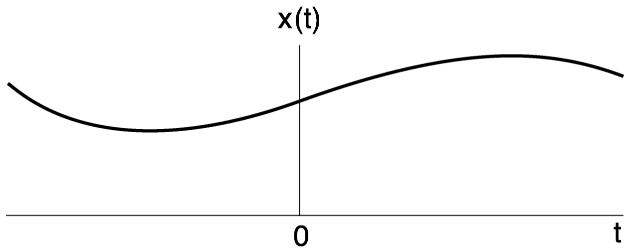
Outline of the course

- Tema 1: Discrete signals and systems in the temporal domain
  - 1.1 Recall of the signals and systems in the continuous time
  - 1.2 Signals in discrete time
  - 1.3 Systems in discrete time
  - 1.4 Convolution in the discrete time
- Tema 2: Discrete signals and systems in frequency domain
- Tema 3: Sampling
- Tema 4: Discrete Fourier Transform
- Tema 5: Zeta transform
- Tema 6: Introduction to the design of the discrete filters

- Comments:
  - Discrete signals can be easily used with computers/digital machines, processors
  - Please, study/recall the geometric series and the arithmetic series

1.1 Recall of signals and systems in continuous time

- ¿What is a signal?
  - Is a ``mathematical model” (a function) which represents a variable of interests, that changes with the time.
  - Examples of signals: radio, volts, temperature, ...

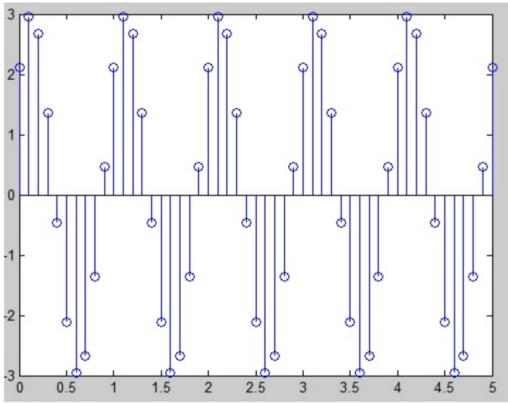


signals and systems in continuous time (CT): where *t takes continuous values*

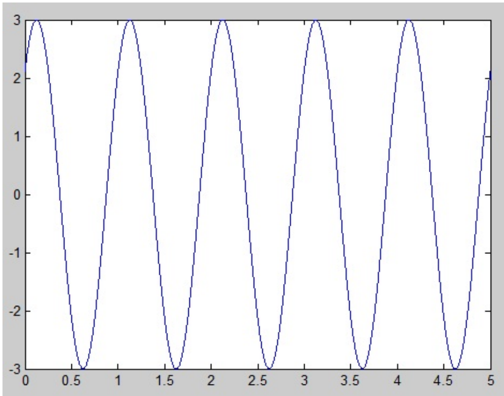
- One-dimensional (temperature in a place) vs. Multidimensional (for instance, an image)

1.1 Recall of signals and systems in continuous time

- Continuous signals vs. discrete signals
  - Continuous: defined for any real values.
    - Example: voice.
  - Discrete: defined for only for certain time values.
    - Example: final prize of stocks (in a stock market), every day.



Discrete Signal

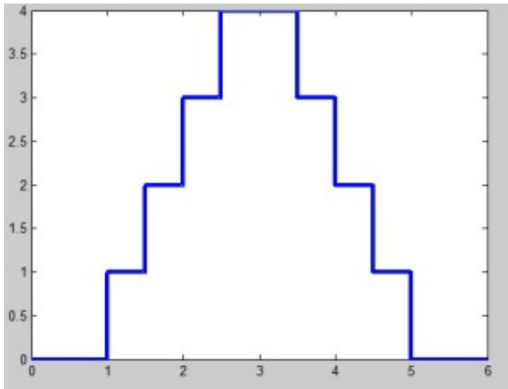


Continuous Signal

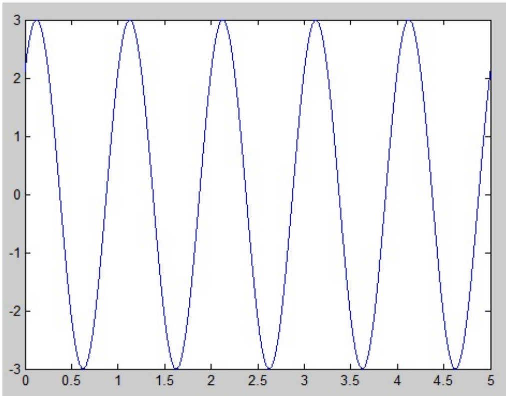
REMARK: here we just look the x-axis…

1.1 Recall of signals and systems in continuous time

- Digital Signals vs. Analog Signals
  - Digital Signals: take only certain values (a finite number of values, in general) within an interval of time.
  - Analog Signals: can take a (infinite) number of continuous values in a (bounded or unbounded) interval.



Señal digital



Señal analógica

1.1 Recall of signals and systems in continuous time

- Deterministic Signals vs. Stochastic Signals
  - Stochastic Signals: contain randomness
  - the definitions, formulas and the treatment change

Operations with Signals

- What can we do?
  - Any mathematical operation.

- Examples:
  - Level/amplitude change: $\Longrightarrow Ax(t)$
  - Translation: $\Longrightarrow x(t-t_0)$
  - Time inversion: $\Longrightarrow x(-t)$
  - Change of scale: $\Longrightarrow x(at)$
  - Derivation $\Longrightarrow \frac{dx(t)}{dt}$
  - integration $\Longrightarrow \int_0^t x(\tau)d\tau$

Special cases of signals

- Real and Complex signals
  - Complex signal

$$x(t) = x_r(t) + jx_i(t)$$

Example of a complex signal: $y(t) = e^{j0.3t} = cos(0.3t) + jsin(0.3t)$

Example of a real signal: $x(t) = cos(0.25t)$

- Odd and even signals: both real signal such that

$$x_e(t) = x_e(-t)$$

$$x_o(t) = -x_o(-t)$$

- Generally, we can write

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Special cases of signals

- Hermitian and anti-hermitian signals:
  - Hermitian signals (if real, then is an even signal):

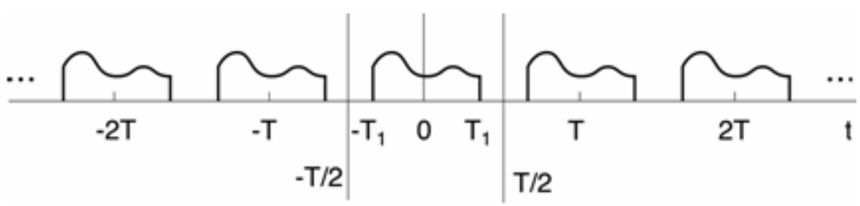
$$x(t) = x^*(-t)\ \forall t$$

  - anti-hermitian signals (if real, then is an odd signal):

$$x(t) = -x^*(-t) \forall t$$

- Periodic signals:

$$CT:\ \exists\, T > 0 \mid x(t) = x(t+T),\ \forall t$$



T is the *fundamental period*; note that the signal is repeating each 2T, 3T, …

Question: if we sum different periodic signals (with different period), the obtained signal is periodic? And if we multiply different periodic signals?

Special cases of signals

- Complex Exponential (CE)

$$x(t) = ce^{st} = re^{j\theta}e^{\sigma t}e^{jw_0 t} = re^{\sigma t}e^{j(w_0 t+\theta)}$$

real

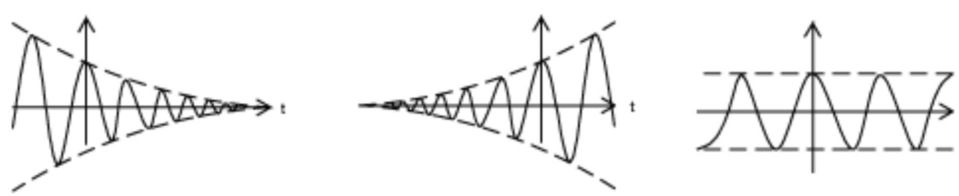
Amplitude

complex

$$c = re^{j\theta}$$
$$s = \sigma + jw_0 \qquad w_0 = 2\pi f$$

Euler Formula: $e^{jw_o t} = cos(w_o t) + jsin(w_o t)$

$$\Re_e\{x(t)\} = re^{\sigma t}cos(w_0 t + \theta)$$

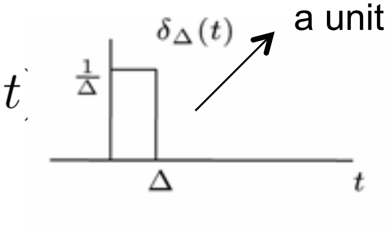


σ<0 σ>0 σ=0

Special cases of signals: delta and step

- Dirac delta or impulse: $\delta(t)$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t$$



Note it has a unit area!

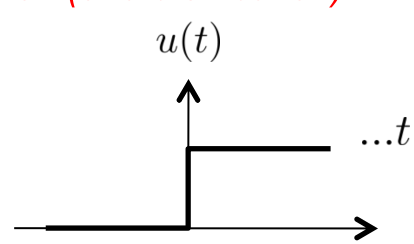
- Properties:

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

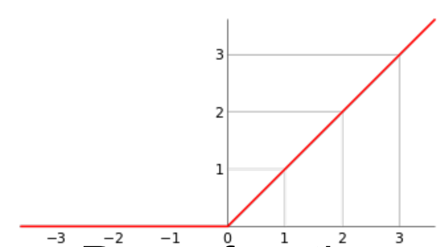
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

- *Note that the <u>Dirac delta in t=0 diverges (takes the values “infinite”);</u> it is not a stirctly function, it is a generalized function (or a distribution)*

- Heaviside step function *u*(*t*):



$$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$



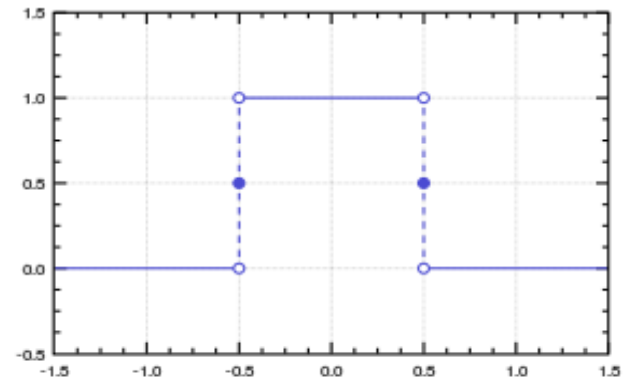
Ramp function

$$u(t) = \int_{-\infty}^{t} \delta(x)dx \qquad u(t) = \frac{d}{dt}\max\{t, 0\}$$

Special cases of signals: rect and sinc

- Unit Rectangle: $p(t)$

$$p(t) = \begin{cases} 1, & \text{si } |t| < 1/2 \\ 0, & \text{si } |t| > 1/2 \end{cases}$$



  - Without unit area: $p_T(t) = p(t/T)$

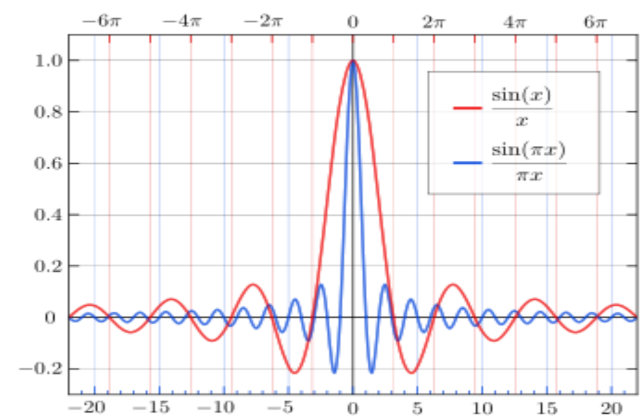
$$p_T(t) = \begin{cases} 1, & \text{si } |t| < T/2 \\ 0, & \text{si } |t| > T/2 \end{cases}$$

- Sinc function: $sinc(t)$

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

  - Without unit area:

$$sinc_T(t) = sinc(t/T) = \frac{sinc(\pi t/T)}{\pi t/T}$$



Zeros at the multiples of 1 (or T) !!!!

* IMPORTANT REMARK: The zeros are at multiples of T !!!

Properties of a signal

- **Mean value** (formulas for a deterministic signal): the definition depends if the signal is periodic or not, etc.
- Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For non-periodic signals

- Mean Power:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

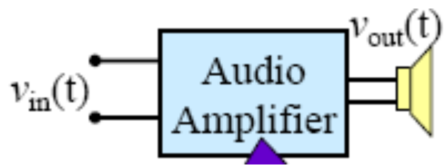
➔ Energy in a unit of time

- Recall:
  - There are *power signals* and *energy signals*:
    - Finite Energy ➔ then the power is zero ➔ energy signal
    - Finite Power ➔ then the energy is infinite ➔ power signal
    - Some signals are neither energy nor power signals.
    - For a periodic signal: if the energy in one period is finite, then it is a power signal

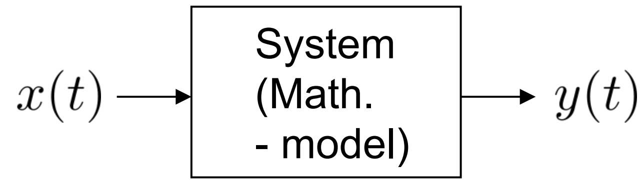
Continuous systems: models

- System: any operation/transformation over the signal

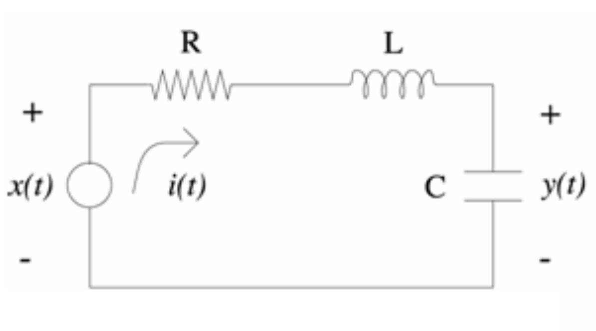
Physical Model



System Model



- Example 1 (CT): circuit RLC, described by differential equations



$$R \;\; i(t) + L\frac{d_i(t)}{dt} + y(t) = x(t)$$

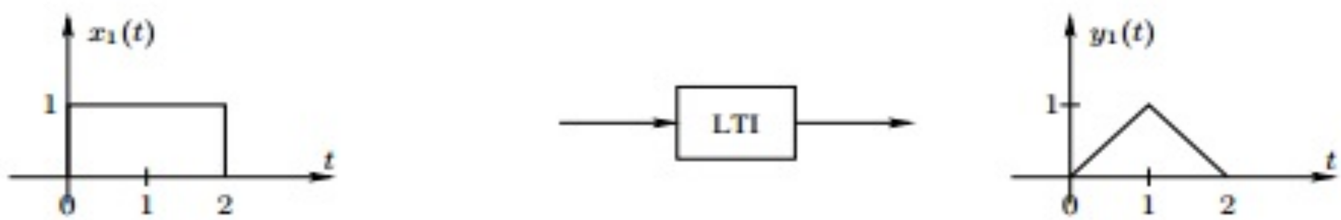
$$i(t) = C\frac{dy(t)}{dt}$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

*Note: different physical models can be represented by the same mathematical model (e.g., differential equations).

Properties of a system (Causality, Linearity, temporal invariance, etc.)

- Why to know?
  - Important practical consequences for the analysis



Causal system

- The output y(t) in a time instant depends only to the values of the input x(t) until this time instant (no from “future” values of x(t)).
- **ALL** physical systems based on real time are causals, sice the time goes only forward….
- For Spatial signals/images, it is not the case. We can go up and down, left and right…
- It is not the case the analysis for recorded signals (we can go in the “future”…).

Causal system

- A system $x(t) \to y(t)$ is causal:

When: $x_1(t) \to y_1(t) \quad x_2(t) \to y_2(t)$

if: $x_1(t) = x_2(t) \quad \forall t \leq t_0$

then $y_1(t) = y_2(t) \quad \forall t \leq t_0$

  – If two input signals are the same until t0, the output signals are the same unitl t0.

- Anti-causal: the output y(t) only depends on the future values of the input x(t).
- Non-causal: the output y(t) only depends on the past values and on the future vaules of the input x(t).

*Note: for linear and invariant systems (LTI) there is another (easier) method to see that.

Are they causal?

- Examples:

$$y(t) = x(t+1)$$
$$y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)d\tau$$
$$y(t) = x(t) - x(t+0.5)$$
$$y(t) = t^2(x(t) - x(t+0.5))$$

Temporal invariance (TI)

- Definition:

Consider: $x(t) \rightarrow y(t)$

then: $x(t-t_0) \rightarrow y(t-t_0)$

Examples: are they temporal invariant?

$$y(t) = sin(x(t))$$
$$y(t) = t \cdot x(t)$$

Periodic input in a TI system

- If the input is periodic, the output will be periodic,

$$x(t+T) = x(t)$$
$$x(t) \rightarrow y(t)$$

Due to the system TI then:

$$x(t+T) \rightarrow y(t+T)$$

Since

$$x(t+T) = x(t)$$

Then the output are the same,

$$i.e., y(t) = y(t+T)$$

the output is also periodic

Linearity

- Many systems are non-linear.
- <u>But in this course, we focus on the LINEAR SYSTEMS</u>
- Why?
  - They model several real systems as well.
  - They are analytically tractable.

Linearity

- Definition:

$$x_1(t) \rightarrow y_1(t) \qquad x_2(t) \rightarrow y_2(t)$$
$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

  - For linear systems: zero input → zero output

Examples:

$$y(t) = Ax(t)$$
$$y(t) = tx(t)$$
$$y(t) = x^2(t)$$
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$
$$y(t) = sin(x(t))$$

Without Memory

- If the output only depends on the input value at the same time instant.
- Mathematical definition:

$$x_1(t_0) = x_2(t_0) \rightarrow y_1(t_0) = y_2(t_0)$$

Examples:

$$y(t) = x(t)$$
$$y(t) = x(t-1)$$
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$
$$y(t) = 2(x(t) - x^2(t))^2$$

- y(t) must just depend on x(t)

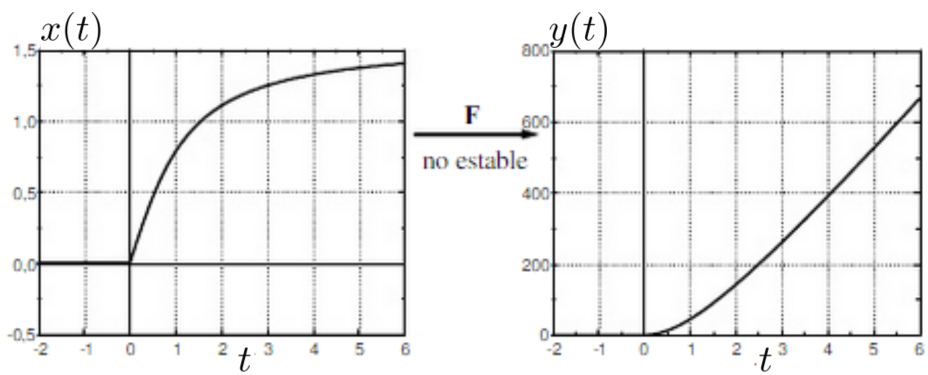
Stability

- Stable system: bounded inputs generate bounded outputs.

$$\forall t, \quad |x(t)| < B_x \quad (B_x \in \Re^+) \Longrightarrow \forall t, \quad |y(t)| < B_y \quad (B_y \in \Re^+)$$

- Example:

  Consider the system: $x(t) \rightarrow y(t) = \int_{-\infty}^{t} f(\tau) d\tau$



The system is not stable, since there exists several possible bounded inputs which produce ouputs that diverge (think in any input signal where the result of the integral is an increasing area)

Exercise: study this system $\quad y(t) = \frac{1}{2T} \int_{-T}^{T} x(t-\tau) d\tau$

Invertibililty

- Definition:

T{} denotes any transformation

$$\exists T^{-1} | \;\; y(t) = T\{x(t)\} \Longrightarrow x(t) = T^{-1}\{y(t)\}$$

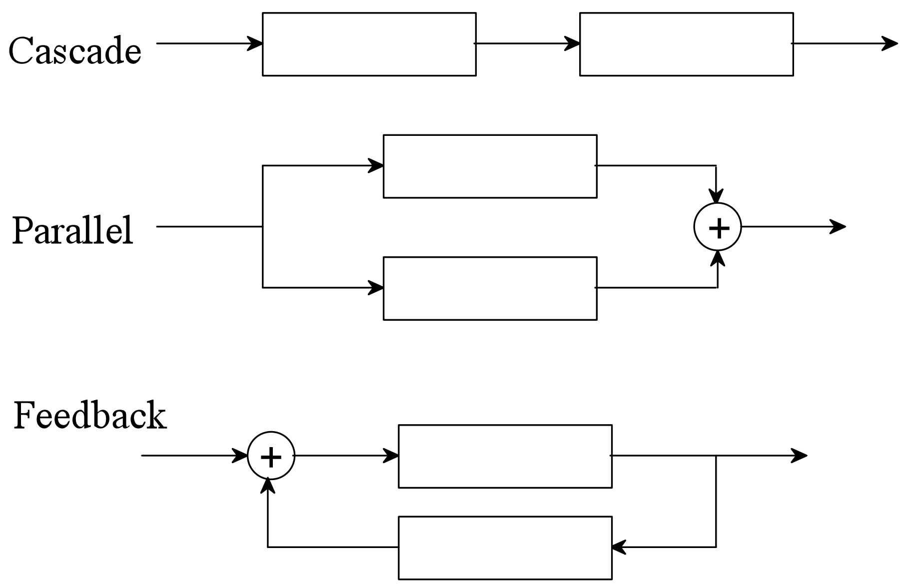
- For instance, if two different entries/inputs, x1(t) and x2(t), produces the SAME output y(t), then the system is not invertible.

Exercise:

$$y(t) = 0$$
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$
$$y(t) = x^2(t)$$
$$y(t) = 2x(t)$$

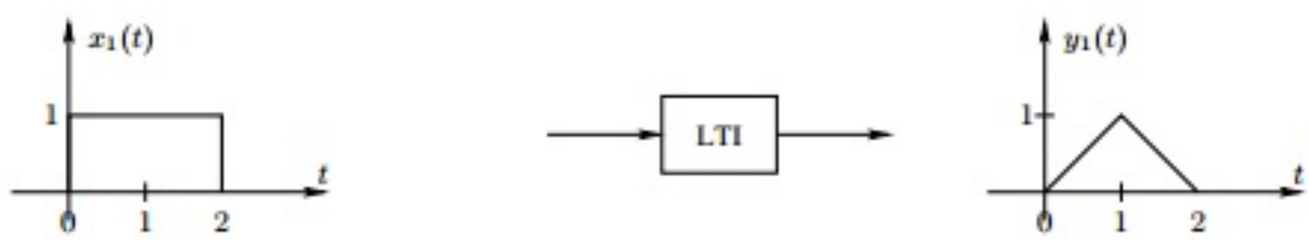
Interconnection among systems

- Represented by a block diagram:

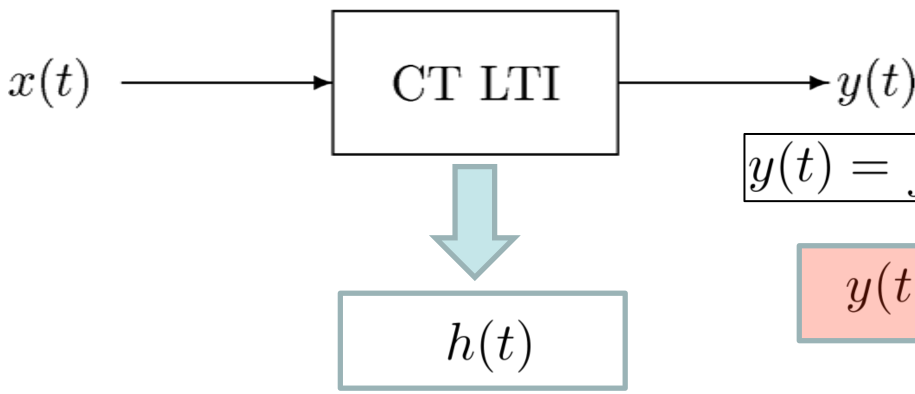


Linear and Temporal Invariant (LTI) systems

- We focus on LTI systems
  - Good models for several real applications
  - they are analytically tractable
  - they are`` simple’’ from a mathematical point of view
  - They are also simple to design (design of filters)
- Main advantages:
  - Outputs can be computed by convolution.
  - The exponential functions are eigenfunctions of the LTI systems.



LTI system in CT: convolution by integral



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

h(*t*)= impulse response (i.e., to the Dirac delta)

$$x(t) = \delta(t) \Longrightarrow y(t) = \int_{-\infty}^{+\infty} \delta(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} \delta(t-\tau)h(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\delta(t-\tau)d\tau = h(t)$$

h(*t*) represents completely the LTI system

Computation of the Convolution

- Calculating a convolution:
  - Convolution with a delta $\delta(t)$ → easy (we obtain the signal $x(t)$)
  - Convolution with exponentials → easy more or less (solution: another exp.)
  - Generic Convolution → more difficult

4 steps:

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$h(\tau) \xrightarrow[\text{Invert}]{} h(-\tau) \xrightarrow[\text{move}]{} h(t-\tau)$$

$$\xrightarrow[\text{Multiply}]{} x(\tau)h(t-\tau) \xrightarrow[\text{Integrate}]{} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

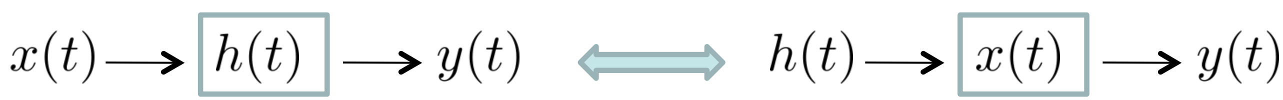
Properties and examples of convolution

- Neutral Element is the unit impulse (Dirac Delta): $\delta(t)$

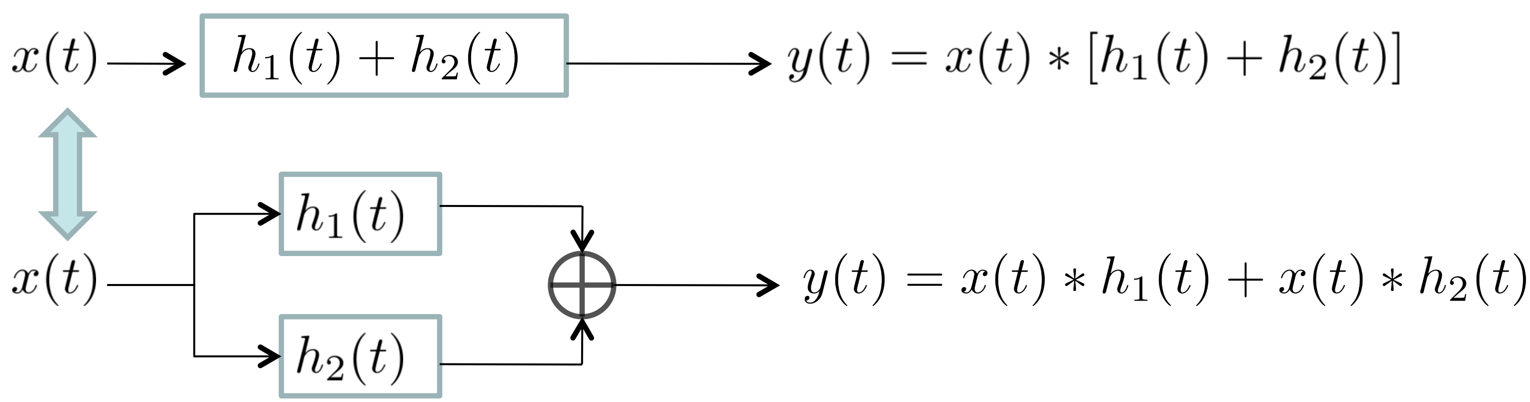
$$x(t) * \delta(t) = x(t) \quad \Longrightarrow \quad x(t) * \delta(t - t_0) = x(t - t_0)$$

- Commutative:

$$x(t) * h(t) = h(t) * x(t)$$



- Distributive:

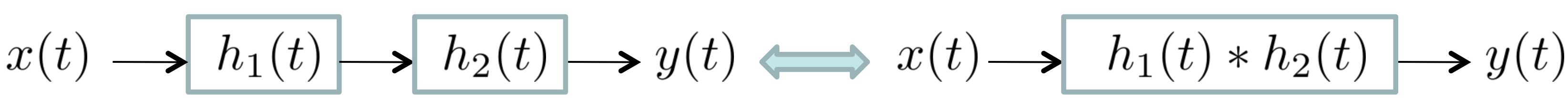


Properties and examples of convolution

- Associative:

$$y(t) = [x(t) * h_1(t)] * h_2(t) \iff y(t) = x(t) * [h_1(t) * h_2(t)]$$

$$y(t) = x(t) * [h_2(t) * h_1(t)] \iff y(t) = [x(t) * h_2(t)] * h_1(t)$$



- Response to the step function:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

*Remark: The equivalences above are valid only if the intermidiate steps are finite. The convolution has a structure of commutative group.

Properties of LTI systems = become properties about *h*(*t*)

- Causality:

$$h(t) = 0, \quad \forall t < 0$$

- Stability:

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \rightarrow h(t)$$

- Without memory:

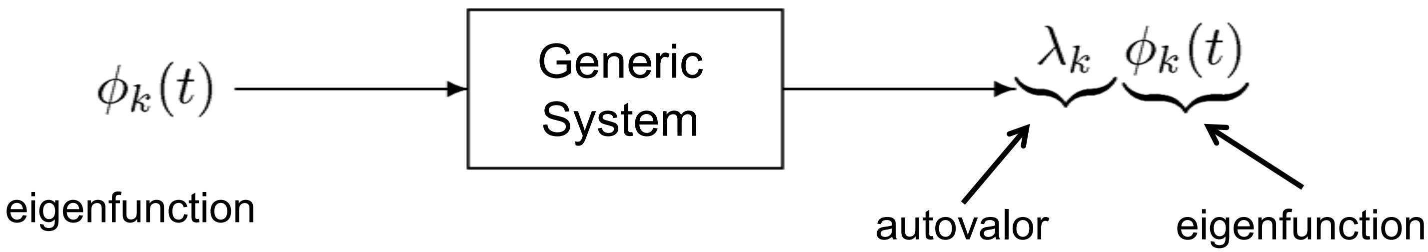
$$h(t) = 0, \quad \forall t \neq 0 \rightarrow y(t) = cte.x(t)$$

- Invertible:

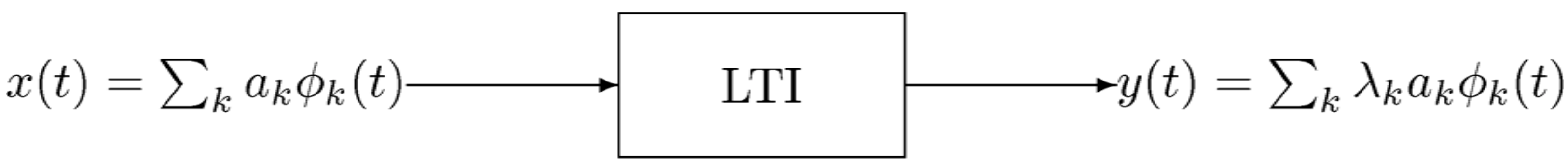
$$h(t) * h_i(t) = \delta(t)$$

Eigenfunctions of LTI systems: exponentials

➢ What is an eigenfunction?



- Output with input an eigenfunction $\rightarrow$ the same eigenfunction multiplied for a scalar number/factor (called eigenvalue)
- For the linearity of LTI systems:



Exponentials as eigenfunctions of LTI systems

Very important slide….

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

s: complex number

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = [\int_{-\infty}^{\infty} h(\tau) e^{s\tau} d\tau] e^{st} = H(s) e^{st}$$

eigenvalue

eigenfunction

H(s) is also complex number, in general

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

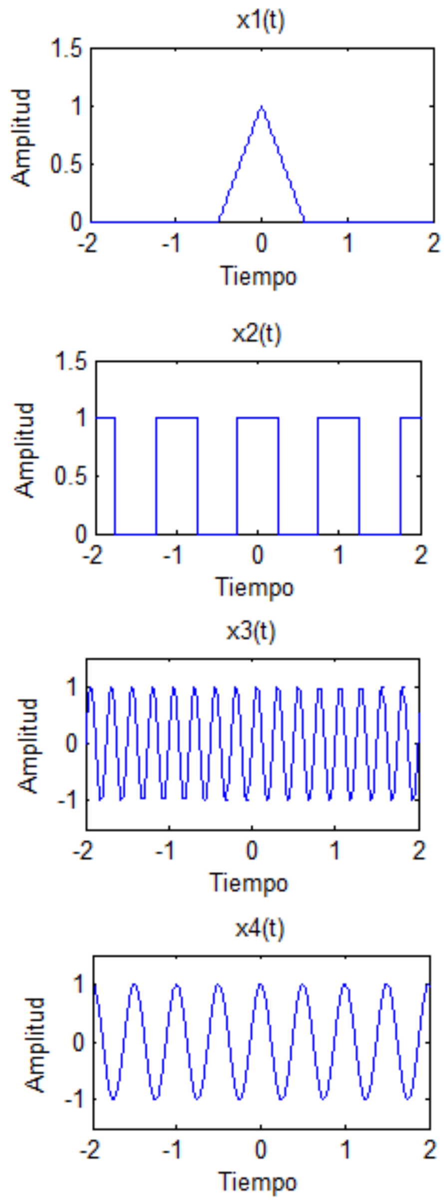
Laplace Transform

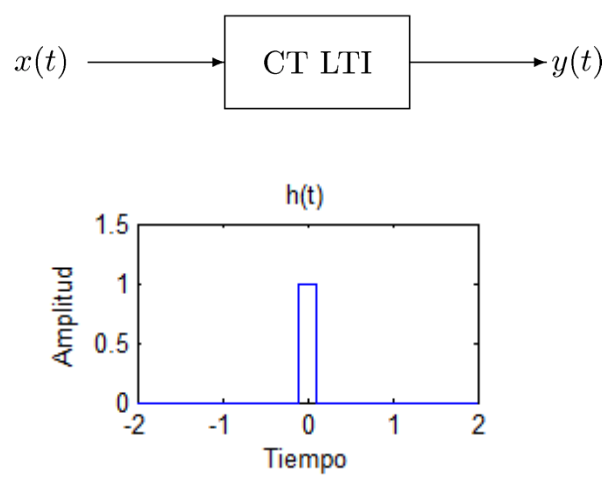
- If x(t) is a combination of exponential complex signals we get :

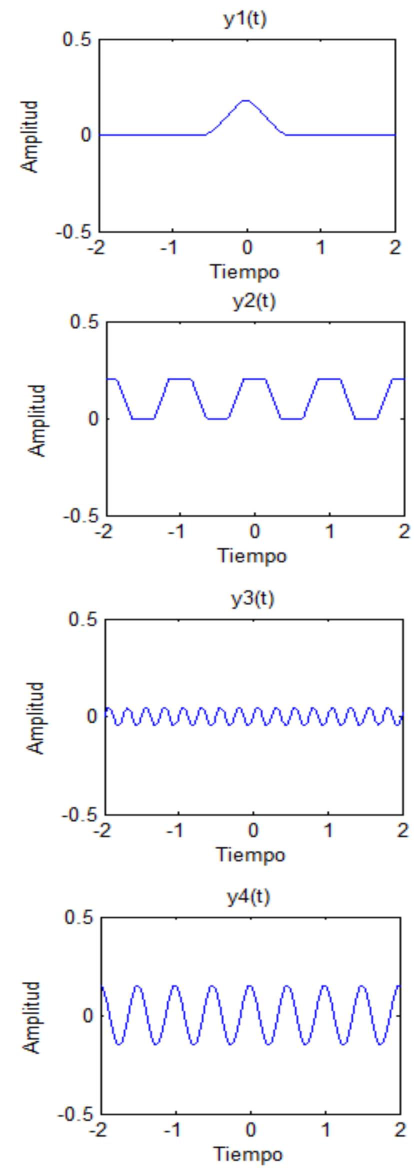
$$x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

Zeros, poles…. We can understand this study

Examples of LTI systems







Matlab code examples: inputs

```
%Señal x1(t)
fs = 10000; t = -2:1/fs:2; w = 1;
x1 = tripuls(t,w);
figure; plot(t,x1); xlabel('Tiempo'); ylabel('Amplitud'); title('x1(t)'); ylim([0 1.5]);
    xlim([-2 2]); axis([xlim ylim]);

%Señal x2(t)
t= -6:1/fs:6; w2 = 0.5; d= -10:w2*2:10;
x2=pulstran(t,d,'rectpuls',w2);
figure; plot(t,x2); xlabel('Tiempo'); ylabel('Amplitud'); title('x2(t)'); ylim([0 1.5]);
    xlim([-2 2]); axis([xlim ylim]);

%Señal x3(t)
t = -6:1/fs:6;
x3 = sin(t*2*4*pi);
figure; plot(t, x3); xlabel('Tiempo'); ylabel('Amplitud'); title('x3(t)'); ylim([-1.5
    1.5]); xlim([-2 2]); axis([xlim ylim]);

%Señal x4(t)
t = -6:1/fs:6;
x4 = cos(t*4*pi);
figure; plot(t,x4); xlabel('Tiempo'); ylabel('Amplitud'); title('x4(t)'); ylim([-1.5
    1.5]); xlim([-2 2]); axis([xlim ylim]);
```

Matlab code examples: h(t) and outputs

```
%Señal h(t)
t= -2:1/fs:2; w3 = 0.1;
h = 0*(t<=-w3) + 1*(t>-w3).*(t<w3) + 0*(t>w3);
figure; plot(t,h); xlabel('Tiempo'); ylabel('Amplitud'); title('h(t)'); ylim([0 1]);
    xlim([-2 2]); axis([xlim ylim]);

%Señal y1(t)
y1 = (1/fs)*conv(x1, h); th = -4:1/fs:4;
figure; plot (th, y1); xlabel('Tiempo'); ylabel('Amplitud'); title('y1(t)'); ylim([-0.5
    0.5]); xlim([-2 2]); axis([xlim ylim]);

%Señal y2(t)
y2 = (1/fs)*conv(x2, h); th = -8:1/fs:8;
figure; plot (th, y2); xlabel('Tiempo'); ylabel('Amplitud'); title('y2(t)'); ylim([-0.5
    0.5]); xlim([-2 2]); axis([xlim ylim]);

%Señal y3(t)
y3 = (1/fs)*conv(x3, h); th = -8:1/fs:8;
figure; plot(th, y3); xlabel('Tiempo'); ylabel('Amplitud'); title('y3(t)'); ylim([-0.5
    0.5]); xlim([-2 2]); axis([xlim ylim]);

%Señal y4(t)
y4 = (1/fs)*conv(x4, h); th = -8:1/fs:8;
figure; plot(th, y4); xlabel('Tiempo'); ylabel('Amplitud'); title('y4(t)'); ylim([-0.5
    0.5]); xlim([-2 2]); axis([xlim ylim]);
```

Exponential functions (real and complex)

- Very important for the applications
- Recall that the solution of a linear differential equation (LDE) with constant values can be expressed as combination of exponentials (LDE equivalent to Convolution !!).
- Recalling Complex Exponential functions

$$e^{st}, \quad \text{if} \quad s = jw \rightarrow e^{jwt} \rightarrow cos(wt) + jsin(wt)$$

Signals Expressed as sum of complex exponentials?

- For PERIODIC SIGNALS: Fourier SERIES (FS)
- FOR NON-PERIODIC SIGNALS: Fourier TRANSFORM (FT)
- There exists a Generalized FT for periodic signals considering delta functions…and assuming that some integrals that do not exist, converge… people accept that since there is a perfect match with FS.
- Direct and inverse equations of FT (sintesis and analysis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw \rightarrow \text{Direct/ sintesis}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jtw}dt \longrightarrow \text{Inverse/ analysis}$$

- Application to LTI systems; compute the FT of h(t) ➔ frequency responce

For the future and for the examen

- Please study this topic! (previous slides):
  - properties
  - Convolution
  - special signals (delta, step, ramp, exponential)

- Fourier:
  - Recall the main transforms
  - Direct and inverse equations of FT

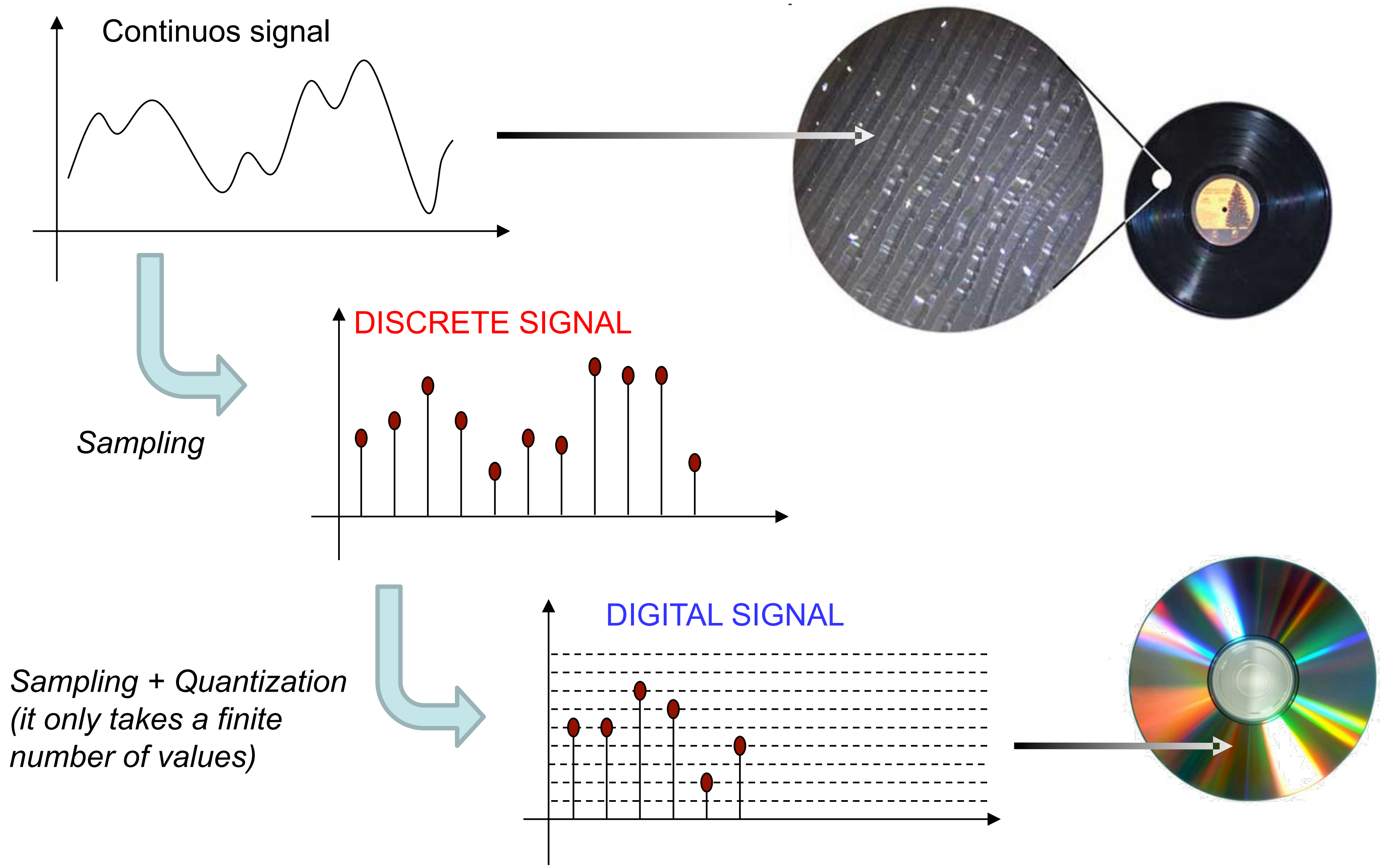
Where we are

- Tema 1: Discrete signals and systems in the temporal domain
  - 1.1 Recall of the signals and systems in the continuous time
  - 1.2 Signals in discrete time
  - 1.3 Systems in discrete time
  - 1.4 Convolution in the discrete time

1.2 Signals in discrete time

- We will see:
  - Discrete signals versus continuous signals and digital signals
  - classifications
  - signals of interests
  - Operations

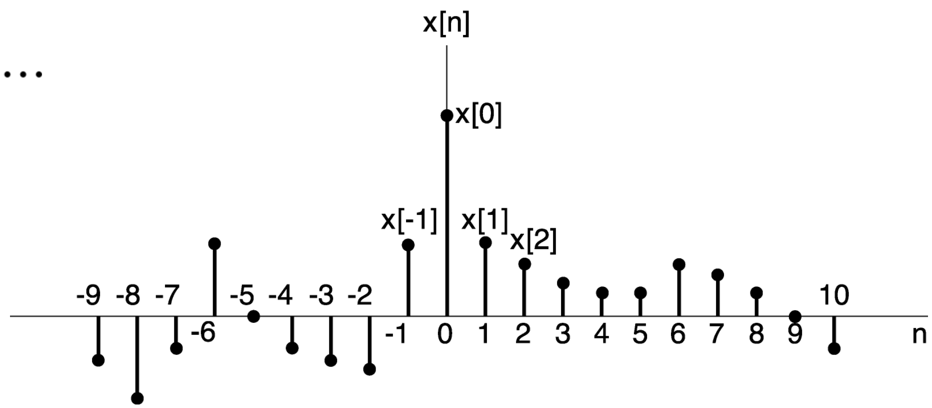
Analog/continuos versus Digital



Mathematical definition/notation for a discrete signal

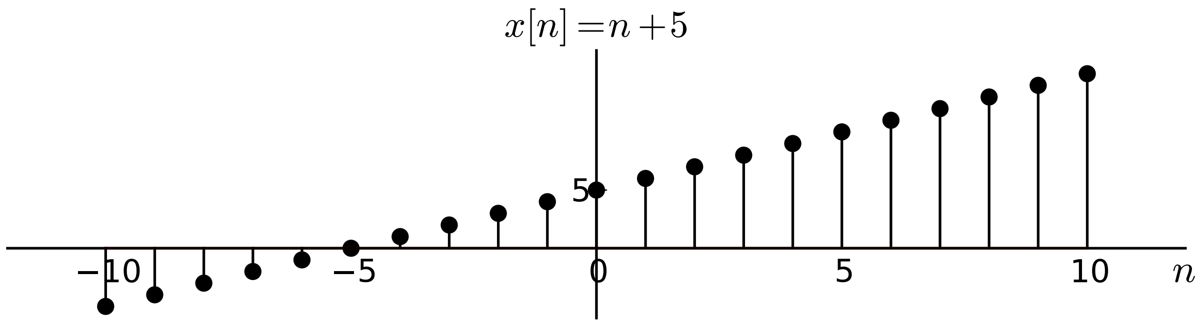
- A discrete signal is a *sequence of real numbers* and is denoted as*:*

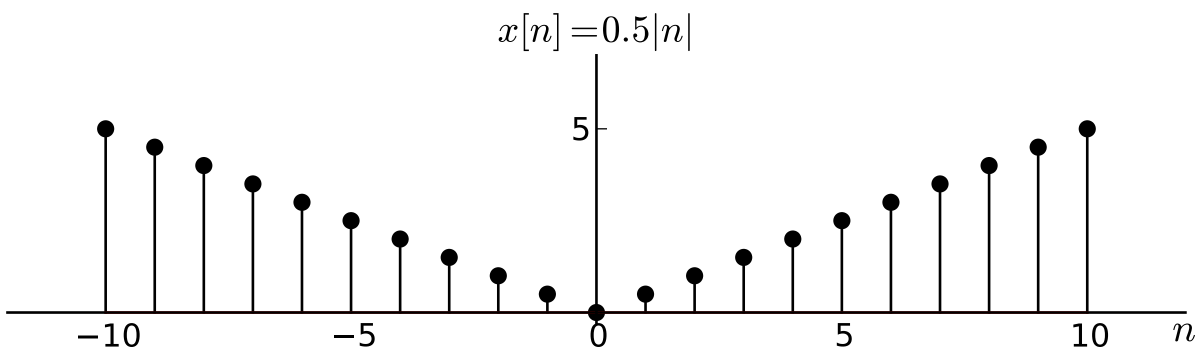
$$x[n], y[n], z[n]...$$



- In a Discrete Signal:
  - The independent variable (*n*) takes integer values, i.e., discrete
  - The dependent variable (*x*, *y*, *z*…) takes real values, i.e., continouos

Examples:





Basic parameters

- Area, mean value, energy, power:

AREA

$$A_x = \sum_{n-\infty}^{\infty} x[n]$$
$$= \ldots + x[-1] + x[0] + x[1] + \ldots$$

ENERGY

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
$$= \ldots + |x[-1]|^2 + |x[0]|^2 + |x[1]|^2 \ldots$$

MEAN VALUE

$$\bar{x} = \langle x[n] \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$$

POWER

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

*Remark: with the notation |.| we have denoted the module of a vector (or a complex number), then the definition is valid also for complex signal.

Basic parameters

- We can also define the energy in a finite interval $-N \leq n \leq N$ as

$$E_N = \sum_{n=-N}^{N} |x[n]|^2$$

- Then the energy of the signal, E, is:

$$E = \lim_{N \to \infty} E_N$$

- Also the power can be expressed as function of E_N ,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} E_N$$

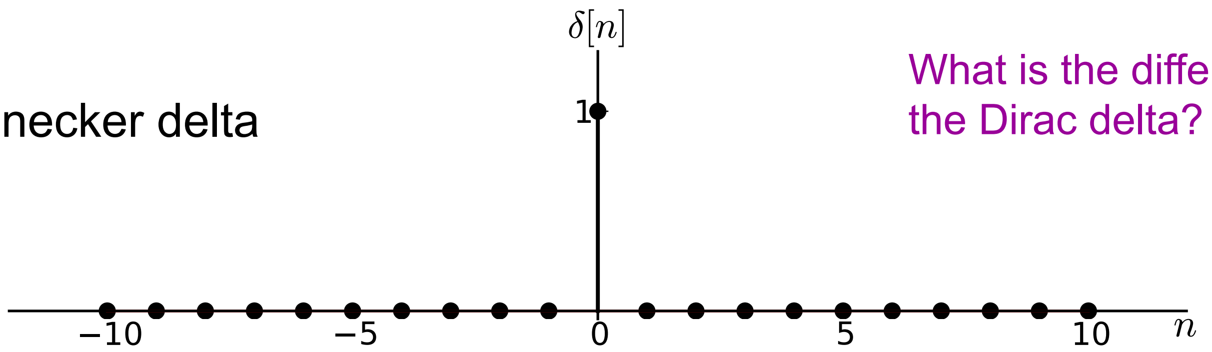
Signals in discrete time (DT): classification

- A discrete signal can be:
  - Energy signal: (finite energy and zero power) $E_x > 0, P_x = 0$
  - Power signal: $E_x = \infty, P_x < \infty$
  - Signal with infinite power: $E_x = \infty, P_x = \infty$

- We focus on energy ang power signals.

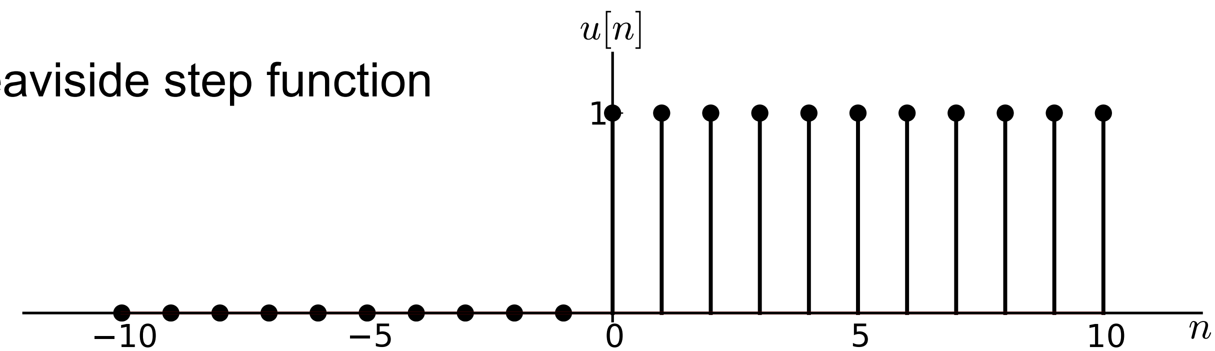
Basic signals: delta and step functions

Kronecker delta

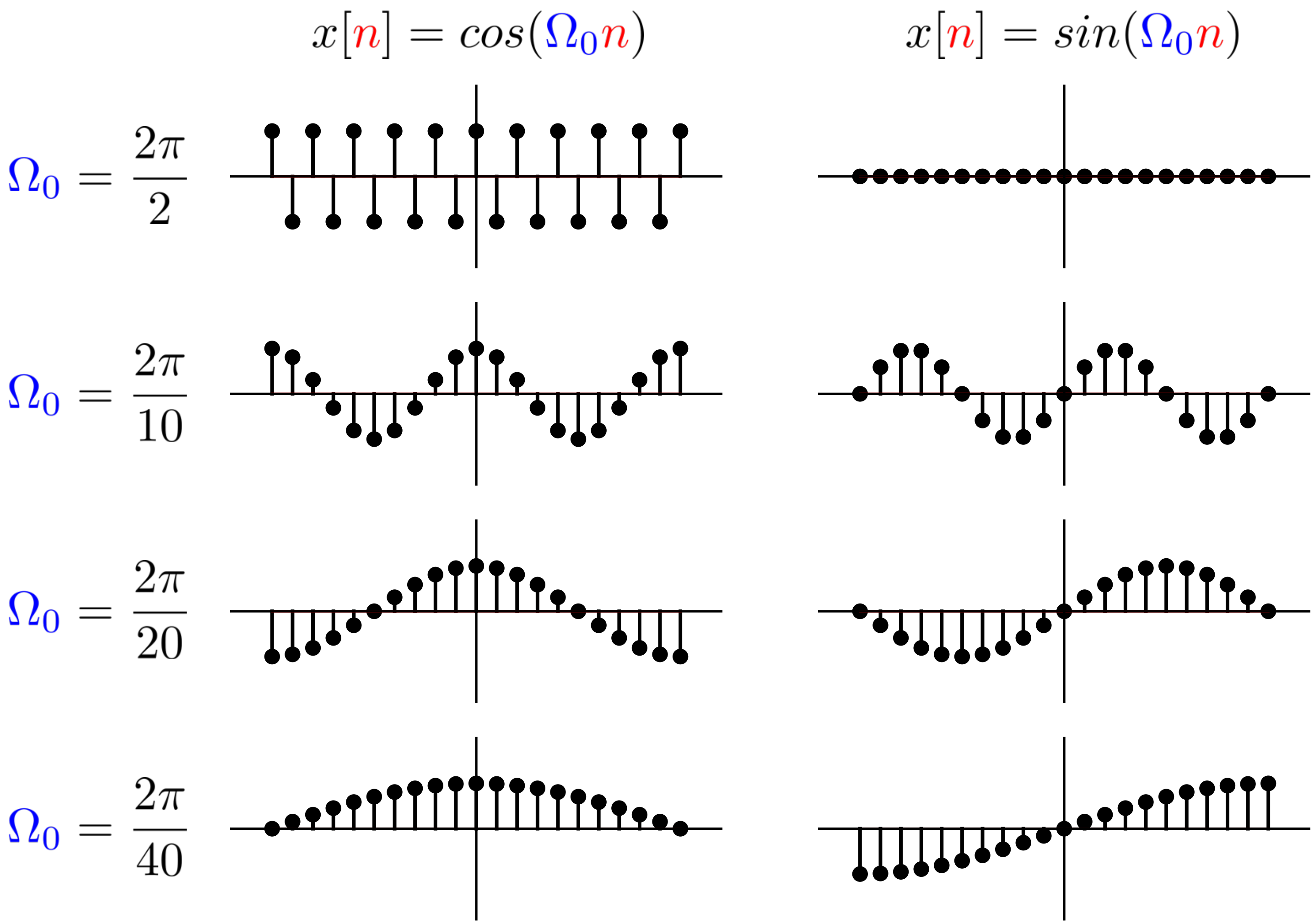


What is the difference with the Dirac delta?

Heaviside step function



Basic Signals: sin and cos



Basic Signals: sin and cos

- Properties of the sinusoidal signals in DT:
  - Two sinusoidal signals with angular frequency of

$$\Omega_0, \quad \Omega_1 = \Omega_0 + 2\pi$$

are identical.

  - They are periodic if and ony if the angular frequency can be expressed as:

$$\Omega_0 = 2\pi \frac{m}{N}$$

$$\cos(3n) \Rightarrow$$ NON-PERIODIC !!!!!!

Where N and m are integers without common factors. In this case the period is *N*.

Basic signals: Real exponential/power function

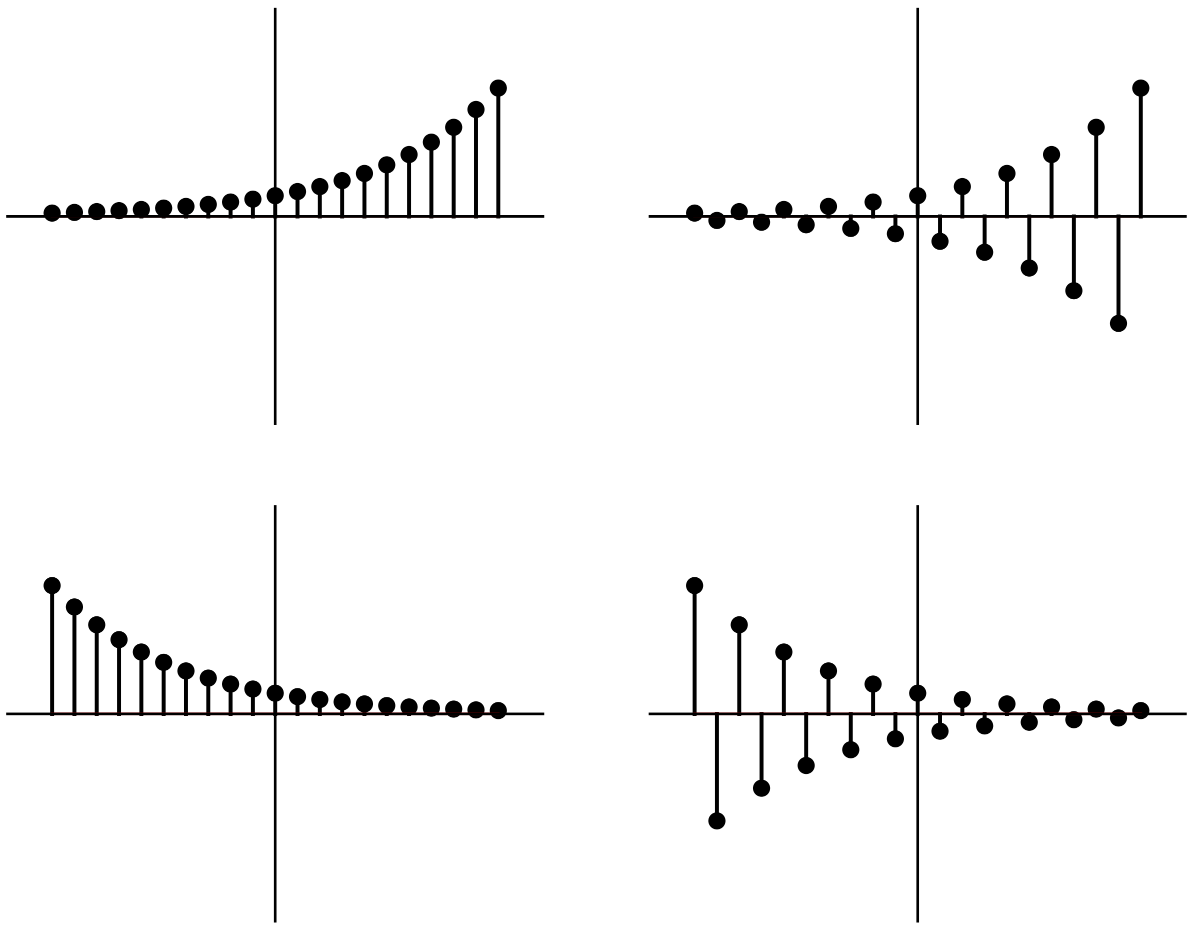
$$x[n] = Ca^n$$

$a > 1$

$a < -1$

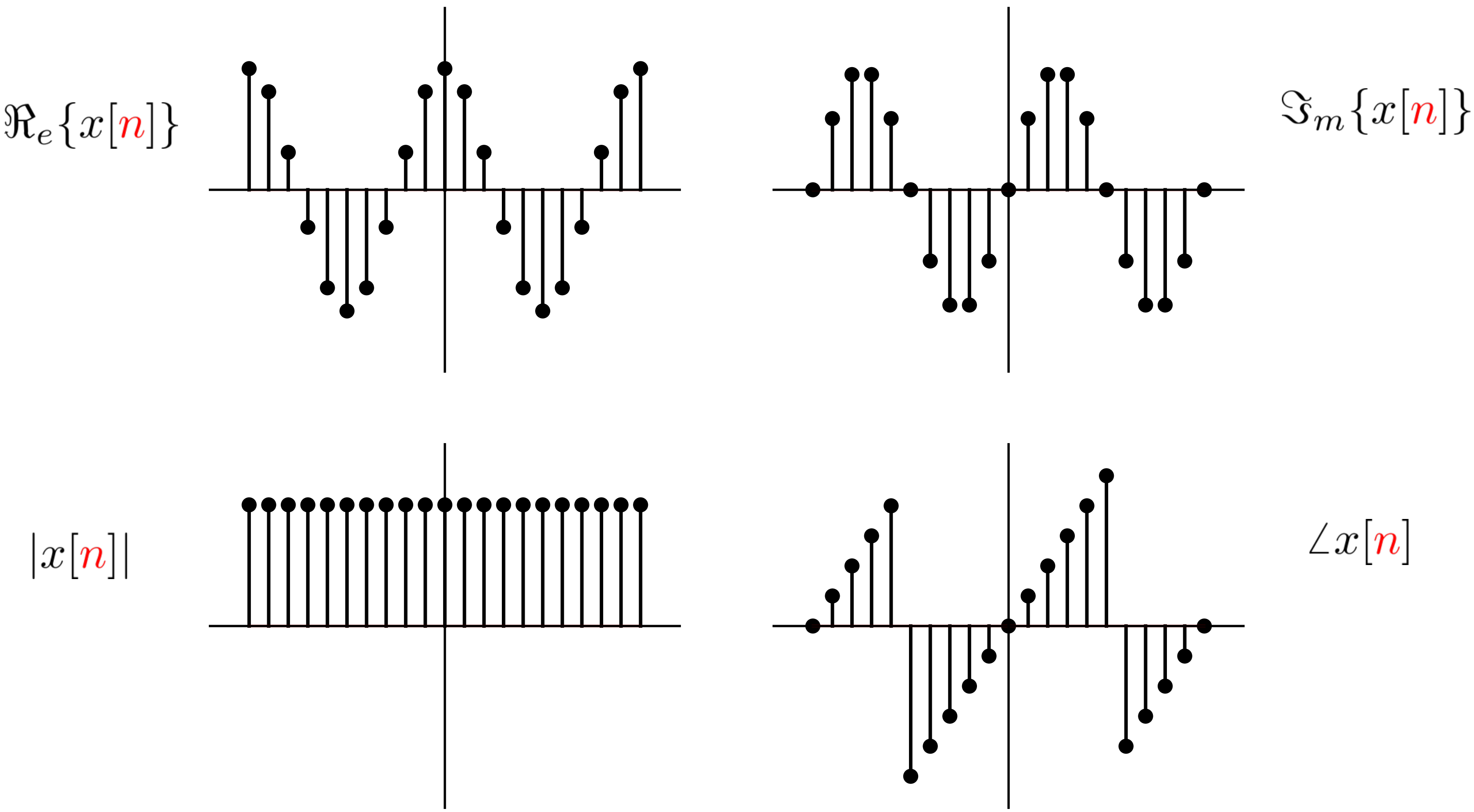
$0 < a < 1$

$-1 < a < 0$



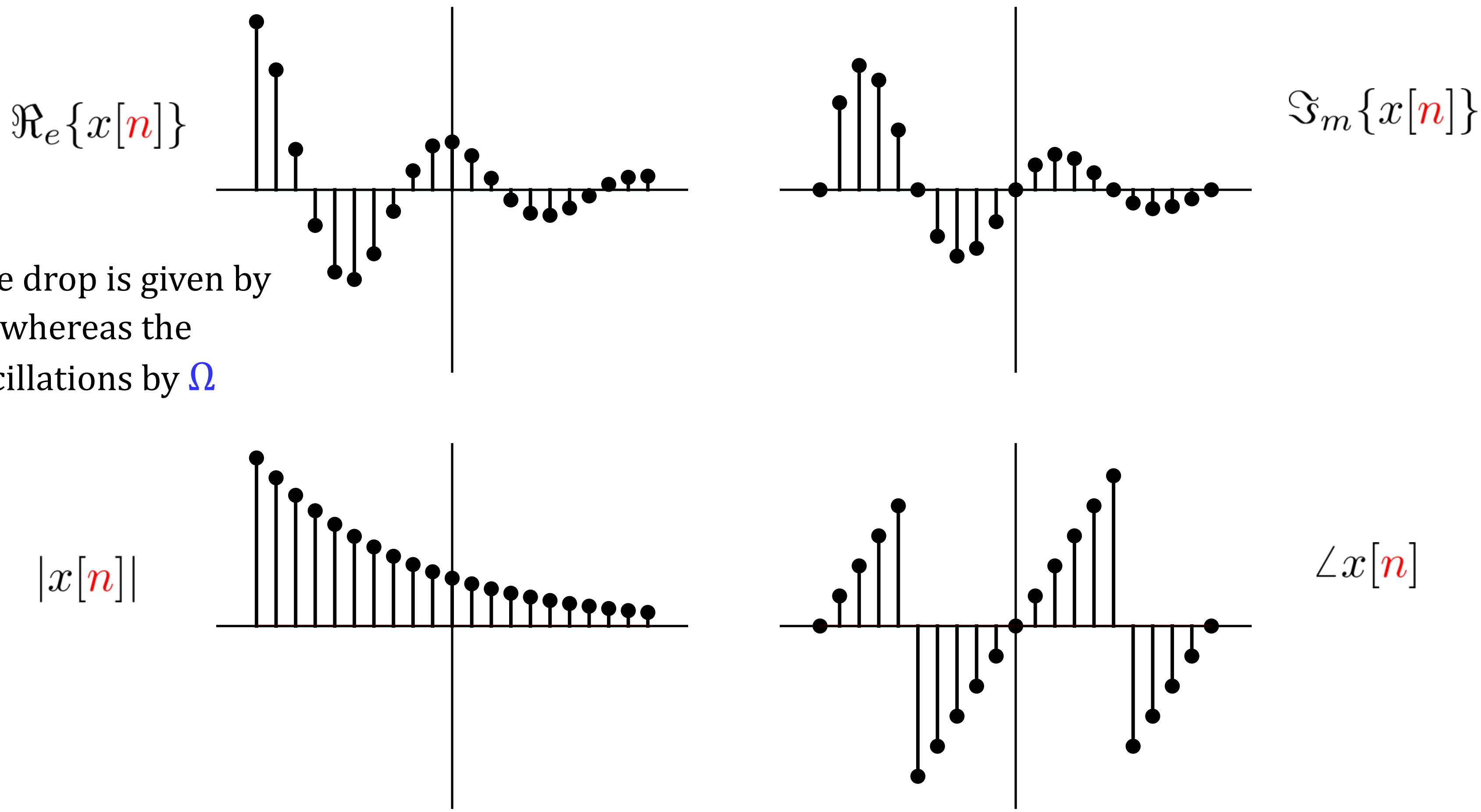
Basic signals: Complex exponential

$$x[n] = e^{j\Omega_0 n}$$



Basic signals: Complex exponential with "damping"/envelope

$$x[n] = e^{(a+j\Omega)n} = e^{an}e^{j\Omega n}$$

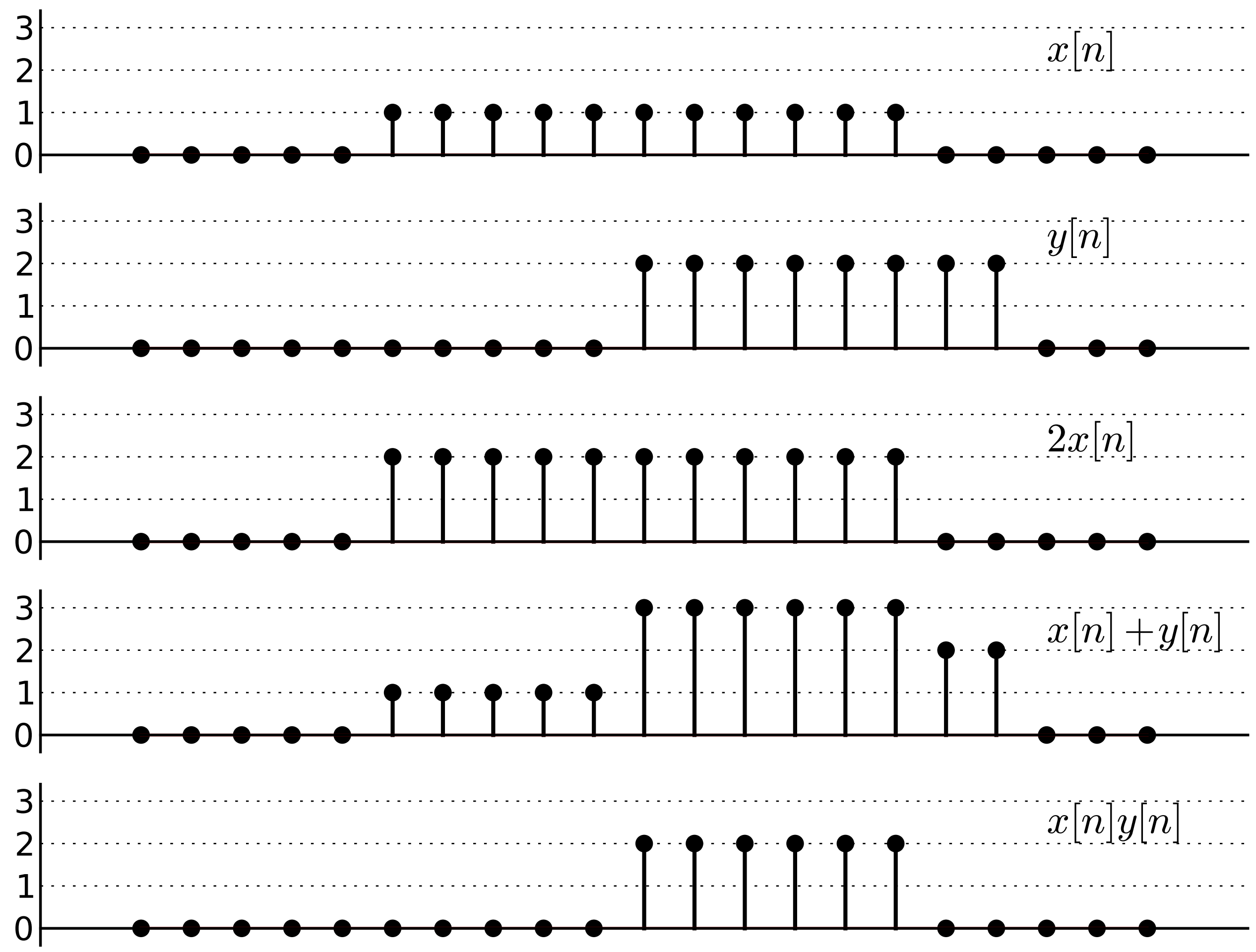


The drop is given by a, whereas the oscillations by Ω

Basic operations about the dependent variable

- Change of scale of y[n]: $y[n] = K \cdot x[n]$
- Sum: $y[n] = x_1[n] + x_2[n]$
- Product: $y[n] = x_1[n] \cdot x_2[n]$

Basic operations about the dependent variable



Basic operations about the independent variable

- Translation/movement:

$$y[n] = x[n + n_0] \rightarrow \begin{cases} n_0 < 0 \rightarrow \text{To the right} \\ n_0 > 0 \rightarrow \text{To the left} \end{cases}$$

  - The value n_0 must be an integer

- Symetric signal with respt to the the y-axis:

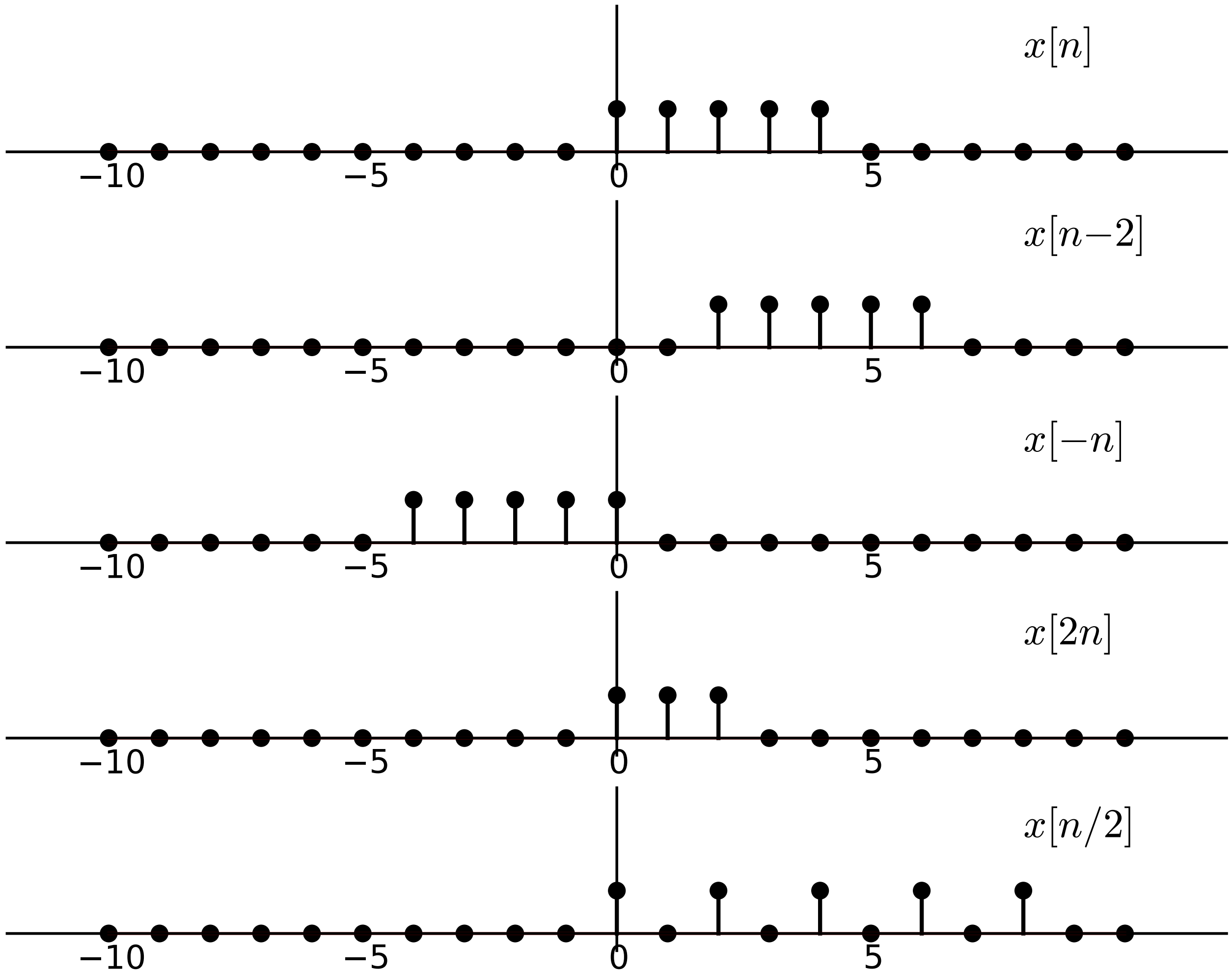
$$y[n] = x[-n]$$

- Change of scale: (rational values in DT)

$$y[n] = x[an] \quad \begin{cases} \text{expansion} & 0 \leq a < 1 \\ \text{contraction} & a > 1 \end{cases}$$

See other slides….

Basic operations about the independent variable



Basic operations about the independent variable

➢ IMPORTANT: en DT, the scale change produces the following consequences:

- During **compression**, we **lose samples**
- During **expansion**, we have to **add new samples** (typycally zeros)
- REMARK: in the topic “SAMPLING”, we will see how to do it without having any problems/issues

Difference and sum

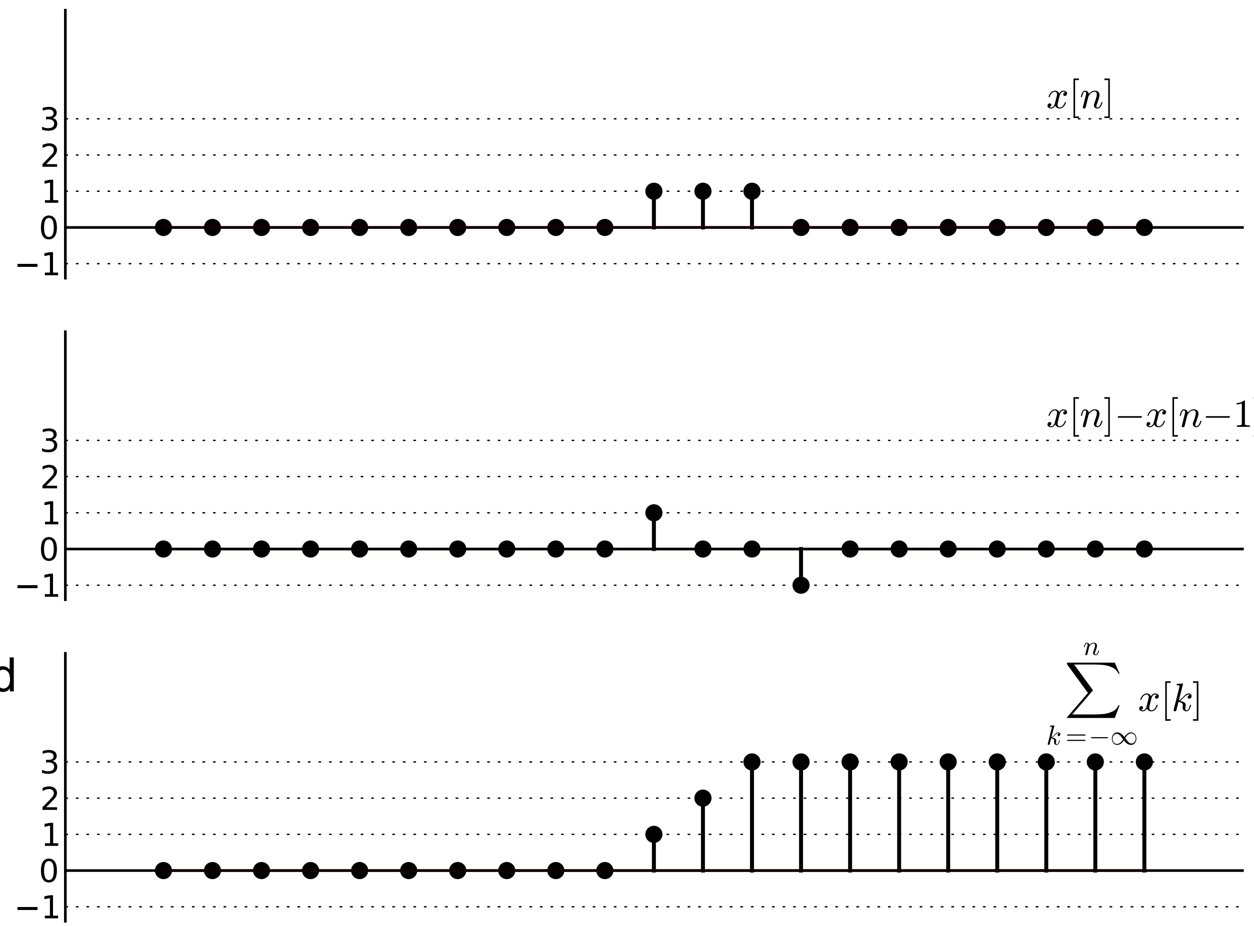
Original Signal:

Difference (related to derivative in CT):

$$y[n] = x[n] - x[n-1]$$

Sum (also called “accumulator”, related to the integral in CT):

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$



Basic Properties

- Odd and even signals:
  - *Even signal x*[*n*]: $x[n] = x[-n]$
  - *Odd signal x*[*n*]: $x[n] = -x[-n]$
- Periodicity
  - Signal *x*[*n*] is periodic with periodic N if: $x[n] = x[n-N]$
- Decomposition by Deltas
  - Any discrete signal can be expressed as a train of deltas:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

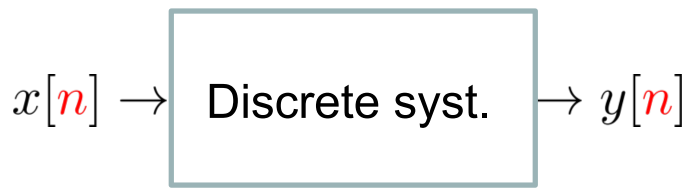
1.3 Systems is discrete time

- We will see:
  - Discrete Systems
  - Properties
  - Interconnections

Definition of the system DT

- As in the CT we have graphically:

$$x[n] \rightarrow y[n]$$



where $x[n]$ and $y[n]$ are the input signal and output signal, respectively.

Basic Properties of a DT system

- Memory:
  - The ouput at any time t depends only to the input at time t.
- Invertible
- Causalidad:
  - Outputs depend only from the present and from the past (no future)
  - For LTI systems, y[n] should NOT depend on x[k] with k>n
- Stability:
  - An input is bouded (finite amplitud) produces a bounded output (finite amplitud)

Basic Properties of a DT system

- **Temporal invariance**:

$$\text{If: } x[n] \rightarrow y[n]$$
$$\text{Then: } x[n-n_0] \rightarrow y[n-n_0]$$

- **Linearity:**.

$$\text{If: } x_1[n] \rightarrow y_1[n] \qquad x_2[n] \rightarrow y_2[n]$$
$$\text{Then: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

- **Interconnections**:
  - seris, parallel, feedback

1.4 LTI systems and convolution in DT

- We will see:
  - Convolution in DT (it is a sum) and its properties
  - output of LTI systems as a convolution
  - (Linear) difference equations with constant coefficients

Definition of LTI systems in DT

- We focus on linear and invariant systems in DT:

$x[n] \rightarrow$ LTI systems $\rightarrow y[n]$

$$\text{If: } x[n] \rightarrow y[n]$$
$$\text{Then: } x[n-n_0] \rightarrow y[n-n_0]$$

$$\text{If: } x_1[n] \rightarrow y_1[n] \qquad x_2[n] \rightarrow y_2[n]$$
$$\text{Then: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

Response to the impulse in DT

- Impulso response:

$$\delta[n] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n]$$

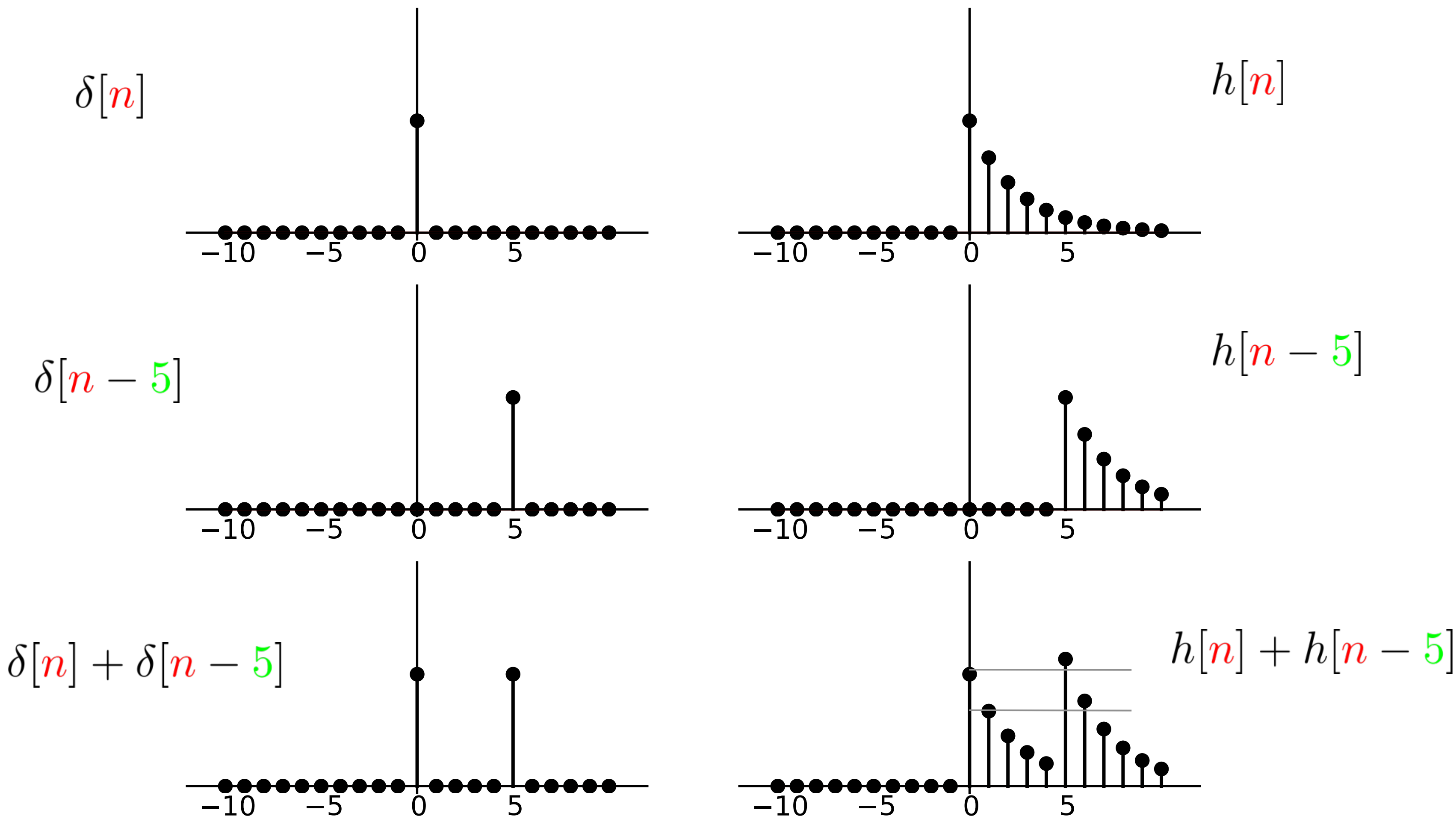
- For the time invariance:

$$\delta[n-n_0] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n-n_0]$$

- For the linearity:

$$\delta[n-n_0]+\delta[n-n_1] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n-n_0]+h[n-n_1]$$

Response to the impulse in DT



Response to the impulse in DT

- Response to train of deltas ➔ sum of response to the impulse (for the linearity)

$$\sum_{k=-\infty}^{\infty} a_k \delta[n-k] \rightarrow \boxed{\text{SLTI}} \rightarrow \sum_{k=-\infty}^{\infty} a_k h[n-k]$$

- Since x[n] can be expressed as a train of deltas, then:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{\text{SLTI}} \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution in DT

- Hence the output y[n] can be obtained as the convolution of h[n] with x[n]:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad y[n] = x[n] * h[n]$$

- Convolution of two signals in DT; notation:

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Computing of the convolution

- For any time instant n:

1) Express in the domain *k*: $h[k]$ $\quad x[k]$

2) Invert: $h[-k]$

3) Move *n* units: $h[n-k]$

4) Multiply: $x[k]h[n-k]$

5) Sum: $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$

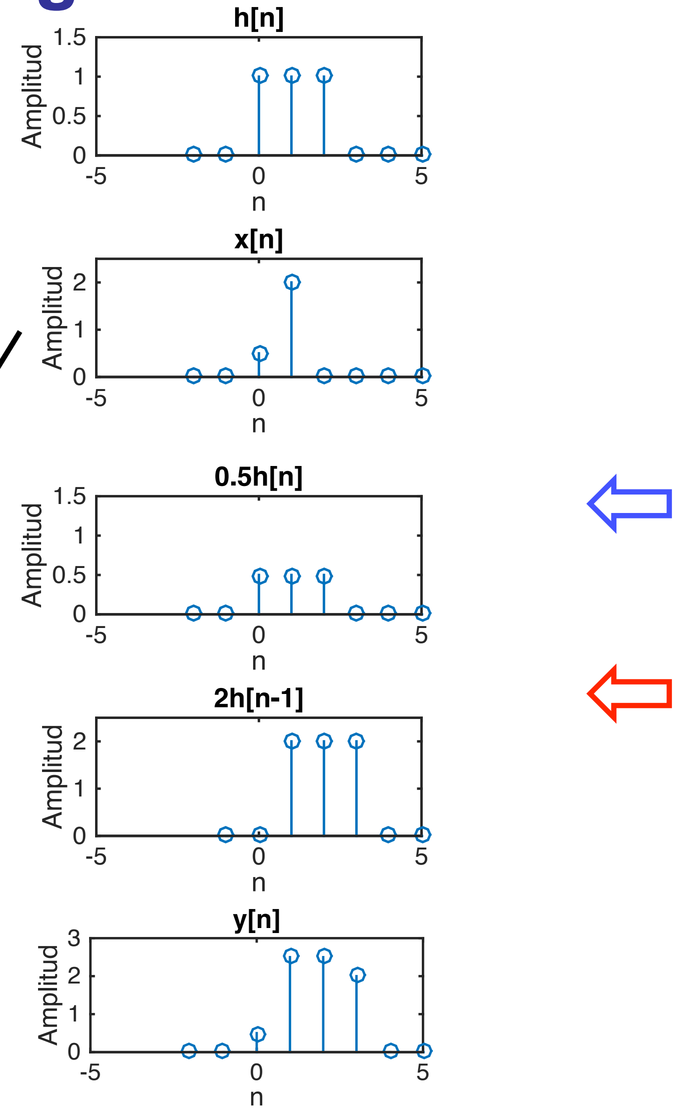
Alternative way of computing a convolution

Alternative way:
1) We can express one signal as sum of deltas
2) Then we make the convolution with the other signal above in step 1)
3) We sum all the signals obtained in step 2)

In the example on the right ➔
We can express x[n] as:

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

$$y[n] = h[n] * x[n] = 0.5h[n] + 2h[n-1]$$



Length of the convolution is 4:
Start: sum of the starts (0+0=0)
End: sum of the ends (2+1=3)
(considering non-zero samples)

Length of the convolution is N+M-1

Properties of LTI systems in DT

- The response to the impulse h[n] provides a complete characterization of the LTI system.
- See below:
  - Causality: $h[n] = 0, \forall n < 0$
  - Stability: $\sum_{-\infty}^{\infty} |h[n]| < \infty$
  - Memory: $h[n] = 0, \forall n \neq 0$
  - Invertibility: $h[n] * h_i[n] = \delta[n]$

Properties of the LTI systems in DT

- Distributive property, parallel systems:

$$y[n] = x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

- Associative property, systems in series:

$$y[n] = x[n] * h_1[n] * h_2[n] = x[n] * h_2[n] * h_1[n]$$

Properties of the LTI systems in DT

- The output y[n] of a LTI system with input a complex exponential, is other complex exponential with the **same frequency** and, generally, different phase and amplitude:

$$x[n] = Ae^{j(\Omega_0 n+\phi)} \rightarrow \boxed{\text{LTI system}} \rightarrow y[n]$$

$$y[n] = A'e^{j(\Omega_0 n+\phi')} = \frac{A'}{A}Ae^{(j\Omega_0 n+\phi)}e^{j(\phi'-\phi)} = \left(\frac{A'}{A}e^{j(\phi'-\phi)}\right)x[n]$$

- The complex exponentials are **eigenfunctions** for the LTI systems.
- The multiplying factor is the correspoding **eigenvalue** (corresponding to frecuencia Ω_0)

$$(A_0 e^{j\phi_0}) = \left(\frac{A'}{A}e^{j(\phi'-\phi)}\right)$$

Properties of the LTI systems in DT

- The output of an LTI systems when the input is a sum of complex exponentials with different frequencies, is the sum of the **same** complex exponentials with different phase and amplitude

$$x[n] = \sum_{k=0}^{K} e^{j\Omega_k n} \rightarrow \boxed{\text{LTI systems}} \rightarrow y[n] = \sum_{k=0}^{K} A_k e^{j\phi_k} \cdot e^{j(\Omega_k n)}$$

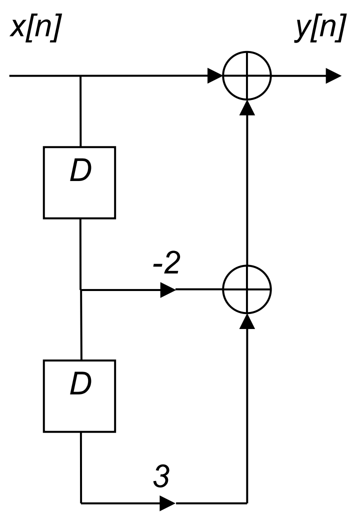
- Then, the output **y[n] NEVER will be contain a frequency that is not contained in the input x[n].**

Linear Difference Equations

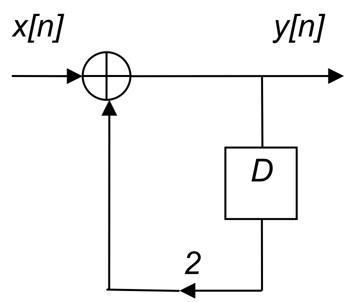
- A LTI systems in DT can be expressed using linear difference equations with constant coefficients.
- Definition: $y[n] = \sum_{p=1}^{P} a_p y[n-p] + \sum_{m=0}^{M} b_m x[n-m]$
- They are ARMA (autoregressive-moving average) filters
- If all a_p=0 ➔ FIR (FINITE IMPULSE RESPONSE) filters
- If all b_m=0 except b_0 ➔ IIR (INFINITE IMPULSE RESPONSE) filters

LTI systems by Difference Equations

- Examples:



$$y[n] = x[n] - 2x[n-1] + 3x[n-2]$$



$$y[n] = x[n] + 2y[n-1]$$