TOPIC 5
ZETA TRANSFORM
PART 3

Zeta Transform of the impulse response of a LTI system

The Zeta Transform of an impulse response of an LTI system is always rational: namely, a fraction of polynomials !

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{L} b_i z^{-i}}{\sum_{r=0}^{R} c_r z^{-r}}$$

It can be expressed as a fraction of polynomials !!!

Zeta Transform of the impulse response of a LTI system: fraction of polynomials !

EXAMPLE:

$$H(z) = \frac{1 - 2z^{-1}}{3 + 4z^{-1} - z^{-2}}$$

It can be expressed as a fraction of polynomials:

$$= \frac{z^2}{z^2} \frac{1 - 2z^{-1}}{3 + 4z^{-1} - z^{-2}}$$

$$= \frac{z^2 - 2z}{3z^2 + 4z - 1}$$

Zeta Transform of the impulse response of a LTI system: fraction of polynomials !

$$H(z) = \frac{N(z)}{D(z)}$$

→ **polynomials !!!**

zeros of $N(z)$: zeros of the Zeta transform $H(z)$

zeros of $D(z)$: poles of the Zeta transform $H(z)$

Zeta Transform of linear combination of exponentials: fraction of polynomials !

If x[n] is a linear combination of exponentials in discrete time:

Then X(z) is a fraction of polynomials, as well

$$X(z) = \frac{N(z)}{D(z)}$$

polynomials !!!

MAIN PROPERTIES
of the ZETA TRANSFORM

PROPERTY-1 (already seen and proved)

We have already seen that:

$$x[n - n_0] \longrightarrow z^{-n_0} X(z)$$

REGARDING THE ROC, we COULD have:

-An additional (multiple) pole at z=0 (if n0>0).
-An additional (multiple) pole at z=Infinity (if n0<0).

RECALL THAT THE POLE ARE NOT CONTAINED IN THE ROC.

PROPERTY-2

$$nx[n] \longleftrightarrow -z\frac{dX(z)}{dz},$$

THE ROC DOES NOT CHANGE.

PROOF:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Definition of ZT of nx[n]

PROPERTY – 3:
Initial value Theorem

If x[n] is causal, i.e., x[n]=0 for n<0, then:

$$x[0] = \lim_{z \longrightarrow \infty} X(z)$$

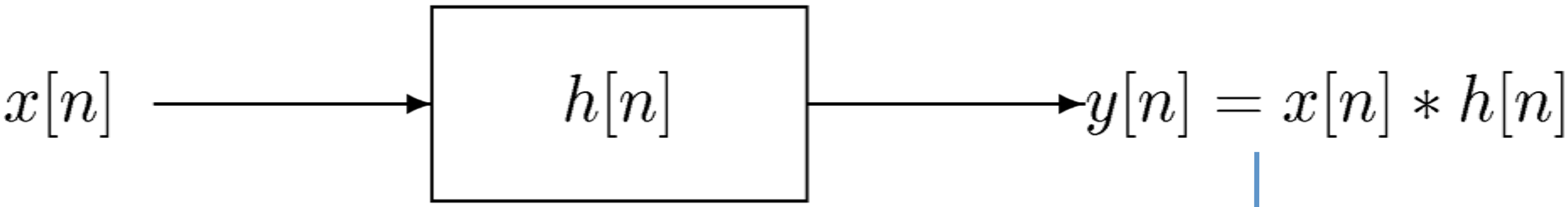
PROPERTY – 4:
Final value Theorem

If the poles of (z-1)X(z) are inside the unit circles, then:

$$x[\infty] = \lim_{z \to 1} (z-1) X(z)$$

(related to the residue computations and theorem…)

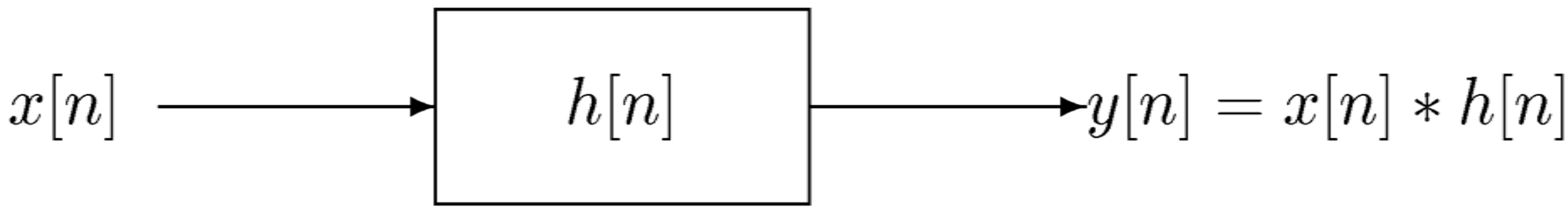
PROPERTY – 5: Convolution in time, product in the transformed domain…



$$Y(z) = H(z)X(z)$$

The ROC of Y(z) is different, in general, from the ROC of H(z) and the ROC of X(z).

PROPERTY – 6:
STABILITY of the LTI system



Recall that,
an LTI system is stable if:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$$

ROC de H(z) must contain the circle of radius 1 !

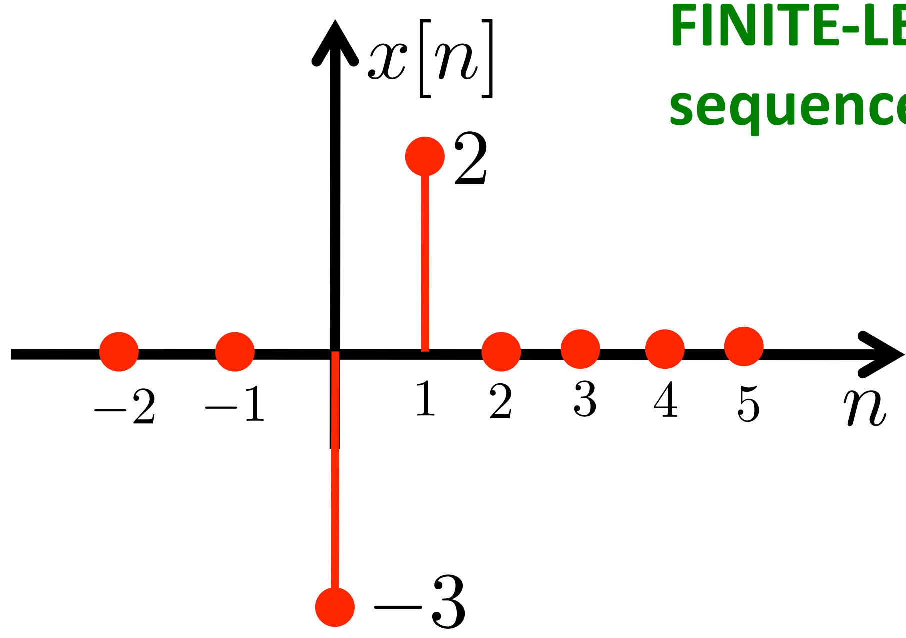
Think to
the definition
of Zeta Transform H(z)….

Namely, STABILITY: when exists the FT of h[n] !

Example 10

$x[0] = -3, x[1] = 2,$

otherwise 0



FINITE-LENGTH sequence

Easy to express with deltas

$$x[n] = -3\delta[n] + 2\delta[n-1]$$

Example 10

$$x[n] = -3\delta[n] + 2\delta[n-1]$$

Using the result of example 7:

$$x[n] = \delta[n - n_0] \longrightarrow X(z) = z^{-n_0}$$

$$X(z) = -3 + 2z^{-1}$$

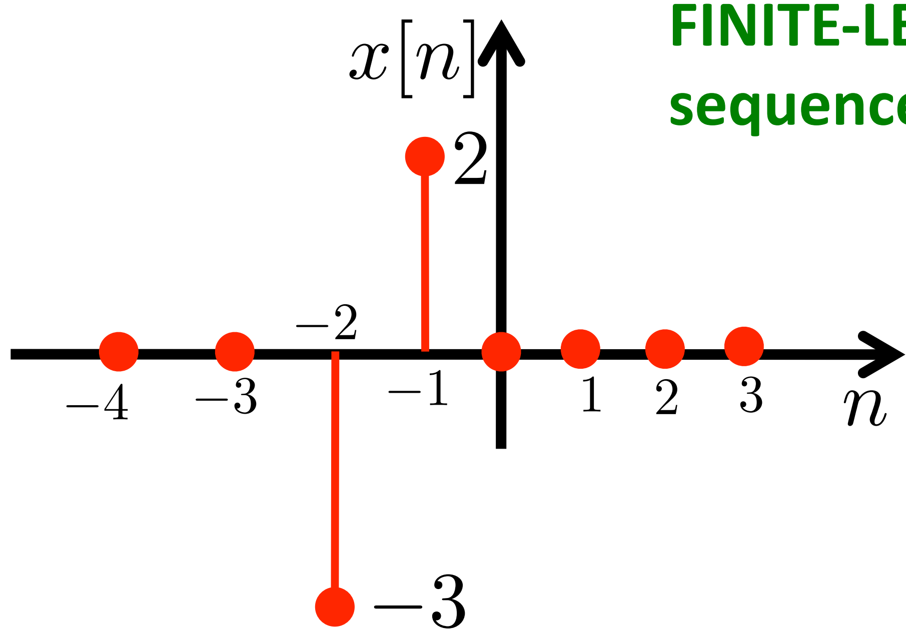
$$X(z) = \frac{-3z + 2}{z}$$

ROC: all the complex plane except z=0! (i.e., r=0)

Example 11

$x[-2] = -3, \quad x[-1] = 2$

otherwise 0



FINITE-LENGTH sequence

Easy to express with deltas

$$x[n] = -3\delta[n+2] + 2\delta[n+1]$$

Example 11

$$x[n] = -3\delta[n+2] + 2\delta[n+1]$$

$$X(z) = -3z^2 + 2z$$

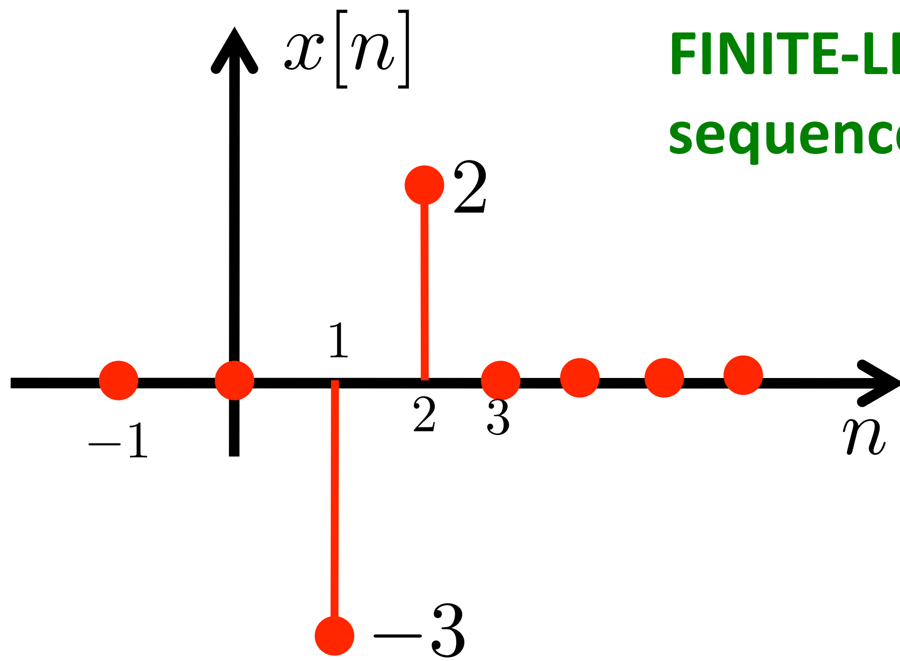
ROC: all the complex plane

except |z|=r=Infinity!

Example 12

$$x[1] = -3, \quad x[2] = 2$$

$$\text{otherwise } 0$$



FINITE-LENGTH sequence

Easy to express with deltas

$$x[n] = -3\delta[n-1] + 2\delta[n-2]$$

Example 12

$$x[n] = -3\delta[n-1] + 2\delta[n-2]$$

$$X(z) = -3z^{-1} + 2z^{-2}$$

$$X(z) = \frac{-3z+2}{z^2}$$

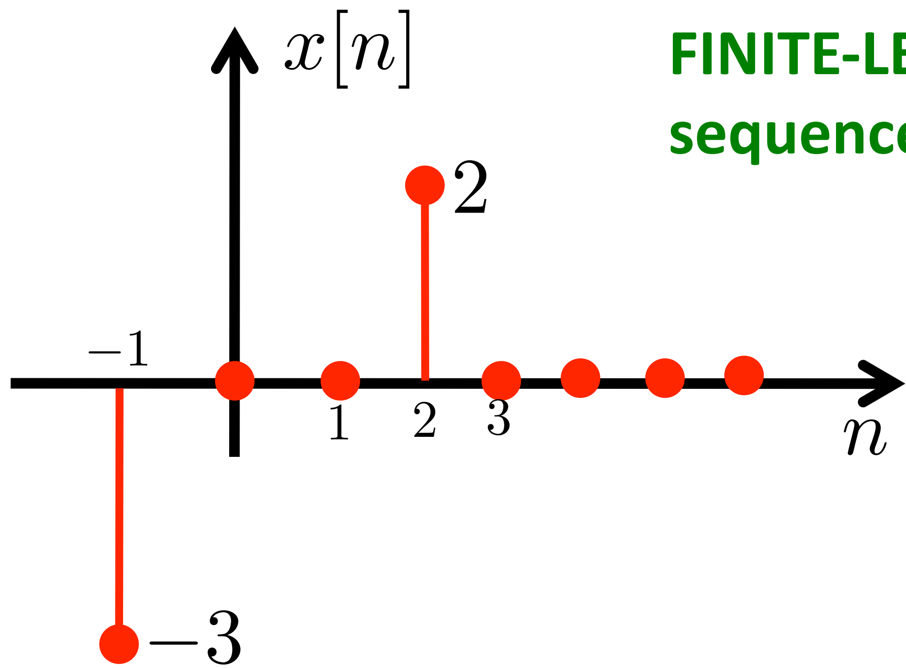
ROC: all the complex plane except z=r=0 (pole multiple of degree 2)!

Infinity is a zero of X(z)…

Example 13

$$x[-1] = -3, \quad x[2] = 2$$

otherwise 0



FINITE-LENGTH sequence

Easy to express with deltas

$$x[n] = -3\delta[n+1] + 2\delta[n-2]$$

Example 13

$$x[n] = -3\delta[n+1] + 2\delta[n-2]$$

$$\Downarrow$$

$$X(z) = -3z + 2z^{-2}$$

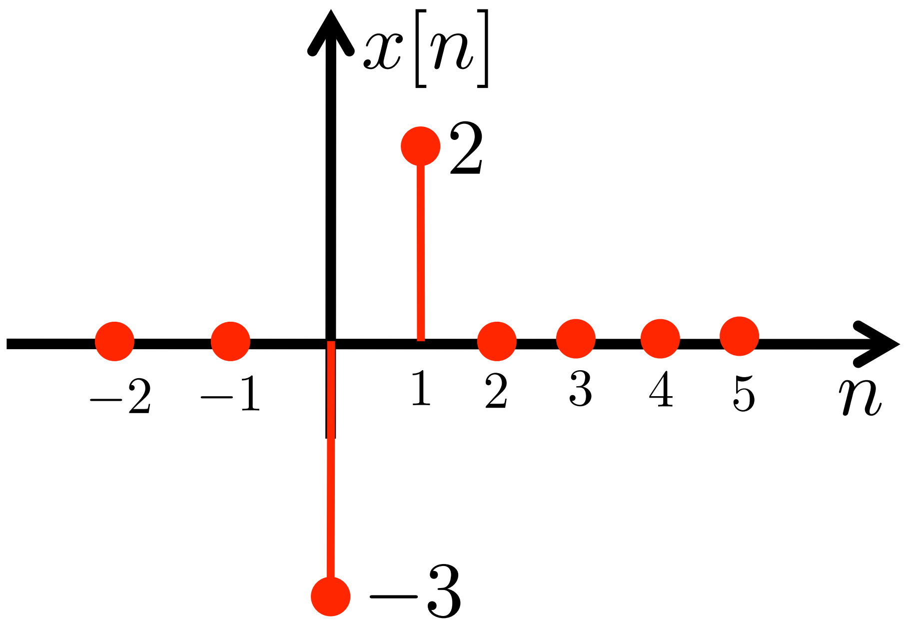
$$X(z) = \frac{-3z^3 + 2}{z^2}$$

ROC: all the complex plane except z=r=0 (pole multiple of degree 2) and |z|=r=Infinity (it is another pole)

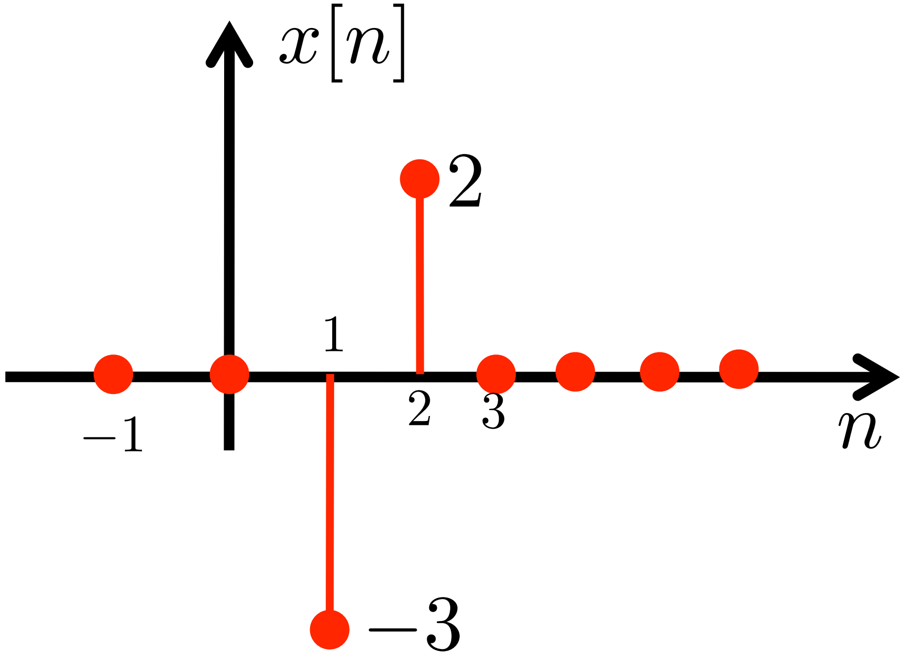
Right-sided sequence-signal

Example of these sequences:

Of the Example 10:

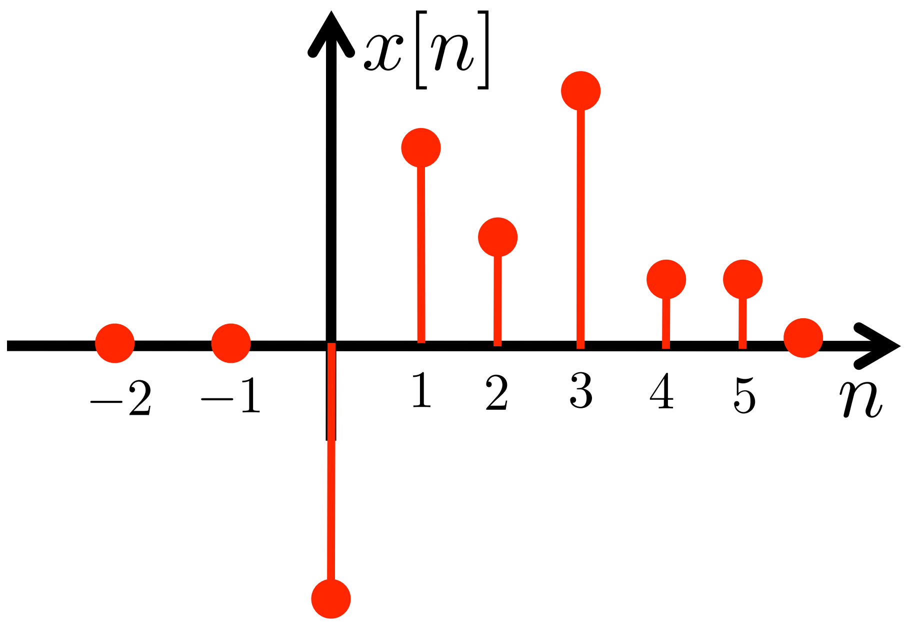


Of the example 12:



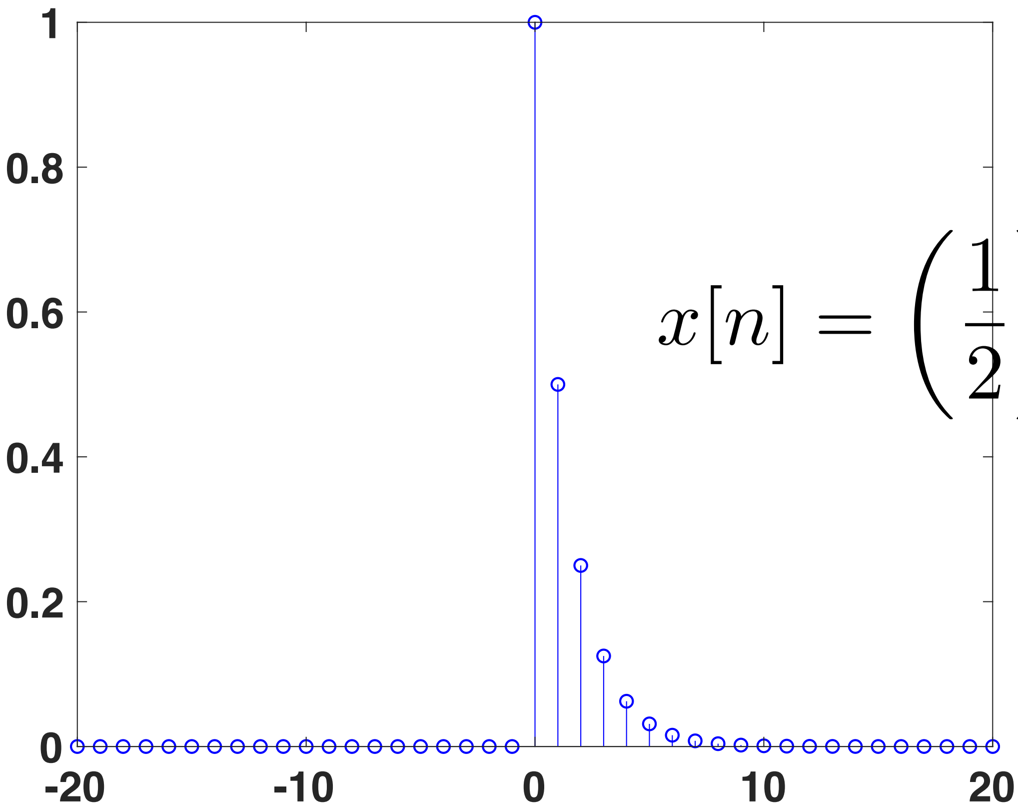
Right-sided sequence-signal

Other examples of right-sided sequences:



Right-sided sequence-signal

Other examples of right-sided sequences, x[n]=(1/2)^n u[n]



$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Right-sided sequence-signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

X(z) and ROC:

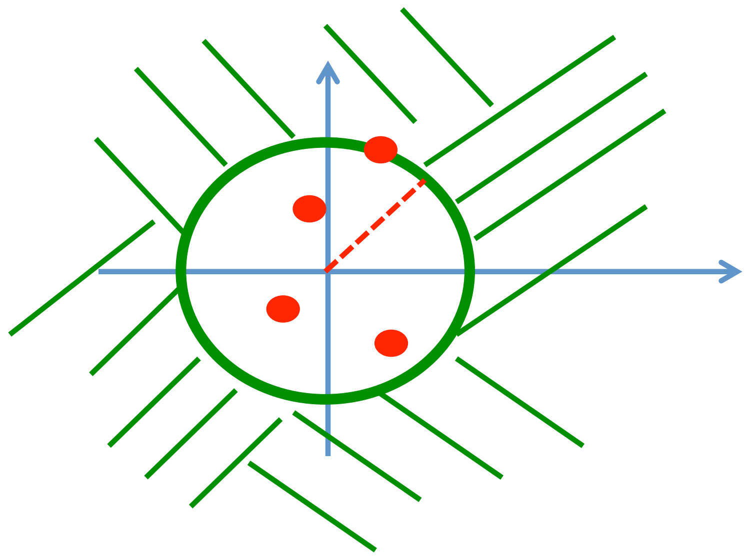
$$X(z) = \frac{z}{z - \frac{1}{2}}$$

$$|z| > \frac{1}{2}$$

$$r > \frac{1}{2}$$

ROCs for right-sided sequences!

Poles in red dots !



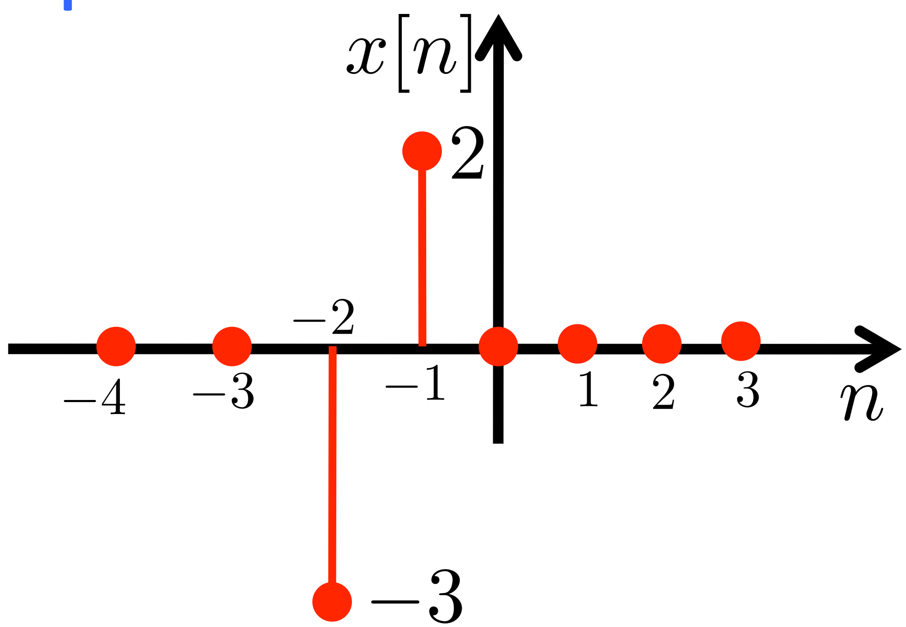
ROC “outside” !

Determined by the pole with biggest module.

LEFT-sided sequence-signal

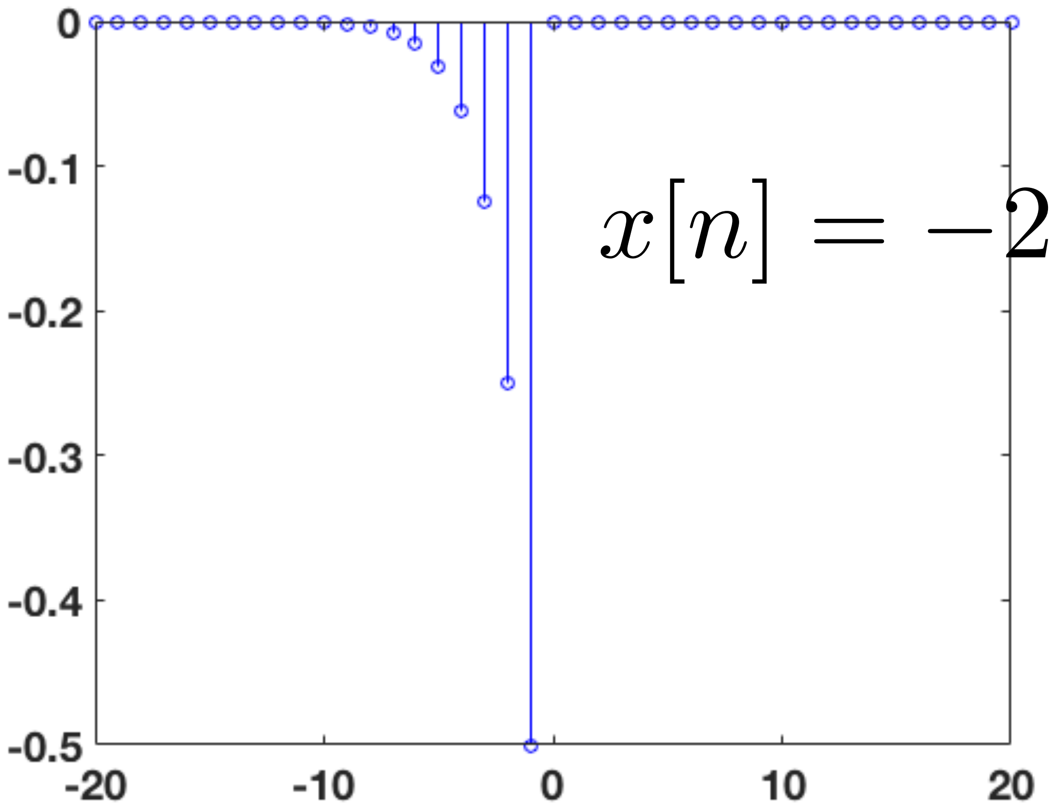
Example of these sequences:

Of the Example 11:



LEFT-sided sequence-signal

Another example of right-sided sequences, x[n]=-2^(n) u[-1-n]



$$x[n] = -2^n u[-n-1]$$

LEFT-sided sequence-signal

$$x[n] = -2^n u[-n-1]$$

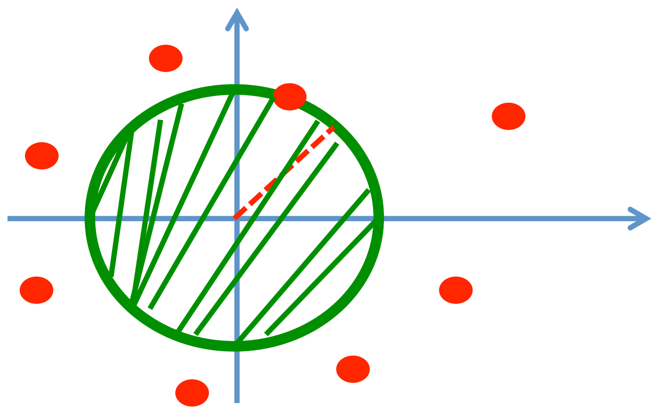
X(z) and ROC:

$$|z| < 2$$

$$X(z) = \frac{z}{z-2}$$

ROCs for LEFT-sided sequences!

Poles in red dots !



ROC "inside" !

Determined by the pole with smallest module.

Recalling "Causality" for an LTI system…

Recall:

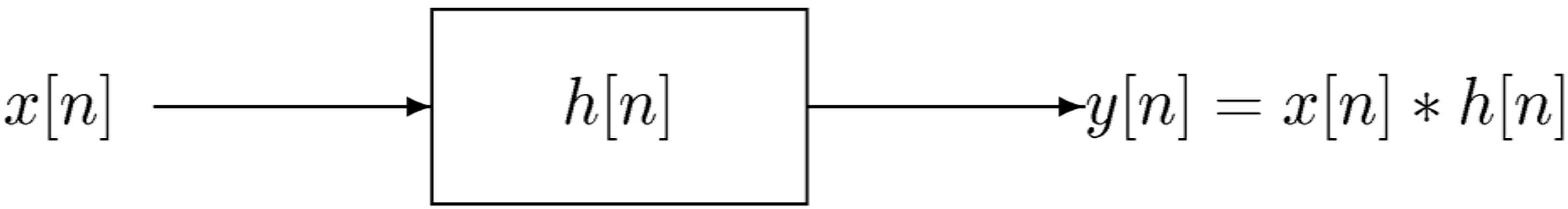
$$h[n] = 0, \quad \forall n < 0$$

h[n] is the impulse response:

It must a right-sided sequence

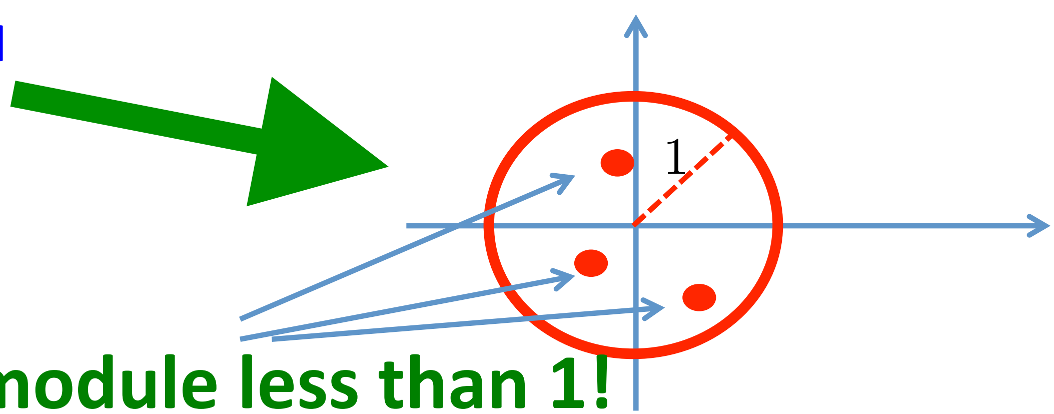
PROPERTY – 7:

STABILITY of the CAUSAL LTI system



All poles within the circle of radius 1 !

Due to the previous properties of right-sided sequences, then:



Poles with module less than 1!

PROPERTY – 8:

CAUSAL LTI system? When M<= N

$$H(z) = \frac{N(z)}{D(z)}$$

polynomials !!!

$$M: \text{ degree of } N(x)$$
$$N: \text{ degree of } D(x)$$

$$M \leq N \longrightarrow h[n] = 0, \quad \forall n < 0$$

PROPERTY – 8:

CAUSAL LTI system? When M<= N

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$\Downarrow \text{ No poles at } \infty, \text{ if } M \leq N$$

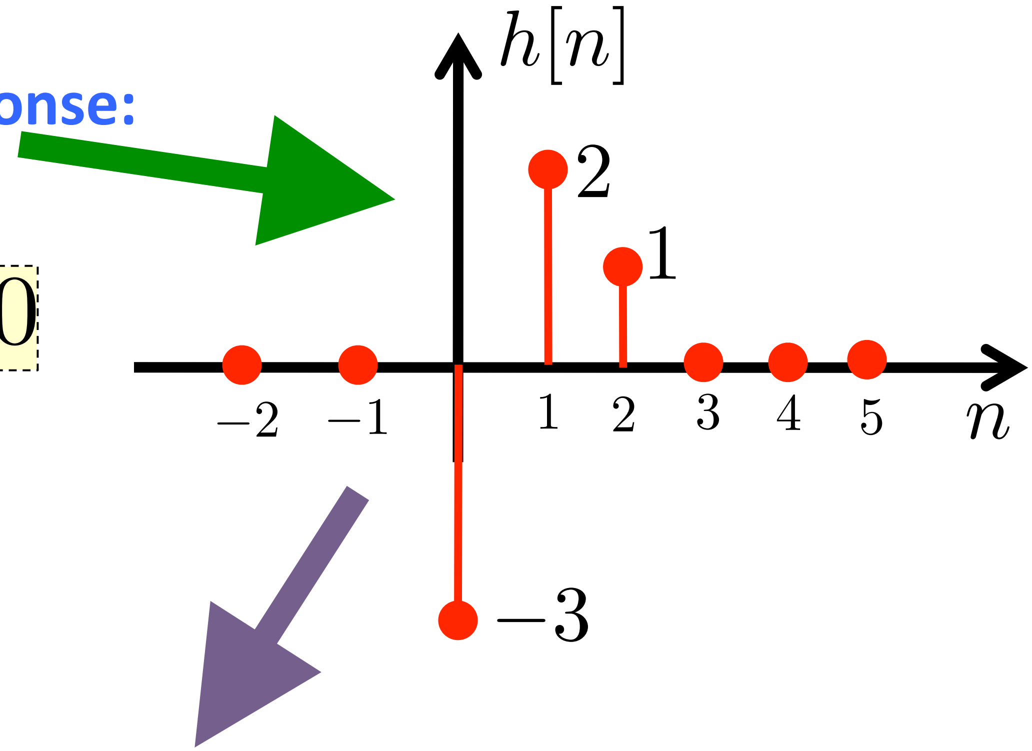
$$M \leq N \longrightarrow h[n] = 0, \quad \forall n < 0$$

Why it is important CAUSAL LTI system?

Consider this impulse response:

Note that:

$$h[n] = 0, \quad \forall n < 0$$



$$h[n] = -3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

Why it is important CAUSAL LTI system?

$$y[n] = x[n] * h[n]$$

$$y[n] = x[n] * (-3\delta[n] + 2\delta[n-1] + \delta[n-2])$$

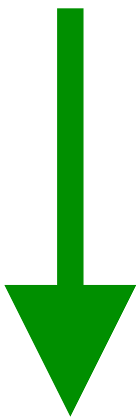
$$y[n] = -3x[n] + 2x[n-1] + x[n-2]$$

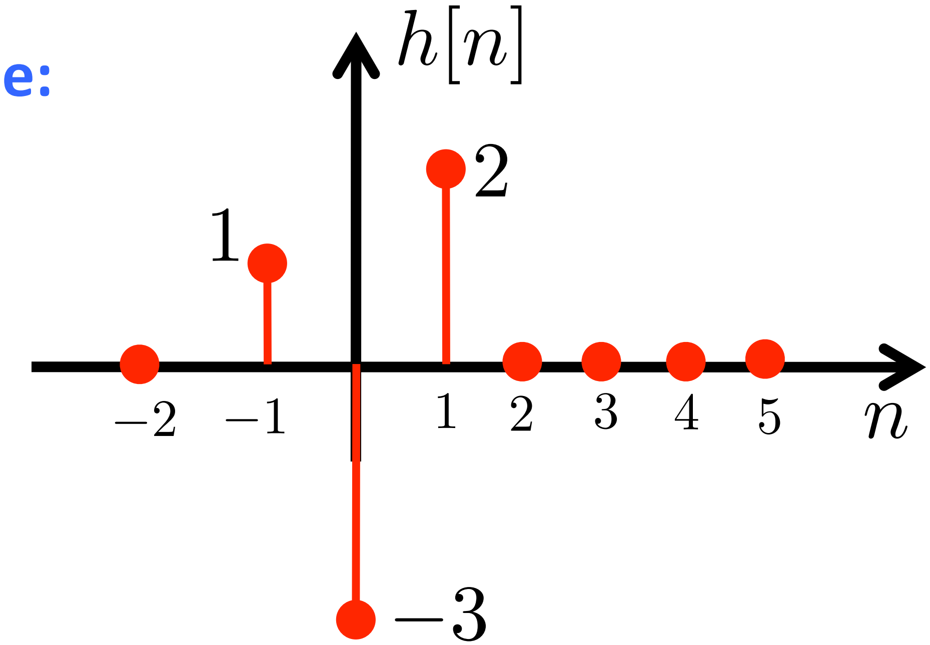
The output of the system ONLY depends on the PRESENT and the PAST of the input !! Not on the future!

EXAMPLE with a NON-CAUSAL LTI system

Consider this impulse response:

Note that:





$$h[n] = -3\delta[n] + 2\delta[n-1] + \delta[n+1]$$

EXAMPLE with a NON-CAUSAL LTI system

$$y[n] = x[n] * h[n]$$

$$y[n] = x[n] * (-3\delta[n] + 2\delta[n-1] + \delta[n+1])$$

$$y[n] = -3x[n] + 2x[n-1] + x[n+1]$$

The output of the system depends ALSO on the FUTURE of the input!!

X(z) of a NON-CAUSAL LTI system

$$h[n] = -3\delta[n] + 2\delta[n-1] + \delta[n+1]$$

↓

$$H(z) = -3 + 2z^{-1} + z$$

$$H(z) = \frac{-3z + 2 + z^2}{z}$$

ROC: all the complex plane, except 0 and Infinity

Note that: $M > N$

$$\text{degree } N(z) > \text{degree } D(z)$$