is depicted in "grey"

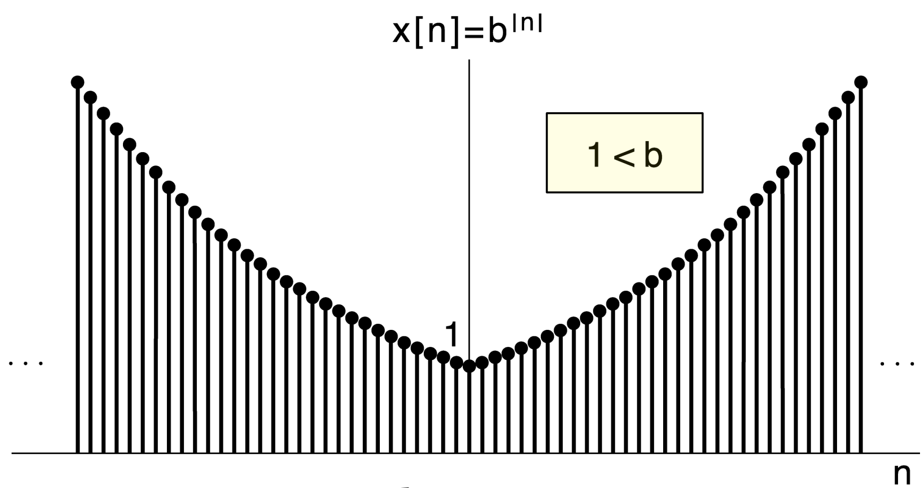


Example 3

$$x[n] = b^{|n|}, \quad b > 0$$



$$0 < b < 1$$



$$b > 1$$

Example 3

$$x[n] = b^{|n|}, \quad b > 0$$

$$x[n] = b^{n}u[n] + b^{-n}u[-n-1]$$

FROM Example 1 and 2 (replacing "a" with "b"):

$$b^{n}u[n] \longleftrightarrow \frac{1}{1 - bz^{-1}}, \quad |z| > b$$

ROC

ROC

$$b^{-n}u[-n-1] \longleftrightarrow \frac{-1}{1 - b^{-1}z^{-1}}, \quad |z| < \frac{1}{b}$$

Example 3: ROC

ROC of this example:
we have to satisfy SIMULTANEOUSLY
the two ROCs below (of the previous examples)

ROC-1: $|z| > b$ ROC-2: $|z| < 1/b$

The only way to obtain that is: $b < 1$

In this case, the ROC is:

$$b < |z| < 1/b$$

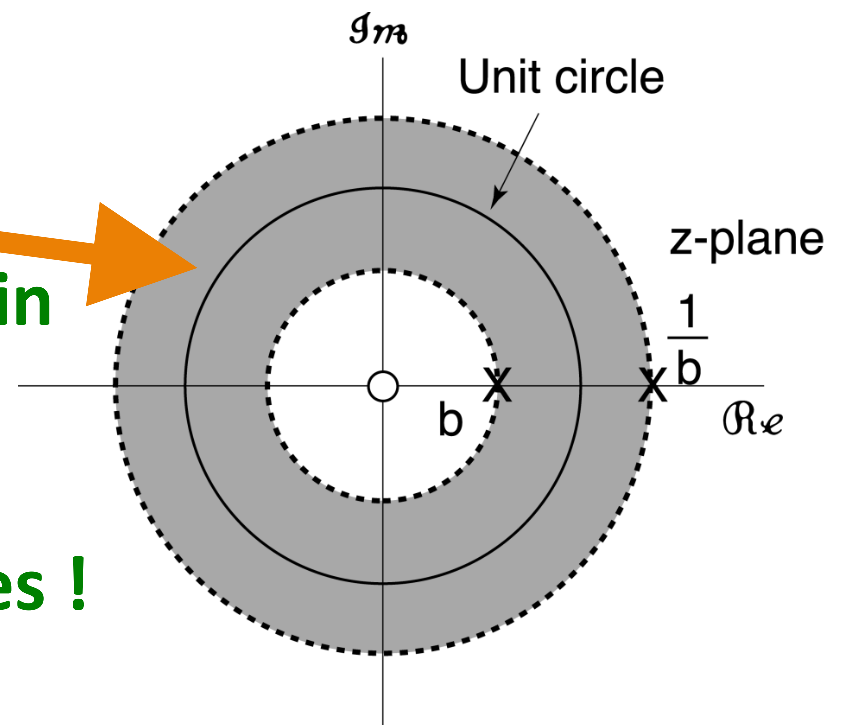
Solution of Example 3 for b<1

Solution for b<1:

$$X(z) = \frac{1}{1-bz^{-1}} + \frac{-1}{1-b^{-1}z^{-1}} \quad , \quad b < |z| < \frac{1}{b}$$

...it can be also written with a unique fraction....

✓The FT always exists! The unit circle is always within The ROC !!



✓ b and 1/b are the two poles !

✓ we have a “DONUT” ROC !

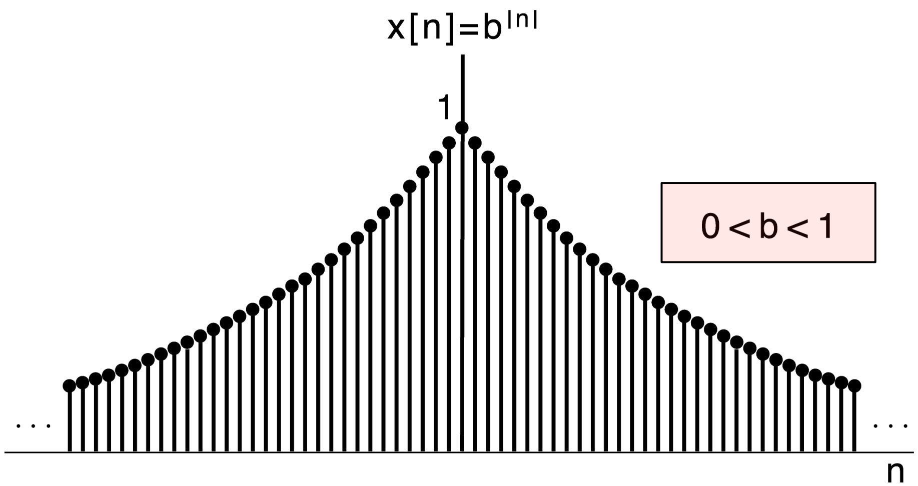
Solution of Example 3 for b>1

Solution for b>1:

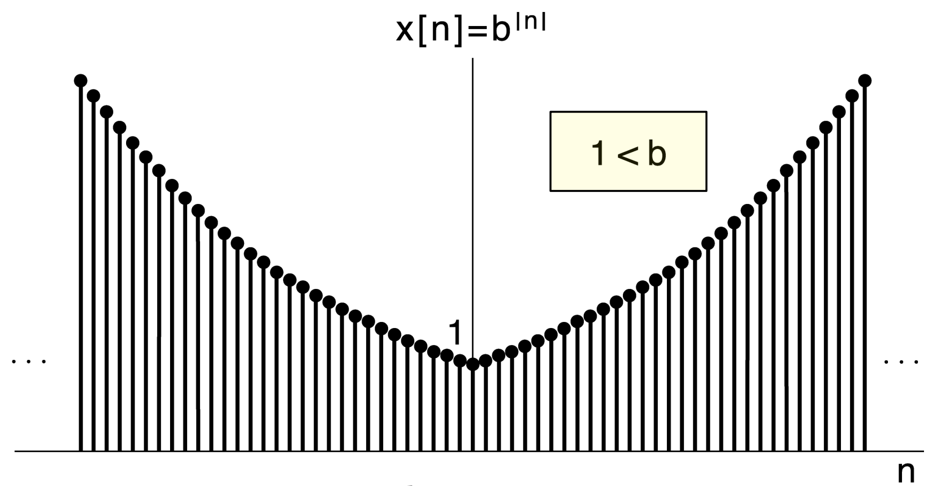
- ✓ No solution.
- ✓ The ROC is empty.

Recalling the signal…

$$x[n] = b^{|n|}, \quad b > 0$$



$$0 < b < 1$$



$$b > 1$$

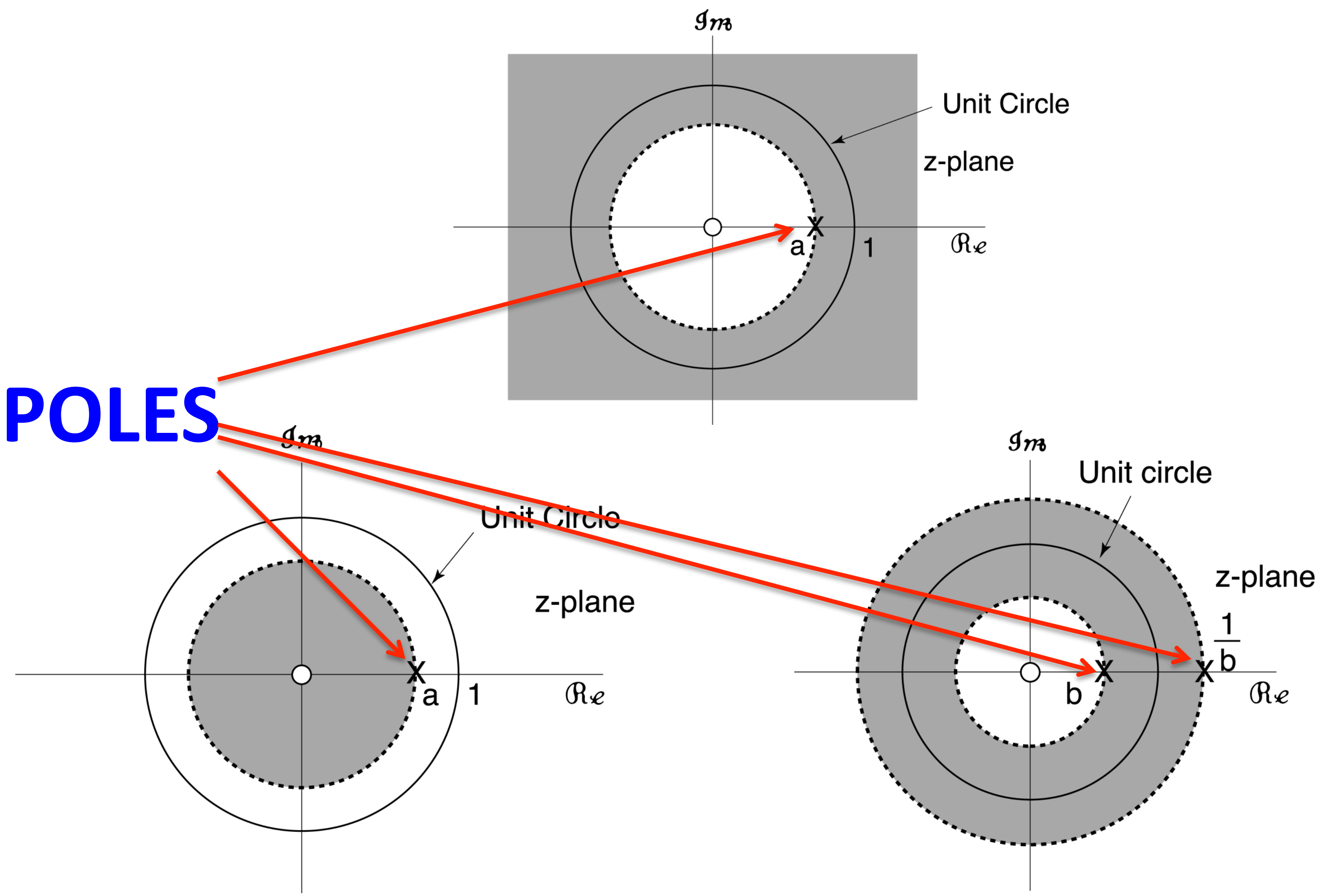
The result "makes sense": for b>1, the signal is unbounded in both sides (infinite energy in both size), and the factor "r" in ZT cannot help simultaneously both sides to converge.

RECALL: about ROCs

The ROCs are always “circular pieces/ portions” of the complex plane, possibly infinite pieces.

A ROC does not contain poles (never!)

RECALL: about ROCs



About ROCs

The ROCs always are defined considering the module of z, i.e., $|z|=r$, which is the variable r !!

Example 4

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n]z^{-n}$$

$$X(z) = 1$$

ROC: all the complex plane !!!!

Example 5

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n}$$

$$x[n] = \delta[n-1] \longrightarrow X(z) = z^{-1}$$

ROC: all the complex plane but except zero !! There is a pole at 0.

Example 6

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n}$$

$$x[n] = \delta[n+1] \longrightarrow X(z) = z$$

ROC: all the complex plane (but except infinity!! There is a pole at Infinity).

**** Generally, also "infinity" is included in this analysis....**

Example 7

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n-n_0]z^{-n}$$

$$x[n] = \delta[n-n_0] \longrightarrow X(z) = z^{-n_0}$$

$$n_0 > 0$$

ROC: all the complex plane but except zero !!

There is a (multiple) pole at 0 (with multiplicity n0, i.e., n0 coincident poles at 0).

Example 8: property

$$q[n] = x[n - n_0]$$

$$Q(z) = \sum_{n=-\infty}^{\infty} q[n] z^{-n}$$

$$Q(z) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}$$

Change of variable: $k = n - n_0$

$n = k + n_0$

Example 8: property

$$Q(z) = \sum_{k=-\infty}^{\infty} x[n] z^{-k-n_0}$$

$$Q(z) = z^{-n_0} \underbrace{\sum_{k=-\infty}^{\infty} x[k] z^{-k}}_{X(z)}$$

$$Q(z) = z^{-n_0} X(z)$$

ROC: note that we adding a (multiple) pole at 0 (or at Infinity of n0<0) to the possible poles of X(z). If X(z) has a (multiple) zero at 0 this pole is removed.

Example 8: property

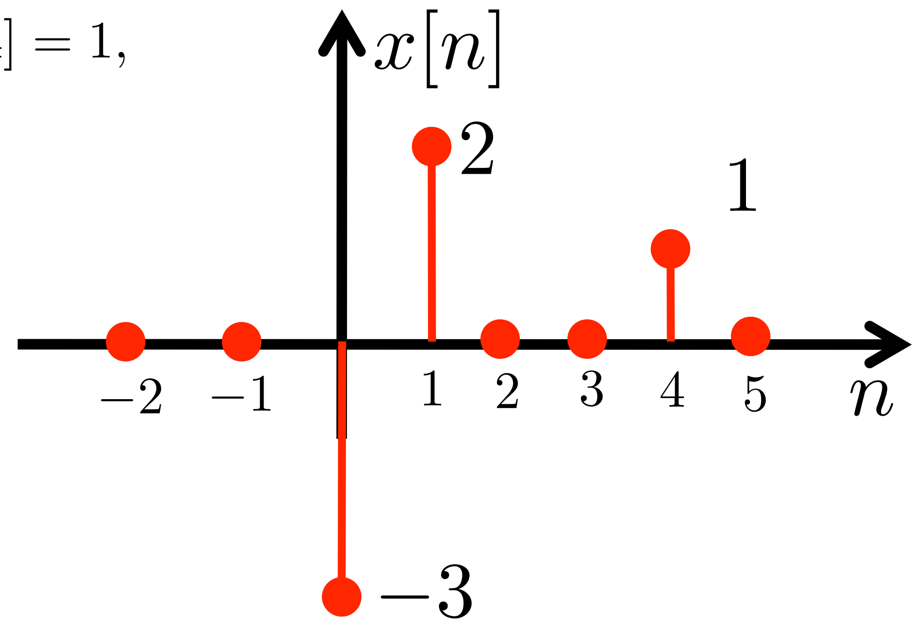
$$q[n] = x[n - n_0] \longrightarrow Q(z) = z^{-n_0} X(z)$$

Shift in time → multiplication by a power of z in the transformed domain

$$x[n - i] \longrightarrow z^{-i} X(z)$$

Example 9

$x[0] = -3, x[1] = 2, x[4] = 1,$
otherwise 0



Easy to express with deltas

$$x[n] = -3\delta[n] + 2\delta[n-1] + \delta[n-4]$$

Example 9

$$x[n] = -3\delta[n] + 2\delta[n-1] + \delta[n-4]$$

Using the result of example 7:

$$x[n] = \delta[n-n_0] \longrightarrow X(z) = z^{-n_0}$$

$$X(z) = -3 + 2z^{-1} + z^{-4}$$

$$X(z) = \frac{-3z^4 + 2z^3 + 1}{z^4}$$

ROC: all the complex plane except 0! We have a (multiple) pole at 0. At infinity, the pole at 0 help us....

ZT of finite length signals

For instance:

$$x[n] = \sum_{j=0}^{L-1} a_n \delta[n-j]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{L-1} x[n]z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots + x[L-1]z^{-(L-1)}$$

$$= \frac{x[0]z^{L-1} + x[1]z^{L-2} + x[2]z^{L-3} + \ldots + x[L-1]}{z^{L-1}}$$

ROC= ? →

ROCs of ZT
of finite length signals

If x[n] has finite length, the ROC is all the complex plane except POSSIBLY the points z=0 and/or z=Infinity.

The ROC is all the complex plane just for x[n]=Delta[n].

Zeta Transform for LTI systems

The Zeta Transform of an impulse response of an LTI system is always rational: namely, a fraction of polynomials !
(see proof in the next slides)

For the proof we need to use the property:

$$x[n-i] \longrightarrow z^{-i}X(z)$$

Zeta Transform of the impulse response of a LTI system

The output of a LTI system can be expressed with the convolution sum and:

$$\sum_{i=0}^{L} b_i y[n-i] = \sum_{r=0}^{R} c_r x[n-r]$$

LINEAR DIFFERENCE EQUATION WITH COSTANT COEFFICIENTS (AND INITIAL CONDITIONS = 0)

Zeta Transform of the impulse response of a LTI system

$$\mathcal{Z}\left\{\sum_{i=0}^{L} b_i y[n-i]\right\} = \mathcal{Z}\left\{\sum_{r=0}^{R} c_r x[n-r]\right\}$$

$$\sum_{i=0}^{L} b_i \mathcal{Z}\left\{y[n-i]\right\} = \sum_{r=0}^{R} c_r \mathcal{Z}\left\{x[n-r]\right\}$$

$$\sum_{i=0}^{L} b_i z^{-i} Y(z) = \sum_{r=0}^{R} c_r z^{-r} X(z)$$

Zeta Transform of the impulse response of a LTI system

$$\sum_{i=0}^{L} b_i z^{-i} Y(z) = \sum_{r=0}^{R} c_r z^{-r} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{R} c_r z^{-r}}{\sum_{i=0}^{L} b_i z^{-i}}$$

It can be expressed as a fraction of polynomials !!!

The Zeta Transform of an impulse response of an LTI system is always rational: namely, a fraction of polynomials !

In LTI systems: ROCs and poles

When X(z) is rational (a fraction of polynomials, i.e., for impulse response of LTI systems) the poles "define/determine" the ROC.