TOPIC 5
ZETA TRANSFORM
PART 5
“back in time domain”

Inverse Zeta Transform

- Almost never used in practice.
- Synthesis equation.

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

LINE INTEGRAL – CURVE INTEGRAL (in a "close" "circular" curve)

ALTERNATIVES ? ➔

PARTIAL FRACTION DECOMPOSITION (DECOMPOSITION FOR “SIMPLE FRACTIONS” or PARTIAL FRACTION EXPANSION)

We will see JUST FEW special cases…

..even without part-fraction decomposition...

$$X(z) = z^5 + 2z - 1$$

$$\downarrow$$

$$x[n] = \delta[n+5] + 2\delta[n+1] - \delta[n]$$

ROC: all the complex plane, except Infinity....

…other very simple case…

$$X(z) = \frac{z^5 + 2z - 1}{z^3} = z^2 + 2z^{-2} - z^{-3}$$

↓

$$x[n] = \delta[n+2] + 2\delta[n-2] - \delta[n-3]$$

ROC: all the complex plane,
except zero and Infinity….
(are multiple poles? Yes !)

PARTIAL FRACTION DECOMPOSITION

We ONLY consider the case where:

- **Degree Numerator <= Degree Denominator**
- **The poles are all real and different**
- **The poles are “simple” (degree 1)**

PARTIAL FRACTION DECOMPOSITION

$$\frac{P(x)}{Q(x)} \qquad Q(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

deg $P < n$,

$$\frac{P(x)}{Q(x)} = \frac{c_1}{x-\alpha_1} + \frac{c_2}{x-\alpha_2} + \cdots + \frac{c_n}{x-\alpha_n}$$

How to find the constant c_i? →

PARTIAL FRACTION DECOMPOSITION

$$\frac{P(x)}{Q(x)}$$

$$Q(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$$

$\deg P < n,$

$$\downarrow$$

$$\frac{P(x)}{Q(x)} = \frac{c_1}{x - \alpha_1} + \frac{c_2}{x - \alpha_2} + \cdots + \frac{c_n}{x - \alpha_n}$$

How to find the constant c_i? →

Come back to Example 14

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z-\frac{1}{4})(z-\frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

How can we obtain the constant A , B?

$$X(z) = \frac{A(1-\frac{1}{3}z^{-1}) + B(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

Come back to Example 14

$$X(z) = \frac{A(1 - \frac{1}{3}z^{-1}) + B(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$X(z) = \frac{A - A\frac{1}{3}z^{-1} + B - B\frac{1}{4}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$X(z) = \frac{A + B + (-A\frac{1}{3} - B\frac{1}{4})z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

Come back to Example 14

$$X(z) = \frac{A + B + (-A\frac{1}{3} - B\frac{1}{4})z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

We desire that $$= \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$\begin{cases} A + B = 3 \\ -\frac{A}{3} - \frac{B}{4} = -\frac{5}{6} \end{cases}$$

Come back to Example 14

$$\begin{cases} A + B = 3 \\ -\frac{A}{3} - \frac{B}{4} = -\frac{5}{6} \end{cases}$$

$\downarrow$

$$A = 1$$

$$B = 2$$

Example 14 - problem

$$X(z)=\frac{3z^2-\frac{5}{6}z}{(z-\frac{1}{4})(z-\frac{1}{3})}=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}=\frac{A}{1-\frac{1}{4}z^{-1}}+\frac{B}{1-\frac{1}{3}z^{-1}}$$

We obtain the values: A = 1, B = 2

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$x[n] = x_1[n] + x_2[n]$$

Example 14 - problem

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$x[n] = x_1[n] + x_2[n]$$

We are able to invert these ZT (depending on the ROC !!!) since we have already seen in the first two examples of ZT we did….

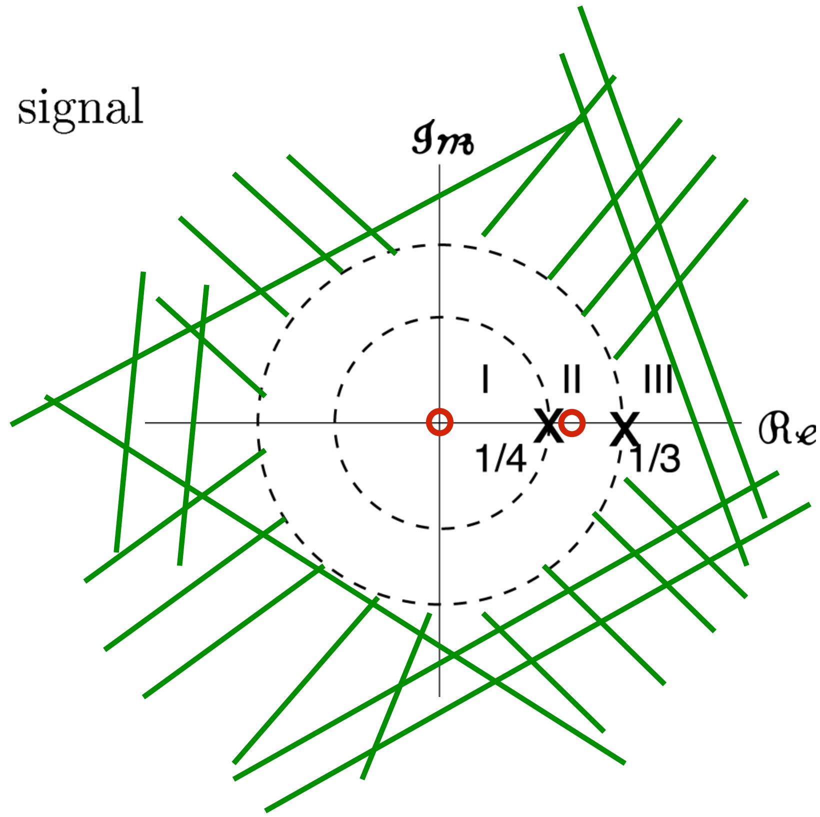
Example 14 - problem

3 POSSIBLE SIGNALS IN TIME !!! 3 POSSIBLE SOLUTIONS

ROC III: $|z| > \frac{1}{3}$ - right-sided signal

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2 \cdot \left(\frac{1}{3}\right)^n u[n]$$



Example 14 - problem

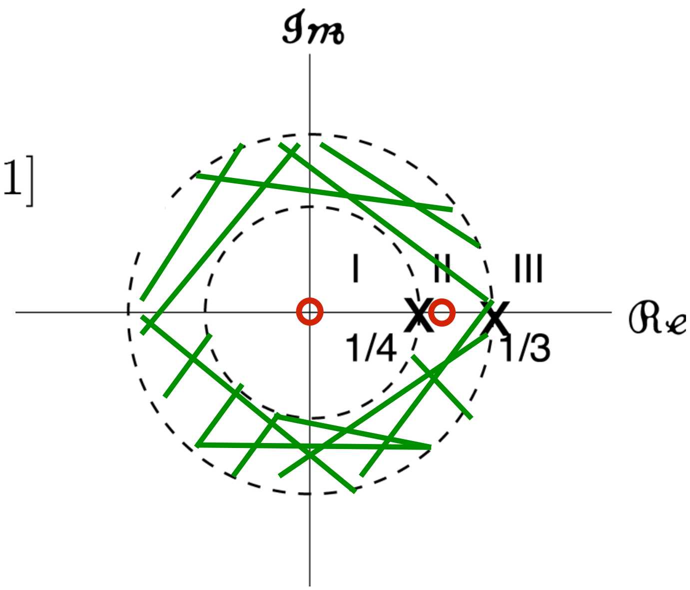
3 POSSIBLE SIGNALS IN TIME !!! 3 POSSIBLE SOLUTIONS

ROC II:

$$\frac{1}{4} < |z| < \frac{1}{3} \qquad \text{- two-sided signal}$$

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$



Example 14 - problem

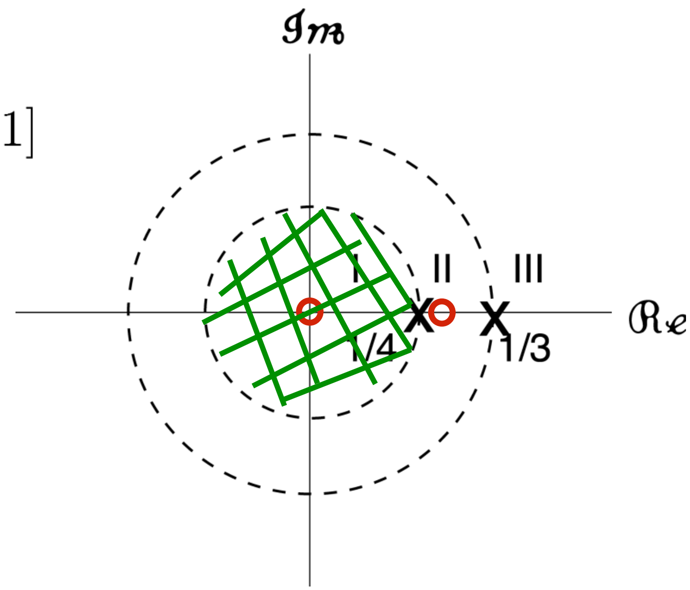
3 POSSIBLE SIGNALS IN TIME !!! 3 POSSIBLE SOLUTIONS

ROC I:

$$|z| < \frac{1}{4} \quad \text{- left-sided signal}$$

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$



Example 14.5

Considering AGAIN the following ZT :

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z - \frac{1}{4})(z - \frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}$$

But in this case we assume the ROC:

$$|z| > \frac{1}{3}$$

How many possible signal x[n] we can have?

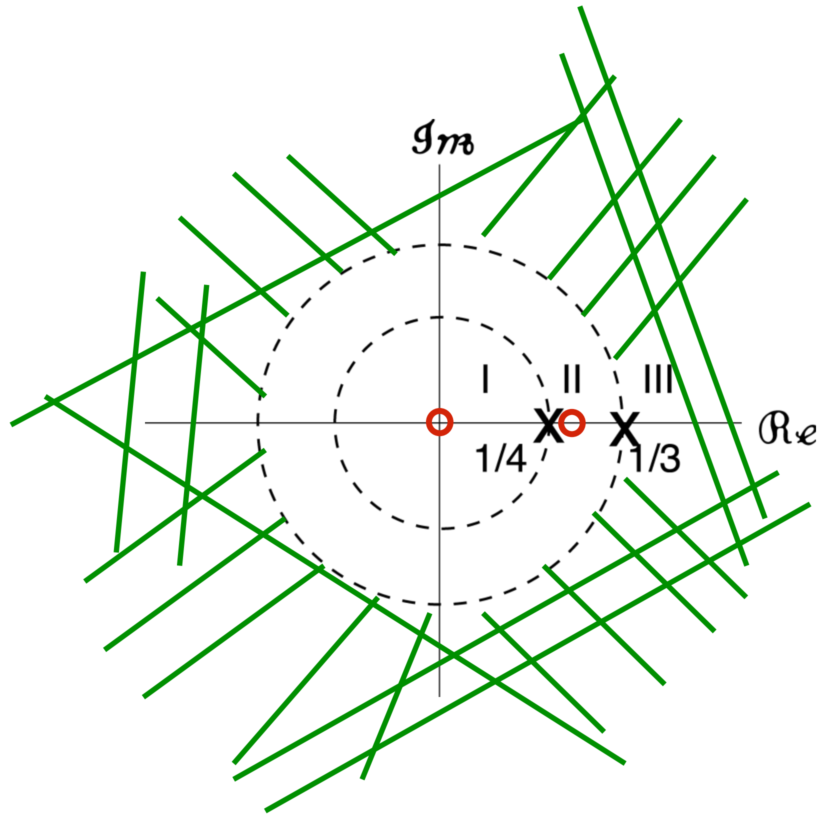
Example 14.5

ONLY ONE POSSIBLE SOLUTION !!!

$$|z| > \frac{1}{3} \quad \text{- right-sided signal}$$

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2 \cdot \left(\frac{1}{3}\right)^n u[n]$$



Example 14.6

Considering the signal in the time domain:

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

How many possible Zeta Transforms we can have?

Example 14.6

ONLY ONE !!!

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z - \frac{1}{4})(z - \frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

With ROC:

$$|z| > \frac{1}{3}$$