TOPIC 5
ZETA TRANSFORM
PART 6

If Zeta Transform H(z) is rational (fraction of polynomials) with real coefficients, then…

The zeros and poles are real or complex conjugates (including infinity)

Coeficientes de $H(z)$ reales $\Rightarrow$ ceros y polos reales ó complejos conjugados

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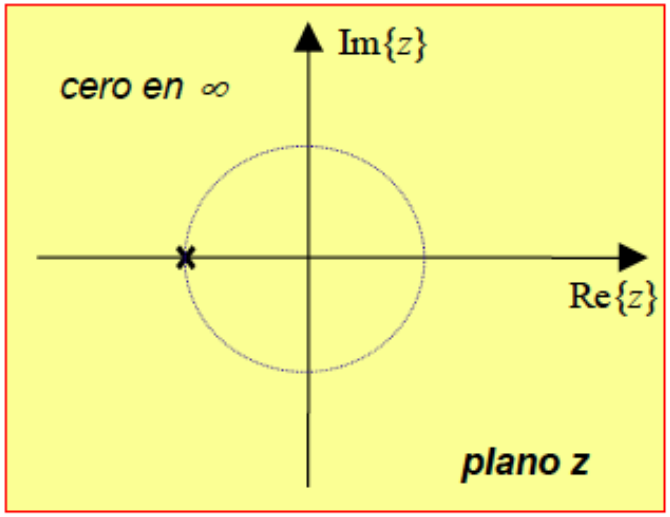
Example:

$$H(z) = \frac{1}{1+z}$$

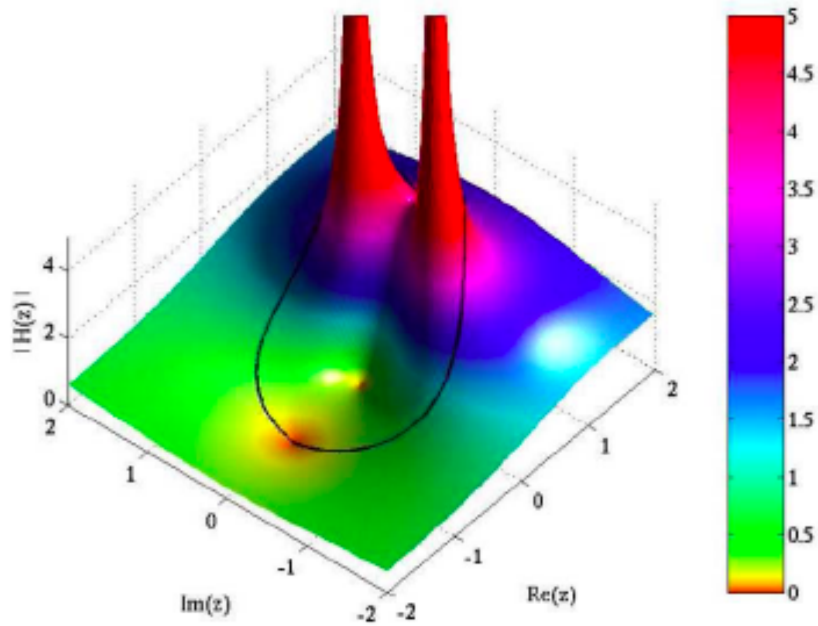


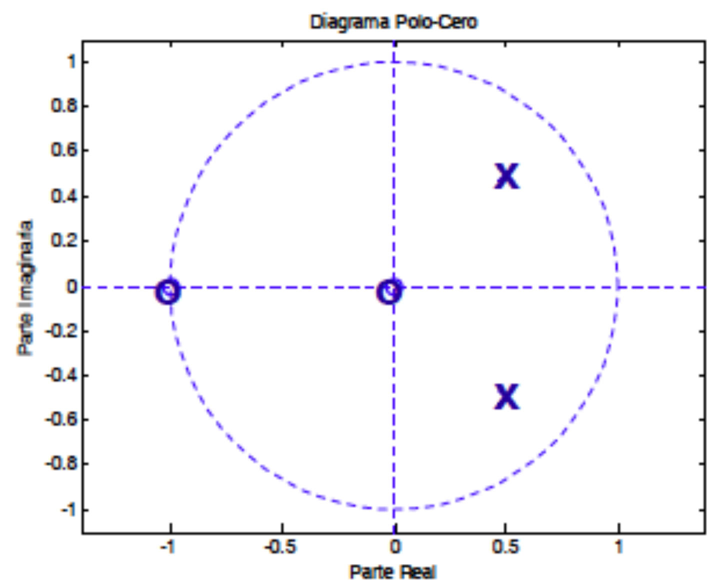
Examples

$$H(z) = \frac{1 + z^{-1}}{1 + z^{-1} + 0.5z^{-2}}$$

c_1=0; c_2=–1;

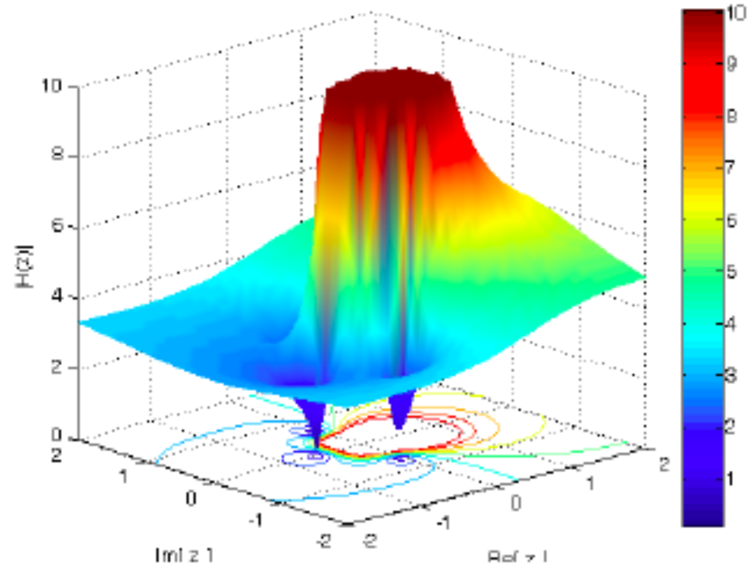
$p_{1,2}$=0.5(1± j)

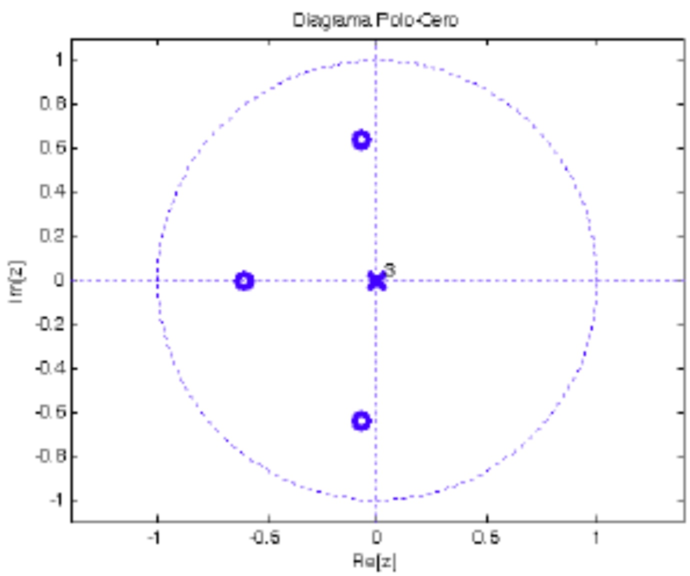


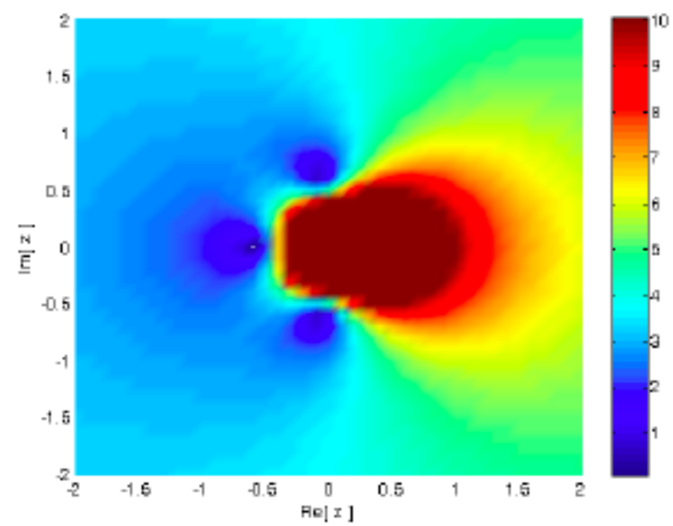


Examples

$$H(z) = \frac{4z^3 + 3z^2 + 2z + 1}{z^3}$$







Examples of ZT with ROCs

Señal	Transformada	ROC
$\delta[\mathrm{n}]$	1	$\forall z$
$u[\mathrm{n}]$	$\frac{1}{1-z^{-1}}$	z|>1
$\delta[\mathrm{n-m}]$	z^{-m}	$\forall z - \{0, \infty\}$
$a^n u[\mathrm{n}]$	$\frac{1}{1-az^{-1}}$	z|>|a|
$na^n u[\mathrm{n}]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	z|>|a|
$[\cos(\omega_o n)]u[n]$	$\frac{1-[\cos(\omega_o)]z^{-1}}{1-[2\cos(\omega_o)]z^{-1}+z^{-2}}$	z|>1
$[\mathrm{sen}(\omega_o n)]u[n]$	$\frac{1-[sen(\omega_o)]z^{-1}}{1-[2\cos(\omega_o)]z^{-1}+z^{-2}}$	z|>1
$[r^n\cos(\omega_o n)]u[n]$	$\frac{1-[r\cos(\omega_o)]z^{-1}}{1-[2r\cos(\omega_o)]z^{-1}+r^2z^{-2}}$	z|>r
$[r^n sen(w_o n)]u[n]$	$\frac{1-[rsen(w_o)]z^{-1}}{1-[2r\cos(w_o)]z^{-1}+r^2z^{-2}}$	|z|>r

Other example of problem "back in time"

$$X(z) = \frac{1}{\left(1-az^{-1}\right)\left(1-bz^{-1}\right)}$$

$$= \frac{a^{-1}b^{-1}}{\left(z^{-1}-a^{-1}\right)\left(z^{-1}-b^{-1}\right)}$$

$$= \left(\frac{a}{a-b}\right)\frac{1}{\left(1-az^{-1}\right)} + \left(\frac{b}{b-a}\right)\frac{1}{\left(1-bz^{-1}\right)}$$

La TZ^{-1} depende de la ROC_X. Si suponemos que $|a|>|b|$, entonces:

$$ROC_X = \begin{cases} |z|>|a|>|b| & x[n] = \left(\frac{a}{a-b}\right)a^n u[n] + \left(\frac{b}{b-a}\right)b^n u[n] & \textit{(sec. derecha)} \\ |a|>|z|>|b| & x[n] = \left(\frac{b}{b-a}\right)b^n u[n] - \left(\frac{a}{a-b}\right)a^n u[-n-1] & \textit{(sec. bilateral)} \\ |a|>|b|>|z| & x(n) = -\left(\frac{a}{a-b}\right)a^n u[-n-1] - \left(\frac{b}{b-a}\right)b^n u[-n-1] & \textit{(sec. izquierda)} \end{cases}$$

There are 3 possible sequences in time !!

Recalling the relationship with Fourier Transform and more clarifications…

Setting r=1

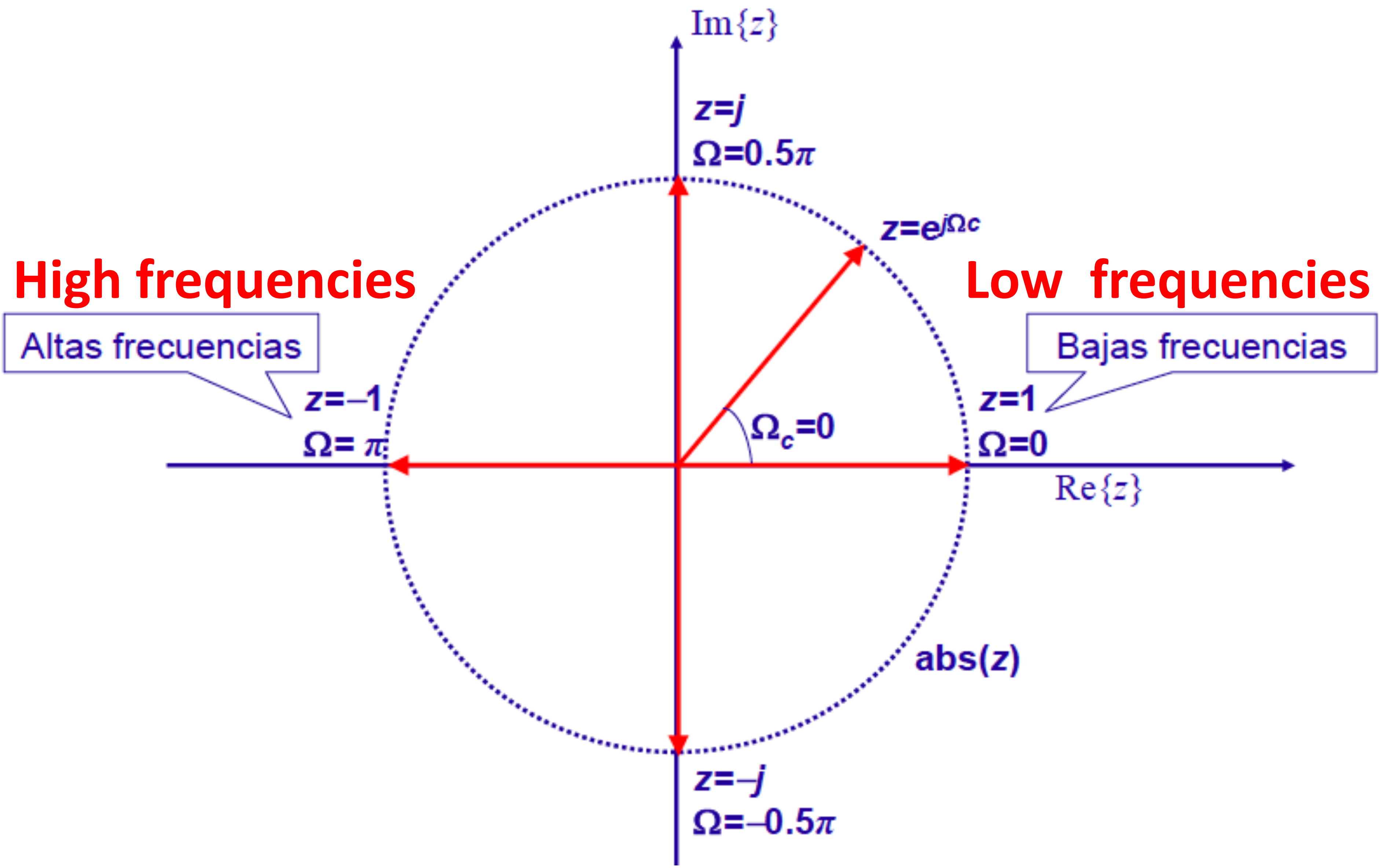
Si la expresión algebraica de la función de sistema es de tipo racional, podemos expresar en función de los polos p_i y los ceros c_i
Suponiendo raíces simples:

$z = e^{j\Omega}$

$$H(z)=\frac{N(z)}{D(z)}=k\frac{\prod_i (z-c_i)}{\prod_i (z-p_i)}=k\frac{\prod_i (1-c_i z^{-1})}{\prod_i (1-p_i z^{-1})}\Rightarrow$$

$$H(\Omega)=\frac{N(\Omega)}{D(\Omega)}=k\frac{\prod_i (1-c_i\,\mathrm{e}^{-j\Omega})}{\prod_i (1-p_i\,\mathrm{e}^{-j\Omega})}=k\frac{\prod_i (\mathrm{e}^{j\Omega}-c_i)}{\prod_i (\mathrm{e}^{j\Omega}-p_i)}$$

$$|H(\Omega)|=|k|\left|\frac{\prod_i (\mathrm{e}^{j\Omega}-c_i)}{\prod_i (\mathrm{e}^{j\Omega}-p_i)}\right|=|k|\frac{\prod_i |\mathrm{e}^{j\Omega}-c_i|}{\prod_i |\mathrm{e}^{j\Omega}-p_i|}$$



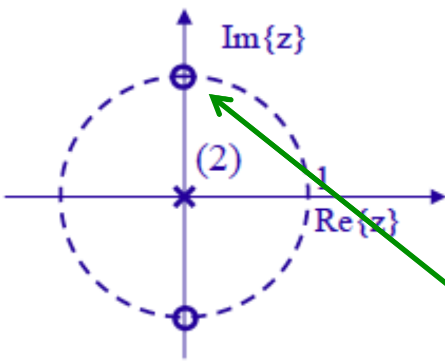
Im{z}
z=j
Ω=0.5π
z=e^jΩc
High frequencies
Low frequencies
Altas frecuencias
Bajas frecuencias
z=–1
Ω= π
Ω_c=0
z=1
Ω=0
Re{z}
abs(z)
z=–j
Ω=–0.5π

Example

2 poles at z=0 !

1 zero at z=j and other zero at z=-j

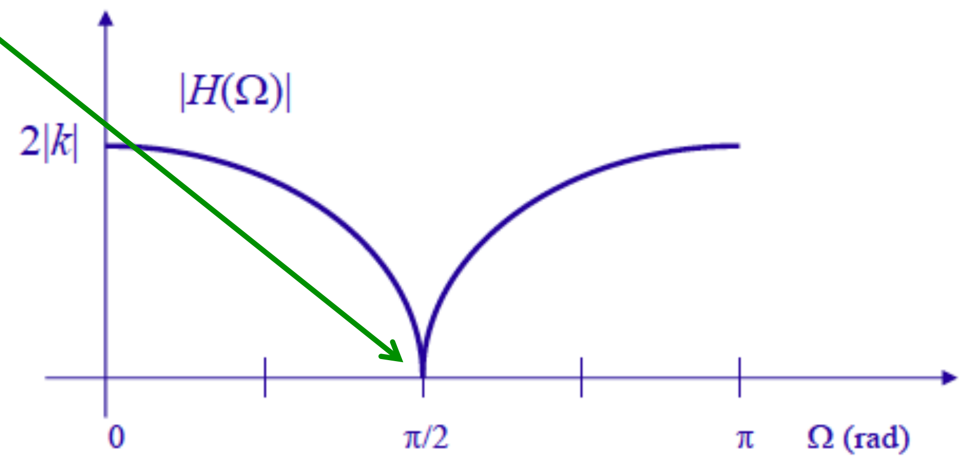
Supongamos el siguiente diagrama de polos y ceros:



$$H(z) = k\frac{\prod_i \left(1 - c_i z^{-1}\right)}{\prod_i \left(1 - p_i z^{-1}\right)} = k\frac{\left(1 - jz^{-1}\right)\left(1 + jz^{-1}\right)}{1^2} \Rightarrow$$

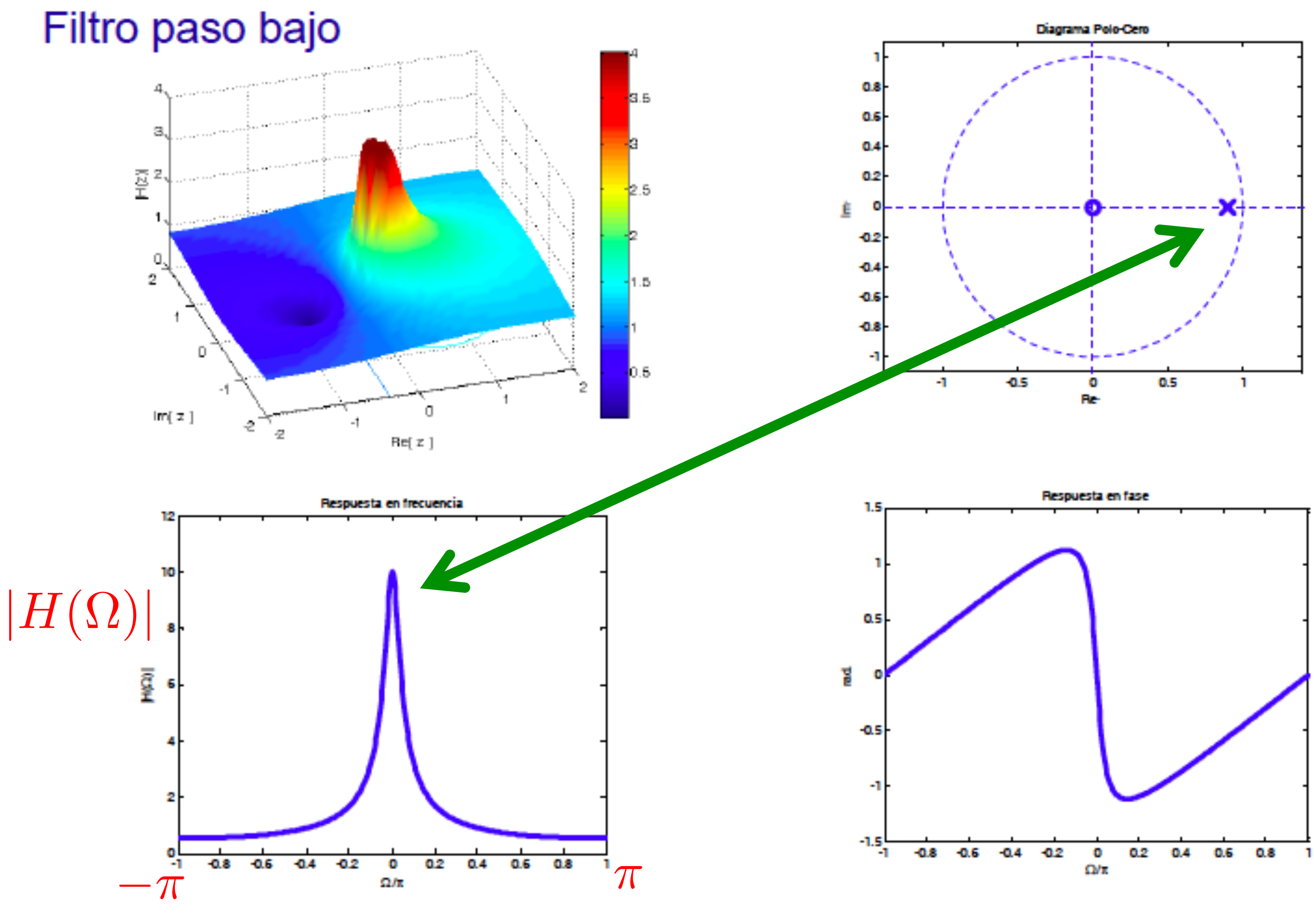
$$H(\Omega) = k\left(1 - j\,\mathrm{e}^{-j\Omega}\right)\left(1 + j\,\mathrm{e}^{-j\Omega}\right) = k\left(1 + \mathrm{e}^{-j2\Omega}\right)$$

Como $H(\Omega)$ es periódica de periodo 2π, basta dibujarla entre 0 y 2π. Para $h[n]$ real (polos y ceros reales o pares complejos conjugados) $|H(\Omega)|$ tiene simetría par, entonces basta dibujar entre 0 y π :

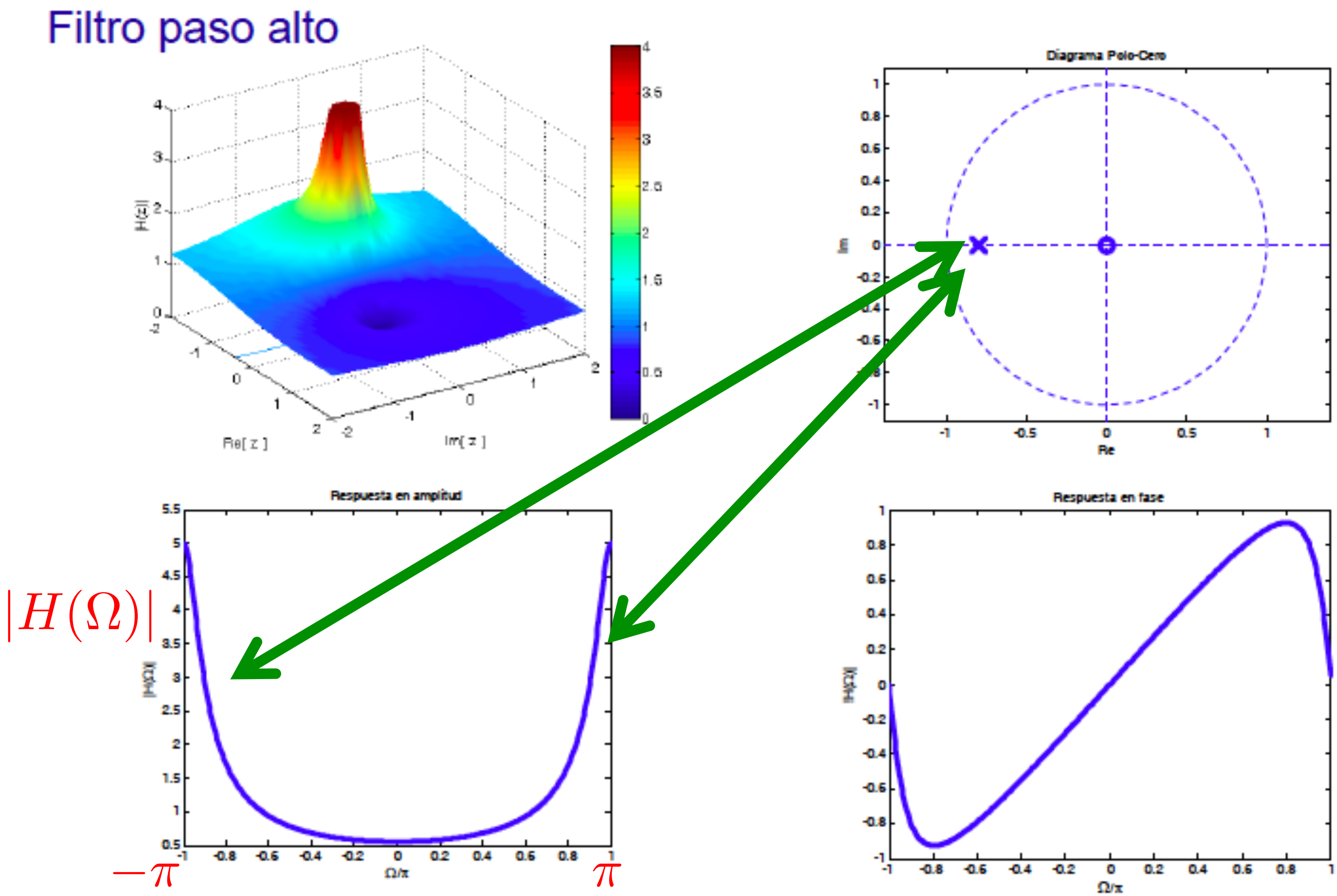


We have to analyze the distances of poles and zeros with the circle |r|=1…

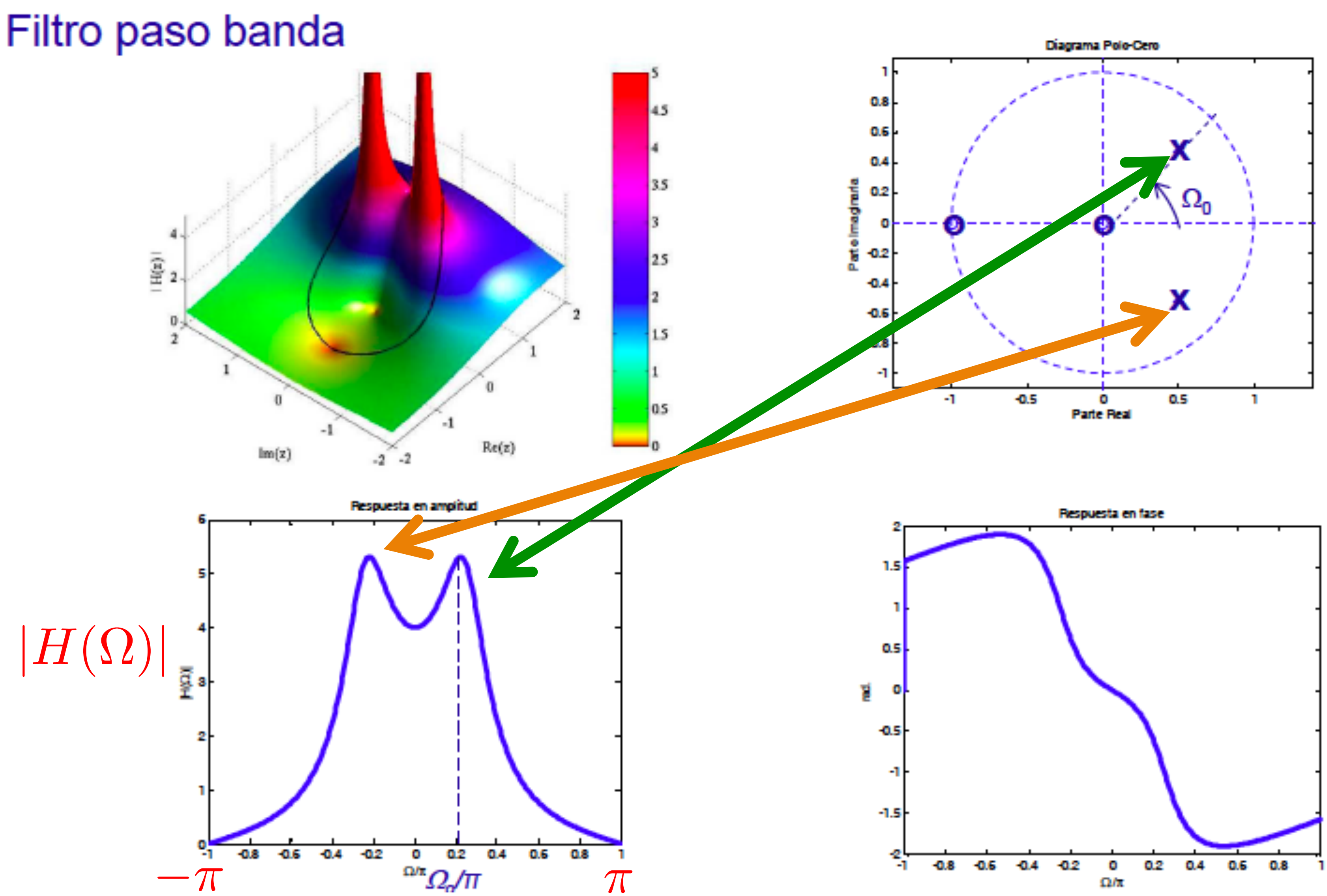
Example: low-pass filter



Example: high-pass filter



Example: band-pass filter

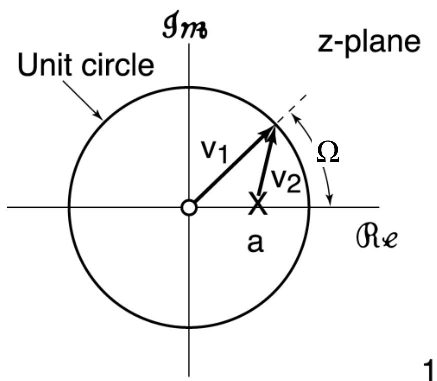


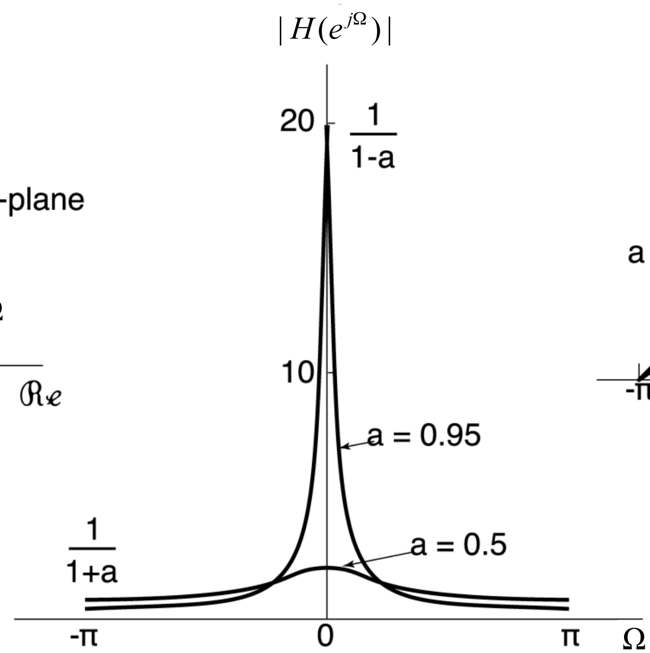
Example again low-pass filter

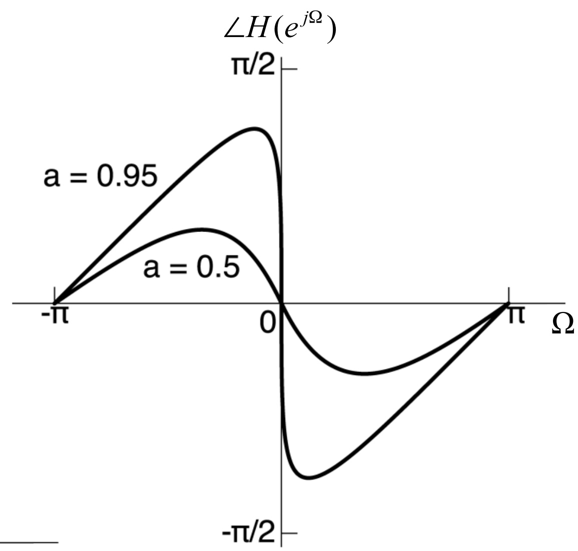
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad , \quad |z| > |a|$$

$$h[n] = a^n u[n] \quad , \quad |a| < 1$$

$$H(\omega) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$







$$H(\Omega) = \frac{v_1}{v_2}, \; |H(\Omega)| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}, \quad \angle H(\Omega) = \angle v_1 - \angle v_2 = \Omega - \angle v_2$$

Other examples
in order to clarify more…

For a better understanding…

Let us consider the signal of type:

$$x[n] = a^{\beta n} u[n - n_0]$$

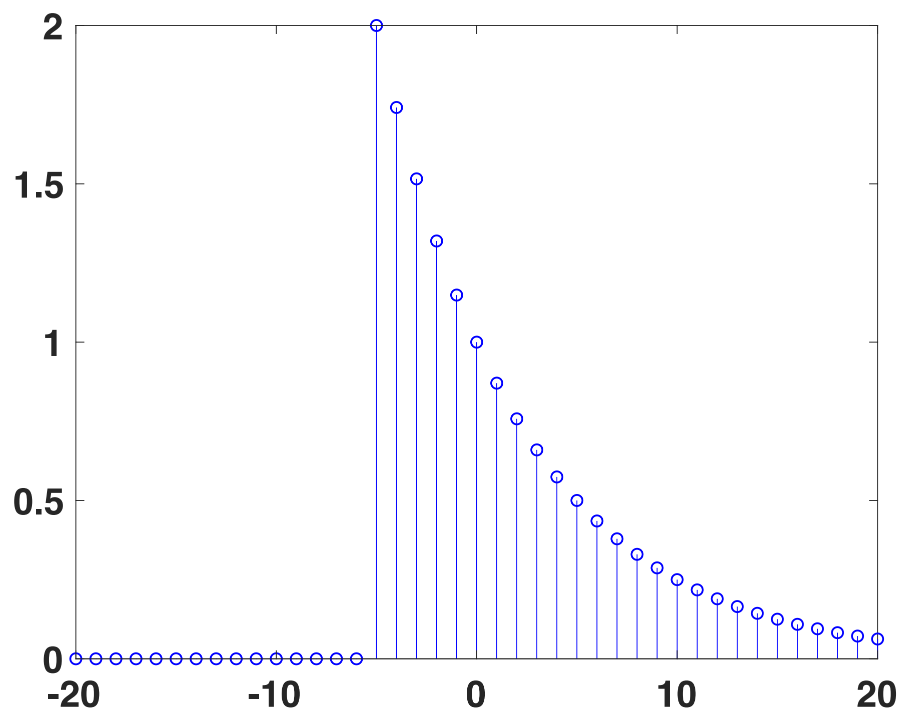
For a better understanding…

$a > 1; \quad a = 2$

$\beta < 0; \quad \beta = -0.2$

$n_0 < 0; \quad n_0 = -5$

$$x[n] = 2^{-0.2n} u[n+5]$$



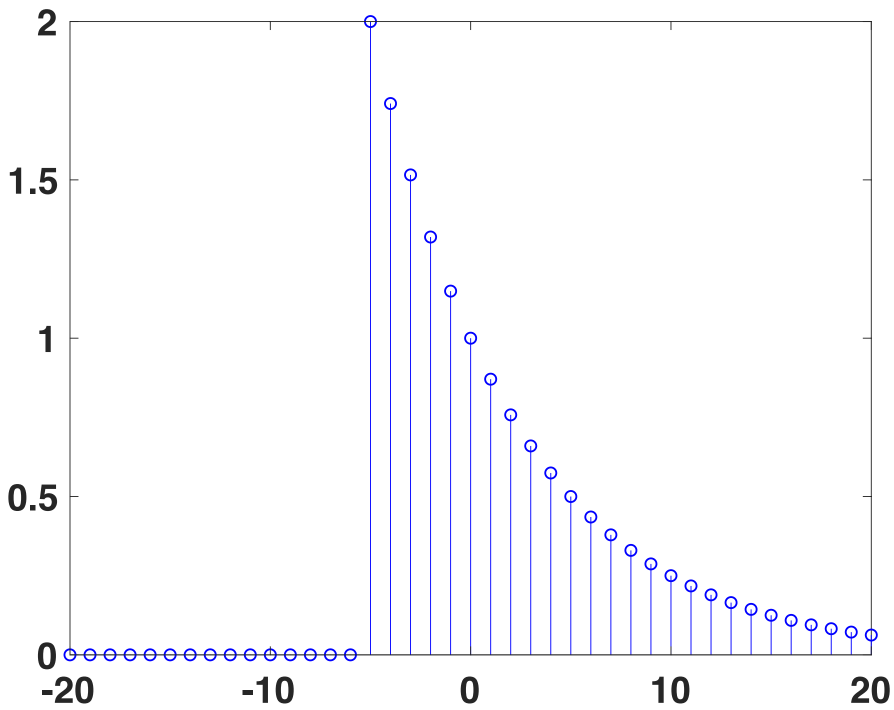
For a better understanding...

$0 < \alpha \leq 1; \quad \alpha = 1/2$

$n_0 < 0; \quad n_0 = -5$

$\beta > 0; \quad \beta = 0.2$

$$x[n] = (1/2)^{0.2n} u[n+5]$$



The same of before !

For a better understanding...

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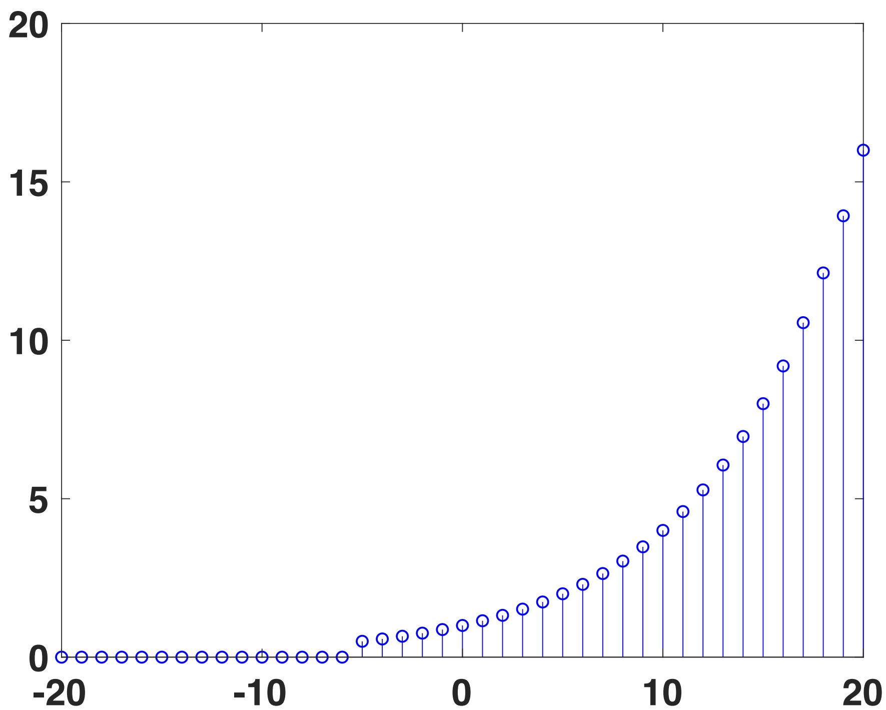

For a better understanding…

$$x[n] = 2^{0.2n}u[n+5]$$

Does the FT exist?

Does the ZT Exist?