# TIME DOMAIN SIGNALS

## LINEAR SYSTEMS WITH CIRCUIT APPLICATIONS

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Biomedical Engineering Degree

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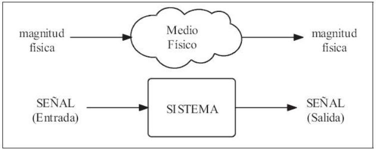
# Introduction

## Signals and Systems (I)




- A physical phenomenon is the variation, transformation of a given physical magnitude into another, due to the interaction with a physical environment.
- The mathematical model that represents the transformation of these physical magnitudes is called **signal** and the mathematical model that represents the effect of the physical environment is called **system**.
- Usually, simply measuring input and output magnitudes we are able to know the effect of the physical envorinment, and therefore, to define mathematically the systems as:

$$y(t) = F(x(t))$$

- The aim of the *Signal and Systems Theory* is to represent mathematically the physical phenomena by defining the systems and determining the input and output signals.

# Introduction

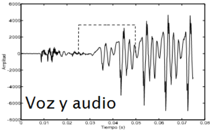
## Signals and Systems (II)

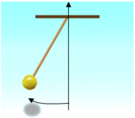
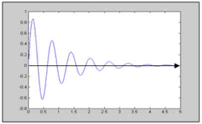
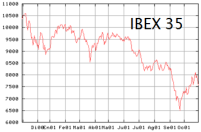
- **Signal** is a mathematical function of one or more independent variables, which contains information about a physical magnitude. As a mathematical function is usually represented as: $u = f(t)$, being time the independent variable. Some examples are:
  - Signal *one-dimensional*. Involving *one* single independent variable, $u = f(t)$: speech recordings, stock market series, *electrocardiogram, ECG*.
  - Signal *two-dimensional*. Involving *two* independent variables, $u = f(x, y)$: gray-scale images.
  - Signal *three-dimensional*. Involving *three* independent variables, $u = f(x, y, t)$: video.
- **System** is the mathematical abstraction that represent a device (equipment) that transform an input signal (the inpunt signal cause the system to respond) into an output signal (is the response of the system)
- Signals are mathematical inputs acting as a: input, output or internal signal, that the systems process or produce.
- For example, in a electric circuit, voltages and currents through the elements of the circuit, as a function of time, are **signals**; whereas the whole circuit is the **system** itself.

# Introduction

## Examples of signals and systems (I)

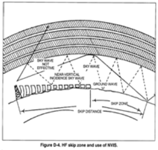
- Signal and systems concepts arise in many fields: communications, aeronautics, circuit design, **biomedical engineering**, power energy...
- Signals are used to represent physical magnitudes: *speech signal* represents acoustic pressure variations, ECG signal represents myocardial cellular electric currents, or the digital signal used in radio communications.
- **Systems** are used to represent the means that process, distort or integrate signals: e.g. microphone, muscles in human body, atomosphere...

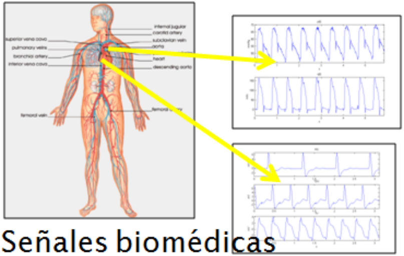


# Introduction

## Signals and Systems examples (II)

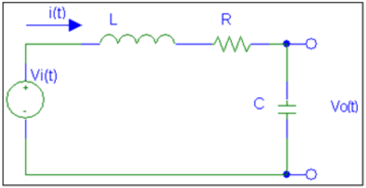


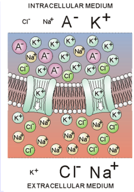



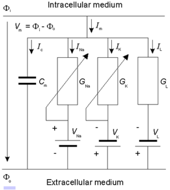
Señales biomédicas

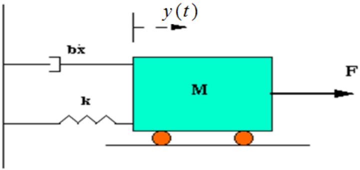
# Introduction

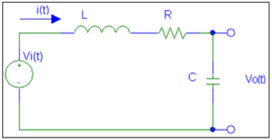
## Signals and Systems examples (III)



# Introduction

## Continuous and Discrete time (I)

- **Continuous time signals**: the independent variable is continuous (real in math sense), and thus these signals are defined for a continuum of values.

$$x(t), t \in \mathbb{R}$$

- **Discrete time signals**: they are defined only at discrete times, and consequently, for these signals, the independent variable takes on only a discrete set of values (integers in math sense). Sometime, they are called **discrete time sequences**, or **sequences**, for short.

$$x[n], n \in \mathbb{Z}$$

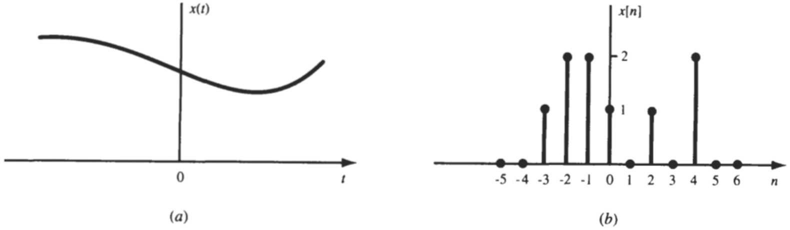



**Fig. 1-1** Graphical representation of (*a*) continuous-time and (*b*) discrete-time signals.

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# Properties of signals

## Real and Complex signals (I)

- A signal, $x(t)$ is real if its value is a real number. $x(t) : \mathbb{R} \to \mathbb{R}$.
- For example, $x(t) = t^2$, or $x(t) = 3$.
- A signal, $z(t)$ is complex if its value is a complex number. $z(t) : \mathbb{R} \to \mathbb{C}$.
- For example, $x(t) = cos(2t) + jsen(5t)$, or $x(t) = \sqrt{t}$.
- To graphically represent a real signal we can use one graph. However, to represent a complex signal we'll need two separate graphs.
- Remember that a complex signal can be in either *rectangular form* or polar form:
  - Rectangular form: real and imaginary part:

$$x(t) = \Re\{x(t)\} + j\Im\{x(t)\} = a(t) + jb(t)$$

  - Magnitude and argument (modulus and phase)):

$$x(t) = |x(t)|\, e^{j\angle\{x(t)\}} = |x(t)|_{\angle\{x(t)\}}$$

- Remember: It is very important to be (very) familiar with Euler's Identity and manipulations with complex numbers.

$$\rho e^{j\phi} = \rho\cos(\phi) + j\rho\sin(\phi)$$

# Properties of signals

## Real and Complex signals (II)

- Complex conjugate of a signal:

$$x^*(t) = \Re\{x(t)\} - j\Im\{x(t)\} = a(t) - jb(t)$$

- Real and imaginary parts can be obtained as:

$$\Re\{x(t)\} = \frac{1}{2}\left[x(t) + x^*(t)\right]; \quad \Im\{x(t)\} = \frac{1}{2j}\left[x(t) - x^*(t)\right]$$

- The magnitude and argument can be obtained as:

$$|x(t)|^2 = x(t) \cdot x^*(t) = (\Re\{x(t)\})^2 + (\Im\{x(t)\})^2$$

$$\angle\{x(t)\} = \operatorname{arc\,tg} \frac{\Im\{x(t)\}}{\Re\{x(t)\}}$$

## Questions

2 (*)Compute and represent real and imaginary part, and magnitude and argument:

- $x_1(t) = \cos(\pi t) + j\,\mathrm{sen}(\pi t)$.
- $x_2(t) = \sqrt{t}$.
- $x_3(t) = e^{-2t}e^{-j2t}$.

# Properties of signals

## Symmetry in real signals

- A real signal is **even** if it is identical with its reflection about the origin, i.e. $x(t) = x(-t)$.
- For example, $x(t) = t^2$, or $x(t) = \cos(\pi t)$, are even signals.
- A real signal is **odd** if it is antisymmetric with its reflection about the origin, i.e., $x(t) = -x(-t)$
- For example $x(t) = t$, $x(t) = \sin(t)$, are odd signals.
- There are signals that have no symmetry, but any signal can be broken into a sum of two signals, one of which is even and one of which is odd.

$$x(t) = x_e(t) + x_o(t)$$

- Even and odd parts can be obtained as:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

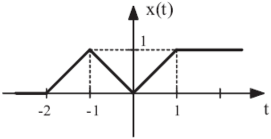
# Properties of signlas

## Questions

3 (*)Study the symmetry of:
1. $x(t) = sen(\pi t)$ .
2. $y(t) = \cos(2\pi t)$.
3. $z(t) = e^{-\alpha t}$, with $\alpha \in \mathbb{R}$.

4 (*)Find the even and odd components of the previous signlas

5 (*)Find the even and odd component of the signal in Figure.



# Propoerties of signals

## Symmetry in complex signals

- **Hermitian** symmetry. A complex signal is hermitian when is conjugate symmetric with its reflection about the origin, i.e., $x(t) = x^*(-t)$.
- For example, $x(t) = e^{jt}$ and $x(t) = j\sin(t)$ are hermitian.
- Additionally:

$$\text{If } x(t) \text{ is hermitian} \Rightarrow \Re\{x(t)\} \text{ is even } \Im\{x(t)\} \text{ is odd.}$$
$$\text{If } x(t) \text{ is hermitian} \Rightarrow |x(t)| \text{ is even and } \angle\{x(t)\} \text{ is odd.}$$

- **Antihermitian** symemtry. A complex signal is **antihermitian** when is antisymmetryc whith its reflection about the origin, i.e., $x(t) = -x^*(-t)$.
- For example, $x(t) = t + j$ and $x(t) = j\cos(t)$ are antihermitian.
- Every complex signal has two components: a **hermitian part** and an **antihermitian part**, that is,

$$x(t) = x_h(t) + x_a(t)$$

- Hermitian and antihermitian components can be computed as:

$$x_h(t) = \frac{1}{2}[x(t) + x^*(-t)]; \quad x_a(t) = \frac{1}{2}[x(t) - x^*(-t)]$$

# Properties of Signals

## Questions

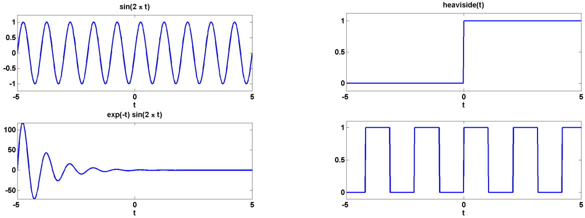
6. Find the hermitian and anti hermitian components:
   1. $x(t) = \cos(\omega_0 t) + j\sin(\omega_0 t)$.
   2. $y(t) = e^{-2t}e^{5jt}$.
7. Show that:
   1. if $x(t)$ is hermitian $\Rightarrow \Re\{x(t)\}$ is even and $\Im\{x(t)\}$ is odd.
   2. if $x(t)$ is hermitian $\Rightarrow |x(t)|$ is even and $\angle\{x(t)\}$ is odd.
8. Study the symmetries $\Re\{x(t)\}$, $\Im\{x(t)\}$, $|x(t)|$ y $\angle\{x(t)\}$, when $x(t)$ is a complex antihermitian signal.

# Properties of signals

## Periodicity

- A signal is said to be **periodic** if we can find a constant time interval $T$, so taht the values of the signal are repetead every $T$. This time interval, $T$ is called **period**.

$$x(t) \text{ is periodic} \Leftrightarrow \exists\, T > 0, T \in \mathbb{R} \text{ so that } x(t) = x(t+T)\ \forall t$$




## Fundamental period

- If $x(t)$ is periodic with period $T$, it is also periodic with periods $2T, 3T, \ldots$.
- We call *fundamental period*, $T_0$, to the smallest value of $T$ for which the equation $x(t) = x(t+T)$ holds.

# Properties of Signals

## Example: Periodic Signals

- We want to find out if $x(t) = \cos(\frac{2\pi}{5}t)$ is a periodic signal. ¿$\exists T$ such that $x(t) = x(t+T)$ $\forall t$?
- To answer, we have to find $x(t+T)$:

$$x(t+T) = \cos\left(\frac{2\pi}{5}(t+T)\right) = \cos\left(\frac{2\pi}{5}t + \frac{2\pi}{5}T\right) =$$
$$= \cos\left(\frac{2\pi}{5}t\right)\cos\left(\frac{2\pi}{5}T\right) - \operatorname{sen}\left(\frac{2\pi}{5}t\right)\operatorname{sen}\left(\frac{2\pi}{5}T\right)$$

- Now, we need to choose an appropriate $T$, so that the previous signal is equivalent to $x(t)$, therefore:

$$\cos\left(\frac{2\pi}{5}T\right) = 1, \quad \operatorname{sen}\left(\frac{2\pi}{5}T\right) = 0$$

  Thereby, every $T = 5\,k$, with $k = 1, 2, \ldots$ is a valid periodof the signal $x(t)$. Thus, we can assure that the signal is periodic.

# Properties of Signals

## Questions

9 Show that if $x_1(t)$ y $x_2(t)$ are periodic signlas with period $T_0$, then the signal $y(t) = x_1(t) + x_2(t)$ is also periodic. ¿Which is its period?

10 Show that if $x_1(t) = x_1(t + T_1)$ y $x_2(t) = x(t + T_2)$ holds, then the signal $y(t) = x_1(t) + x_2(t)$ is periodic. ¿Which is its period?

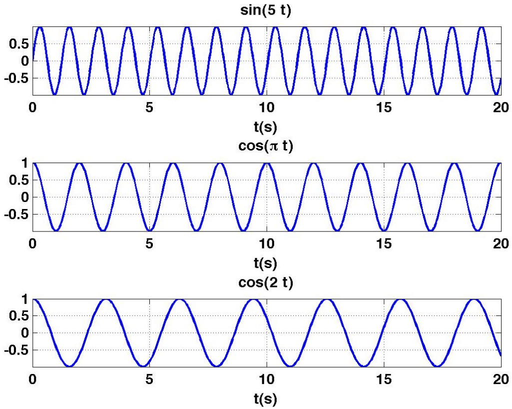
## Questions

11 (*)Study if the following signals are prediocis, if so, find their periods.

- $x_1(t) = \cos(\omega_0 t)$
- $x_2(t) = \sin\left(\omega_0 t + \frac{1}{2}\right)$
- $x_3(t) = e^{j\omega_0 t}$
- $x_4(t) = \cos(10\pi t)$
- $x_5(t) = \sin(10\pi t) + \cos(20\pi t)$
- $x_6(t) = \sin(10\pi t) + \cos(20t)$
- $x_7(t) = \sin(10\pi t)\cos(20\pi t)$

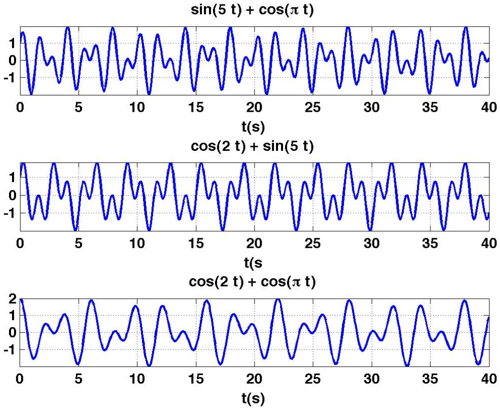
# Properties of Signals

## Example: periodic and non periodic signals



# Properties of Signals

## Example: periodic and non periodic signals



# Properties of Signals

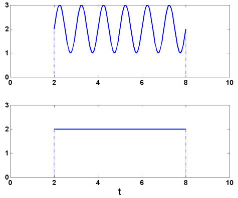
## Average value of a signal in an interval

- We want to characterize a signal with different measurements within an interval, such as: avera value, average power or energy.
- Any of the following measurements can be computed on an interval or on the whole signal.
- To define a time interval we need to specify the begin ($t_B$) and the end ($t_E$) of the interval: $(t_B, t_E)$. This can also be defined as an interval centered at $t_0$ with duration $T$, or the interval $[T, t_0]$.

## Average value in a finite time interval (I)

$$\text{area} = \int_{t_i}^{t_f} x(t)dt = m \cdot T$$

$$m = \frac{\text{area}}{T}$$



# Properties of Signals

## Average value in a finite time interval (II)

- The average value of a signal, also known as *DC level (Direc Current)* in a finite time interval, can be computed as:

$$\langle x(t)\rangle_{[T,t_0]} = \frac{1}{T}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} x(\tau)d\tau; \quad \langle x(t)\rangle_{(t_i,t_f)} = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} x(\tau)d\tau$$

## Total Average Value

- The Total Average Value of signal $x(t)$ is defined as::

$$m_\infty = \langle x(t)\rangle = lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} x(\tau)d\tau\right\}$$

- **Total Average Value for periodic Signals**. Since in periodic signals $x(t) = x(t+T_0)$, the average value in a period would be the same as the total average value $(-\infty,\infty)$, therefore, is easier to compute:

$$m_\infty = \langle x(t)\rangle = \frac{1}{T_0}\int_{\langle T_0\rangle} x(\tau)d\tau$$

# Properties of Signals

## Questions

**12** (*)Find the average value of the following signals:

- $x_1(t) = e^{-t}$, calcular $\langle x_1(t) \rangle_{(2,3)}$.
- $x_2(t) = t^2$, calcular $\langle x_2(t) \rangle_{(1,3)}$.
- $x_3(t) = |\sin(t)|$, calcular $\langle x_3(t) \rangle$.
- $x_4(t) = \cos(\frac{\pi t}{2})$, calcular $\langle x_4(t) \rangle$.
- $x_5(t) = u(t)$, calcular $\langle x_5(t) \rangle_{(-1,3)}$
- $x_6(t) = u(t)$, calcular $\langle x_6(t) \rangle$
- $x_7(t) = e^{j\left(5\pi t - \frac{1}{2}\right)}$, calcular $\langle x_7(t) \rangle_{(-1,3)}$.
- $x_8(t) = e^{j\left(5\pi t - \frac{1}{2}\right)}$, calcular $\langle x_8(t) \rangle$.

# Properties of Signals

## Power and Energy in Signals

- Power and energy are concepts used in physics, for example, in circuits. We are going to define, using the analogy, abstract concepts of **Power** and **Energy** for signals.
- We are going to define the power consumed by a reference resistor $R = 1\Omega$. We can think that $x(t)$ is either $v(t)$ or $i(t)$. The instantaneous power $p(t)$ is defines as:

$$p(t) = |v(t)|^2/R = |i(t)|^2 R$$

- The total energy $E$ and power $P$ consumption is:

$$E = \int_{-\infty}^{\infty} i^2(t)dt \quad joules$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)dt \quad watts$$

# Properties of Signals

## Power of a Signal

- The instantaneous power os a signal is defined as the square magnitude of the signalLa potencia instantánea de una señal se define como el módulo al

$$p_x(t) = |x(t)|^2$$

- The avearge power in a given interval of length $T$, $(t_1, t_2)$:

$$P_T = \langle p_x(t) \rangle_{(t_1,t_2)} = \frac{1}{T}\int_{t_1}^{t_2} p_x(\tau)d\tau = \frac{1}{T}\int_{t_i}^{t_f} |x(\tau)|^2 d\tau$$

- The total average power:

$$P_\infty = \langle p_x(t) \rangle = \lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} p_x(\tau)d\tau\right\} = \lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} |x(\tau)|^2 d\tau\right\}$$

.

- If the signal is periodic, period $T_0$, the average power is computed as:

$$P_\infty = \langle p_x(t) \rangle = \frac{1}{T_0}\int_{\langle T_0 \rangle} p_x(\tau)d\tau = \frac{1}{T_0}\int_{\langle T_0 \rangle} |x(\tau)|^2 d\tau$$

# Properties of Signals

## Energy of a signal

- The energy in circuits can be computed as

$$p_x(t) = \frac{dw(t)}{dt} \Rightarrow w(t) = \int_{-\infty}^{t} p_x(\tau) d\tau$$

- Therefore, the total energy can be computed as:

$$E_\infty = \lim_{T\to\infty} w(t) = \int_{-\infty}^{+\infty} |x(\tau)|^2 d\tau$$

.

## Classification of signals regarding energy and power

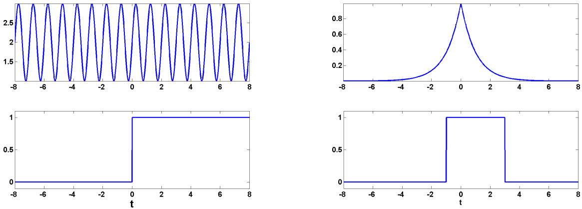
- **Signals with finite energyy** $0 < E_\infty < \infty$.
- For example, signals with limited duration.
- **Signals with finite average power** $0 < P_\infty < \infty$.
- For example, periodic signals.

## Questions

13 Show that any signal with finite energy has zero average power, and also that any signal with finite average power has infinite energy.

# Properties of Signals

**Examples: Signals with finite average power and signals with finite energy.**




## Questiones

**14** (*)Find the average power and eneryg for each of the following signals.

- $x_1(t) = u(t)$
- $x_2(t) = e^{-2t} \cdot u(t)$
- $x_3(t) = e^{j(2t+\frac{\pi}{4})}$
- $x_4(t) = \cos(t)$
- $x_5(t) = (\frac{1}{2})^t \cdot u(t)$
- $x_6(t) = (3+2j)u(t)$

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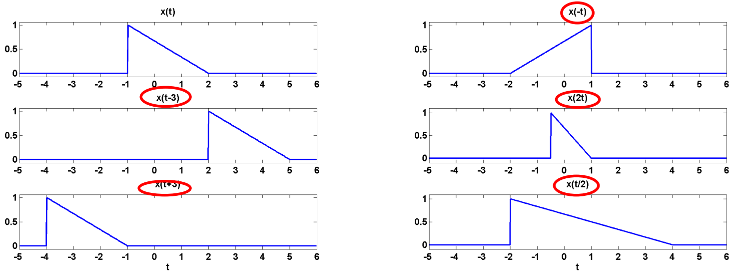
# Transformations of signals

## Transformations of the Independent Variable

- There are 3 basic transformations of the independent variable of a signals:
  - Shifting: $y(t) = x(t \pm a)$, with $a > 0$.
  - Scaling: $y(t) = x(at)$, with $a > 0$.
  - Time-reversal: $y(t) = x(-t)$.



# Transformations of signals

## Shift

- Let's be $a \in \mathbb{R}^+$. the result of add or subtract $a$ to the independent variable is a new shifted signal, $y(t) = x(t \pm a)$.
  - $y_1(t) = x(t-a)$ is a delayed version of $x(t)$ shifted to the right.
  - $y_2(t) = x(t+a)$ is an advanced version of $x(t)$ shifted to the left.
- If $x(t)$ is a song of $\ldots$, $x(t-a)$ means you are going to play the song some time ahead, whereas $x(t+a)$ means that you have already played the song.

## Time reversal

- Multiply by -1 the independent variable (sign change) results in a new signal $y(t) = x(-t)$, which is the same as the original but reflected around $t = 0$. This is the song played backward.

## Scaling

- Let be $a \in \mathbb{R}^+$. Signal $y_1(t) = x(at)$, when $a > 1$, is an accelerated version of $x(t)$, while, with $a < 1$, is a slower version.
- For example, if $x(t)$ is a song, $x(2t)$ is the song played at twice the speed, and $x(t/2)$ is played at half-speed.
- Note that scaling in continuous time does not implyt lost of information. We can always recover original signal from a scaling one, using a new scaling on the transformed signal $a' = 1/a$.

# Transformation of Signals

## Practical advices

- Whener you have several transformations, it is easier (commonly) to start by the time shifting.
- You are only transforming the independent variable.
- It is always a good advice to use intermediate signals, plotting them and keeping the analytical expressions.
- At the end, evaluate always the result in known values of the independent variable

$$y(t) = x(\alpha t + \beta) \Rightarrow y(t)\Big|_{t^*}$$

Where $t^*$ is an easy value where check the transformation.

## Example

- we want $v(t) = x(at + b)$.
  - Start with shifting:

$$z(t) = x(t + b)$$
$$s(t) = z(at) = x(at + b) = v(t) \text{ (OK)}$$

  - Start with scaling:

$$z(t) = x(at)$$
$$r(t) = z(t + b) = x(at + ab) \neq v(t) \text{ (!!)}$$

# Transformation of signals

## Questions

15 How the order affects $v(t) = x(-t+b)$? ¿How the order affects$x(-at)$?

16 Given 15, which is the order to follow when we have transformation of the independent variable?

## Transformations of the dependent variable

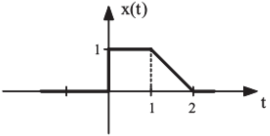
- Let be $a$ a scalar, then $y(t) = ax(t)$ is an amplified $(a > 1)$ or reduced version $(0 < a < 1)$ of the signal $x(t)$ $\forall t$ .
- $y(t) = x(t) + a$ is a new signal that just adds the quantity $a$ to every valued of the signal $s(t)$
- $y(t) = -x(t)$ is just change the sign of every value of the signal $x[t]$.

# Transformation on Signals

## Questions

**17** (*)Let be $x(t)$ the signal in the Figure. Sketch and label properly the following transformations:

- $y_1(t) = x(-t+1)$
- $y_2(t) = x(2t+3)$
- $y_3(t) = x\left(\frac{3}{2}t+1\right)$
- $y_4(t) = -2 \cdot x\left(-\frac{t}{4}+1\right)+3$



# Index

# Basic Signals

### Why we need basic signals

- We are going to study different simple and basic signals that are going to be use as **building blocks** to compose more complex signals.
- Why is that?
  - They are simple signals, so their properties can be studied easily.
  - Almost any signal can be composed as a linear combination of these building blocks.
  - The transformation of a simple signal by a system is easy to study.

# Basic Signals

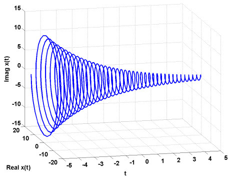
## Continuous-time complex exponential (I)

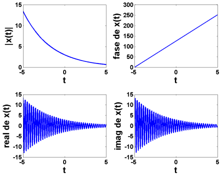
- The more general expression for a continuous-time complex exponential is:

$$x(t) = C \cdot e^{at}$$

where $C, a \in \mathbb{C}$, dados por $C = |C| \cdot e^{j\phi}$ y $a = \sigma + j\Omega$. Therefore,

$$x(t) = |C|e^{j\phi}e^{\sigma t}e^{j\Omega t} = |C|e^{\sigma t}e^{j(\Omega t+\phi)} = |C|e^{\sigma t}(\cos(\Omega t + \phi) + j\sin(\Omega t + \phi))$$




- Depending upon the values of these parameters the complex exponential can exhibit different characteristics.

## Question

**18** Find the magnitude, phase, and real and imaginary partos of $x(t)$, givne by the continuous-time complex exponential. What is the magnitude and phase of $x(t)$ at $t = 0$, and at $t = 1$, and at $t = -\pi$?

# Basic Signals

## Continuous-time complex exponential (II)

- **Real exponential** when $C, a \in \mathbb{R}$. That is, $x(t) = Ce^{\sigma t}$.
  - It can be a growing exponential($\sigma > 0$) or a decaying exponential ($\sigma < 0$).
- **Purely imaginary exponentials**, when $C \in \mathbb{C}$ but $a = j\Omega$ is purely imaginary. In that case,

$$x(t) = Ce^{j\Omega t} = |C| \left(\cos(\Omega t + \phi) + j \sin(\Omega t + \phi)\right)$$

  - It is very easy to establish the relationship between complex exponentias an sinusoidal signals:

$$e^{j\theta} = \cos\theta + j\sin\theta; \quad \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right); \quad \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

  therefore

$$\cos\Omega t = \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right); \quad \sin\Omega t = \frac{1}{2j}\left(e^{j\Omega t} - e^{-j\Omega t}\right)$$

  - In a purely imaginary exponentail (or in a sinusoidal signal for that matter) the frequency can grow infinitely.

# Basic Signals

## Questions

**19** Let be the following sinusoidal signals $x(t) = 100\cos(400\pi t + 60^o)$.

1. What is the maximum amplitude of the signal?
2. What is the frequency in Herz? What is the frequency in $rad/sg$?
3. What is the phase in radianes? What is the phase in degrees?
4. What is the period in milliseconds?
5. What is the first time, after $t = 0$, that $x = 100$?

**20** Show that a purely imaginary exponential signal is always periodic.

**21** (*)Find whether the following signals are periodic or not. If periodic, find the fundamental period.eriódica, especifique su periodo fundamental.

- $x_1(t) = j \cdot e^{10jt}$
- $x_2(t) = e^{(-1+j)t}$
- $x_3(t) = 2\cos(10t + 1) - \sin(4t - 1)$
- $x_4(t) = 1 + e^{j\frac{4\pi t}{7}} - e^{j\frac{2\pi t}{5}}$
- $x_5(t) = [\cos(2t - \frac{\pi}{3})]^2$

**22** (*)Sketch the following signals and indicate, using the plot, whether they are periodic or not.

- $x_0(t) = u(t) - u(t-1)$
- $x_1(t) = \sum_{k=-1}^{2} x_0(t - 2k)$
- $x_2(t) = \sum_{k=-\infty}^{\infty} x_0(t - 2k)$

# Basic Signals

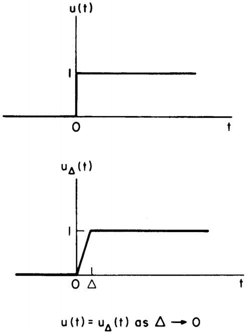
## Continuous-Time Unint Step

- **The continuous-time unit step** is defined as follows::

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0 \end{cases}$$

- Note that the unit step is discontinuous at $t = 0$. To solve this, we define $u(t)$ using an approximation signal:

$$u_\Delta(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{\Delta}, & 0 \leq t \leq \Delta \\ 1, & t > \Delta \end{cases}$$




- Therefore, $u_\Delta(t)$ is a continuous approximation of the unit step and

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

# Basic Signals

## Continuous-Time Unit Impulse

- Also knwon as **Dirac delta**:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

  but **area equal to 1**.

- We can define unit impulse as the **first derivative** of the unit step:

$$\delta(t) = \frac{du(t)}{dt}$$

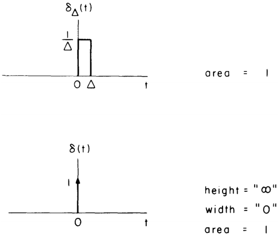
  But this arise some problems, since $u(t)$ is discontinuous at $t = 0$.

- We can use the approximation of the unit step, $u_\Delta(t)$, for which the deritave is well defined:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

- **Note** that $\delta_\Delta(t)$ is a short pulse, of duration $\Delta$ and with unit area for any value of $\Delta$. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintining its unit are. Therefore, at the limit:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



# Basic Signals

## Properties of the unit impulse

1. The area under the function is 1:
$$\int_{-\infty}^{+\infty} \delta(\tau)d\tau = 1$$
2. Scaling property:
$$\delta(at) = \frac{1}{|a|}\delta(t)$$
3. Even property
$$\delta(-t) = \delta(t)$$
4. Sampling property
$$x(t)\delta(t) = x(0)\delta(t)$$
5. Sampling property (ii)
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$
6. Sampling property (iii)
$$x(t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t_0-\tau)d\tau$$
7. Therefore, any continuous-time signal can be decompose as a (infinte) linear combination of shifted and scaled unit impulses
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

# Basic Signals

## Questions

27 Show, and discuss, the meaning of the properties 4. 5 and 6.

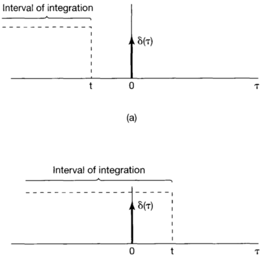
28 Show that the area under the signal $x(t) = A\delta(t)$ is equal to $A$. **Hint:** Use the approximation $\delta_{\triangle}(t)$.

# Basic Signals

## Relationship between unit step and unit impulse (I)

- The relationship between unit step and unit impulse allows to deal with derivative of discontinuities
- Running integral definition:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



# Basic Signals

## Relationship between unit step and unit impulse (II)

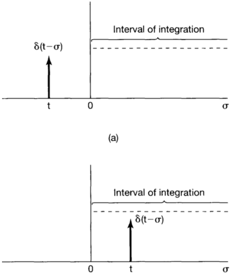
- We can use an alternative definition:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

with $\sigma = t - \tau$

- Or equivanlently:

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$



# Basic Signals

## Relationship between unit step and unit impulse (III)

- **Derivative of discontinuites**

$$\delta(t) = \frac{du(t)}{dt} \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

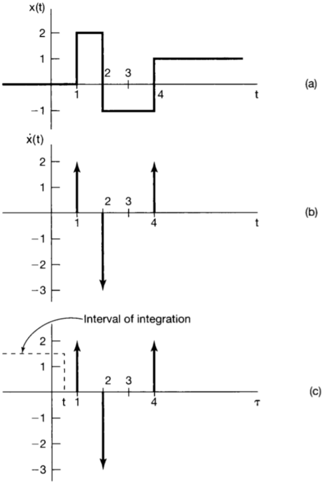



**Figure 1.40** (a) The discontinuous signal $x(t)$ analyzed in Example 1.7; (b) its derivative $\dot{x}(t)$; (c) depiction of the recovery of $x(t)$ as the running integral of $\dot{x}(t)$, illustrated for a value of $t$ between 0 and 1.

# Basic Signals

## Questions

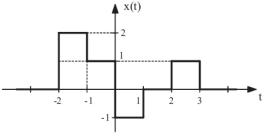
29 (*)Find and sketch the running integral of:

- $x_1(t) = \delta(t) - \delta(t-2)$
- $x_2(t) = -\delta(t+3) + \delta(t-1) + 3\delta(t-3)$

30 (*)Find and sketch the derivative of:

- $x_1(t) = u(t+3) - 2u(t+3) + u(t+6)$
- $x_2(t) = 3u(t) - 2.5u(t-3)$
- $x_3(t) = u(t+1) + e^t u(t-3) - 2u(t)$
- $x_4(t) = \sin(\pi t)u(-t)$

31 Find the analytical expression and sketch the derivative of $x(t)$. Decompose $x(t)$ as a sum of unit steps.



# Basic Signals

## More Basic Signals

- **Rectangular Pulse :**

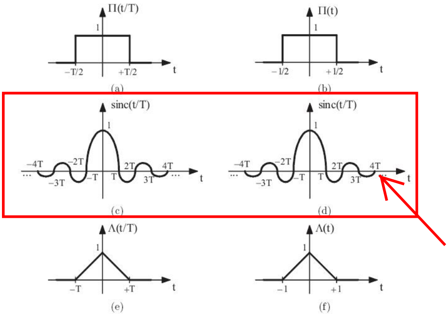
$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1; & -\frac{T}{2} \leq t < +\frac{T}{2} \\ 0; & otherwise \end{cases}$$

- **Sinc** (cardinal sine):

$$sinc\left(\frac{t}{T}\right) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}$$

- **Triangular Pulse:**

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \frac{t}{T} + 1; & -T \leq t \leq 0 \\ -\frac{t}{T} + 1; & 0 \leq t \leq +T \\ 0; & resto \end{cases}$$



# Basic Signals

## Questions

23 (*)Express $x(t) = \Pi(t)$ as a sum of shifted and scaled unit steps.

24 (*)Express $x(t) = u(t)$ as a sum of shifted and scaled rectangular pulses.

25 Sketch the rectangular pulse, the sinc and the triangular pulse for $T = 1$ and $T = 5$.

26 Sketch the following signals.

- $x_1(t) = \sum_{k=-\infty}^{\infty} (2\Lambda(t - 5k) - \Lambda(t - 2 - 5k))$
- $x_2(t) = \sum_{k=-\infty}^{\infty} (\Pi(t - 5k) - \Pi(t - 1 - 5k))$
- $x_3(t) = sinc(t - 5\pi)$

# Index

# Reference

## Reference for this topic


1. Chapter 1. *Signal and Systems*. A. V. Oppenheim, A.S. Willsky. Pearson Educación. 1997, $2^a$ edición.
2. Sections 1.0, 1.1, 1.2, 1.3 y 1.4.

# Index

# Problems

## Problem 1 (*)

Express the following signals as complex exponentials:

1. $x(t) = 2\cos\left(2\pi 60t + \frac{\pi}{4}\right)$.
2. $x(t) = 2\cos\left(t + \frac{\pi}{6}\right) + 4\,\text{sen}\left(t - \frac{\pi}{3}\right)$.

[Sol: (a) $x(t) = e^{j2\pi 60t}e^{j\frac{\pi}{4}} + e^{-j2\pi 60t}e^{-j\frac{\pi}{4}}$. (b) $x(t) = -e^{j\left(t+\frac{\pi}{6}\right)} - e^{-j\left(t+\frac{\pi}{6}\right)}$ ]

## Problem 2 (*)

Fing the magnitude and phase (as a function of $t$), as well as the average power and energy, for the following signals:

1. $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$.
2. $x(t) = \cos(t)$.
3. $x(t) = e^{-2t}u(t)$.

[Sol: (a) $P_\infty = 1, E_\infty = \infty$. (b) $P_\infty = 1/2, E_\infty = \infty$. (c) $P_\infty = 0, E_\infty = 1/4$. ]

# Problems

## Problem 3 (*)

Let be $x(t)$ a signal with $x(t) = 0$ para $t < 3$. For each of the following signals, find the values of $t$ that makes $x(t) = 0$.

1. $x(1 - t)$.
2. $x(t/3)$.
3. $x(3t)$.
4. $x(1 - t) + x(2 - t)$.
5. $x(1 - t) \cdot x(2 - t)$.

[Sol: (a) $t > -2$. (b) $t < 9$. (c) $t < 1$. (d) $t > -1$. (e) $t > -2$. ]

## Problem 4

Find the real part of the following signals and express them in the form $Ae^{-\alpha t} \cos \omega t + \phi$, where $A, \alpha, \omega, \phi$ are real numbers, with $A > 0$ and $-\pi \leq \phi \leq \pi$.

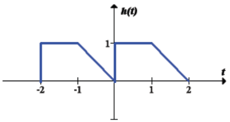
1. $x(t) = -2$.
2. $x(t) = \sqrt{2} e^{j\pi/4} \cos (3t + 2\pi)$.
3. $x(t) = e^{-t} \sin (3t + \pi)$.
4. $x(t) = je^{(-2+j100)t}$.

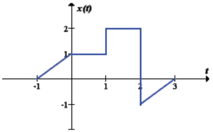
[Sol: (a) $A = 2, \alpha = 0, \omega = 0, \phi = \pi$. (b) $A = 1, \alpha = 0, \omega = 3, \phi = 0$.
(c) $A = 1, \alpha = 1, \omega = 3, \phi = \pi/2$. (d) $A = 1, \alpha = 2, \omega = 100, \phi = \pi/2$. ]

# Problems

**Problem 5 (*)**

Given the signals $x(t)$ and $h(t)$, sketch each of the following signals.




1. $h(t+3)$.
2. $h\left(\frac{t}{2}-2\right)$.
3. $h(1-2t)$.
4. $4h\left(\frac{t}{4}\right)$.
5. $h\left(\frac{t}{2}\right)\delta(t+1)$.
6. $h(t)[u(t+1)-u(t-1)]$.
7. $x(t)h(t+1)$.
8. $x(t)h(-t)$.

# Problems

## Problem 6

Determine whether the following signals are periodic. If they are, find the period.

1. $x(t) = 2\cos\left(3t + \frac{\pi}{4}\right)$.
2. $x(t) = e^{j(\pi t - 1)}$.
3. $x(t) = 2\cos\left(\frac{\pi}{4}t\right) + \sin\left(\frac{\pi}{8}t\right) - 2\cos\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.

[Sol: (a) $T = 2\pi/3$s. (b) $T = 2$s. (c) $T = 16$s. ]

## Problem 7 (*)

Find the derivative of the following signals:

1. $$x(t) = \begin{cases} 0, & t < 1 \\ 2, & 1 \le t < 2 \\ -1, & 2 \le t < 4 \\ 1, & t \ge 4 \end{cases}.$$
2. $x(t) = u(t+2) - u(t-2)$.
3. $x(t) = e^{j\pi t}u(t)$.

# Problems

## Problem 8 (*)

Integrate the following signals computing $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$:

1. $x(t) = \delta(t+2) - \delta(t-2)$.
2. $x(t) = u(t+2) - u(t-2)$.
3. $x(t) = e^{j\pi t} u(t)$.

[Sol: (a) $y(t) = u(t+2) - u(t-2)$. (b) $y(t) = (t+2)u(t+2) + (2-t)u(t-2)$. (c) $y(t) = -\frac{j}{\pi}\left(e^{j\pi t} - 1\right) u(t)$. ]

## Problem 9

Let be $x(t) = \delta(t+2) - \delta(t-2)$. Determine the total energy of $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$.
[Sol: $E_\infty = 4$ J.]

# Problems

## Problem 10 (*)

Let be a periodic signal with period $T = 2$, given by:

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ -2, & 1 \leq t < 2 \end{cases}$$

The derivative of this signal is related with the impulse train with period 2 sec, given by:

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

Determine the values $A_1, t_1, A_2$, y $t_2$, so that

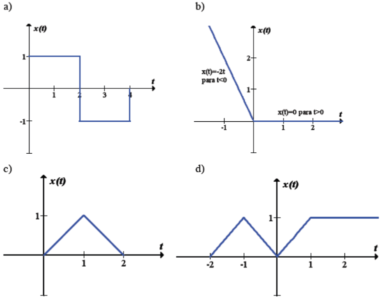
$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2)$$

[Sol: $A_1 = 3, t_1 = 0, A_2 = -3, t_2 = 1.$ ]

# Problems

## Problema 11

Sketch the even and ood part of the following signals:




## Problem 12 (*)

Show that $\delta(2t) = \frac{1}{2}\delta(t)$.

# Problems

## Problem 13 (*)

We define the function $\Phi_{xy}(t)$ of two signals $x(t)$ and $y(t)$ as:

$$\Phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau$$

- What is the relationship between $\Phi_{xy}(t)$ and $\Phi_{yx}(t)$?
- Let's suppose that $x(t)$ is periodic. Is also periodic $\Phi_{xx}(t)$ ? If so, what is the period?
- Find the odd part of $\Phi_{xx}(t)$.

## Problem 14 (*)

Show that $\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_{par}^2(t)dt + \int_{-\infty}^{\infty} x_{impar}^2(t)dt$.