TEMA 2 - SYSTEMS IN THE TIME DOMAIN

LINEAR SYSTEMS WITH CIRCUIT APPLICATIONS

Óscar Barquero Pérez
Departamento de Teoría de la Señal y Comunicaciones - Universidad Rey Juan Carlos
Based on Andrés Martínez and José Luis Rojo slides oscar.barquero@urj.es
(updated September 2, 2018)

Biomedical Engineering Degree

Index

Introduction

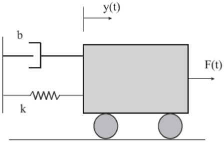
System definition

- A **system** can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

Example: dynamical systems

- A simple dynamical system: a little car on a surface, tide to the wall by a spring.
- Law of forces:

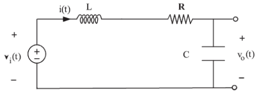
$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = F(t)$$



Introduction

Example: a circuit system

- RLC circuit. The input is $v_i(t)$, an arbitrary signal.
- The output $v_o(t)$ will be a transformation of the input. Is there a equation relating them?



$$LC\frac{d^2v_0(t)}{dt^2} + RC\frac{dv_0(t)}{dt} + v_0(t) = v_i(t)$$

- It is a second order diferential equation. Note the similarity with the mechanical system..
- The signal and systems tools can be used in many applications.

Introduction

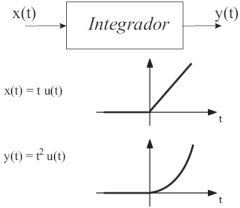
Example: Integrator Systems

- We have an integrator system, which input is the signal $x(t) = tu(t)$. Therefore, for $t < 0$:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} 0d\tau = 0$$

whereas for $t \geq 0$:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{0}^{t} \tau d\tau = \left[\frac{\tau^2}{2}\right]_0^t = \frac{t^2}{2}$$



- The output can be expressed using the unit step signal:

$$y(t) = \frac{1}{2}t^2u(t)$$

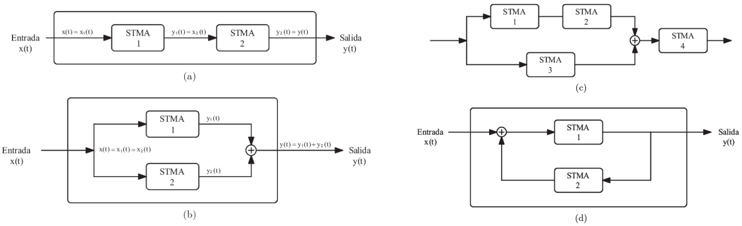
Interconnections of Systems

Innterconnections of Systems (I)

- Many real systems are built as interconnections of different simple subsystems to create a complex system. There are several basic system interconnections:
  1. **Series interconnection.** The output of one systems is the input of the following system.
  2. **Parallel interconneciton** The same input is applied to the interconnected systems, and the output is the sumn of the individual ouputs.
  3. **Combination** We can combine series and parallel interconnections to create more complicated systems.
  4. **Feedback interconnection** The output of the system is feeded back to the input.

Interconexión de sistemas

Interconexiones de sistemas (II)



Cuestiones

1. (*)Find the equation for the series interconnection in Figure (a):

$$(S1) y_1(t) = x_1^2(t); \quad (S2) y_2(t) = e^{x_2(t)}; \quad (S3) y_3(t) = x_3(t-1)$$

2. (*)Find the equation for the following interconnection in Figure (b) $(S1) y_1(t) = x_1^2(t)$, $(S2) y_2(t) = x_2(t+3)$, $(S3) y_3(t) = 2x_3(t)$, y $(S4) y_4(t) = \int_{-\infty}^{t} x_4(\tau) d\tau$.
3. (*)Find the equation for the following feedback interconnection in Figure (d) $(S1) y_1(t) = x_1^2(t)$ y $(S2) y_2(t) = x_2(t-1)$.

Index

System Properties

Memory

- A system is said to be **memoryless** if its output, for each value of t, is dependent only on the input at that same time, that is, $y(t) = f(x(t))$.
- A systems is said to be **with memory** in any other case. That is wheneve the output of the system dependens on *past* and/or *future* values of the input.

Examples

- Memoryless systems:
  - $y(t) = (2x(t) - x^2(t))^2$.
  - A resistor, in which $y(t) = Rx(t)$.
- Systems with memory:
  - A delay system, $y(t) = x(t-2)$.
  - A capacitor $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$.

System properties

Questions

4 (*)Determine if each of the following systems are memoryless or with memory:

1. $y(t) = t \cdot x(t)$.
2. $y(t) = x(t+4)$.
3. $y(t) = \sum_{k=-3}^{0} x(t-k)$.
4. $y(t) = x(-t)$.
5. $y(t) = \cos(3t)x(t)$.
6. $y(t) = x(t) + 0.5y(t-2)$.

System Properties

Invertibility

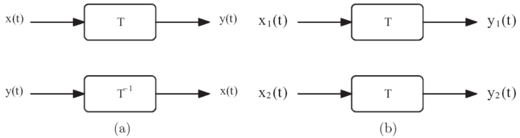
- A system is said to be **invertible** if distinct inputs lead to distinct outputs. In other words, for any known ouput, it is possible to uniquely recover the input that generate this output.

$$T \text{ es invertible } \Leftrightarrow \exists T^{-1} \text{ tal que si } x(t) \xrightarrow{T} y(t) \text{, entonces } y(t) \xrightarrow{T^{-1}} x(t)$$

- Or

$$x_1(t) \neq x_2(t) \Rightarrow y_1(t) \neq y_2(t)$$

- To study the invertibility:
  1. If the system is invertible, then an *invervse system* exists.
  2. It the system is non invertible, then I need to provide a counter-example. That is two distinc inputs produce the same output. The system should *destroy* information.



System properties

Exmaples

- Invertible systems:
  - $T: y(t) = x(t) + 5 \rightarrow T^{-1}: w(t) = v(t) - 5.$
  - $T: y(t) = x^3(t) \rightarrow T^{-1}: w(t) = v^{1/3}(t).$
- Noninvertible systems:
  - $y(t) = 0$. For example, $x_1(t) = 3$ y $x_2(t) = \cos(t)$ produce the sampe output.
  - $y(t) = x^2(t)$. For example, $x_1(t) = 2$ y $x_2(t) = -2$ produce the same output $y(t) = 4$.
- The systems $y(t) = \int_{-\infty}^{t} e^{t-\tau} x(\tau) d\tau$ is invertible. Shoy with subsytems
- The system $y(t) = \int_{-\infty}^{t} \sin(\tau) x(\tau) d\tau$ is noninvertible.
- If a system T is invertible, will be T^{-1} invertible? The answer is no. For example, integrator and the derivative systems.

Questions

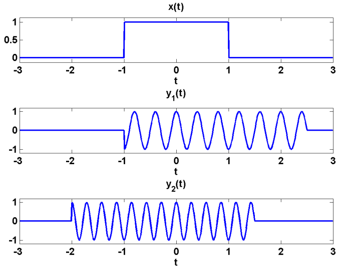
5 (*)Determine if each of the following systems is invertible:

1. $y(t) = x(t-4)$.
2. $y(t) = \cos[x(t)]$.
3. $y(t) = tx(t)$.
4. $y(t) = \frac{dx(t)}{dt}$.

System Properties

Causality

- A system is **causal** is the output at any time depends only on values of the input at the present time (same time) and in the past. Such a system is often referred to as being *physically feasibles* or *nonanticipative*.
- A system is non-causal when the output at any time depends on values of the future.
- A system is *anticausal*if the output at any time depends only on values of the input at the present time and in the future.



System Properties

Exmaples: Causility

- The system $y(t) = x(t) - x(t-1)$ is .
- The system $y(t) = 2x(t+3)$ is .
- The system $y(t) = x(t-1) - x(t+3)$ is .

Questions

6 (*)Study the causality of the following systems:

1. $y(t) = x(-t)$.
2. $y(t) = x(t) \cdot \cos(t+1)$.
3. $y(t) = Ax(t)$.
4. $y(t) = \int_{-\infty}^{t+2} x(\tau) d\tau$.
5. $y(t) = Par\{x(t-1)\}$.

System Properties

Exmaples: Causility

- The system $y(t) = x(t) - x(t-1)$ is causal.
- The system $y(t) = 2x(t+3)$ is .
- The system $y(t) = x(t-1) - x(t+3)$ is .

Questions

6 (*)Study the causality of the following systems:

1. $y(t) = x(-t)$.
2. $y(t) = x(t) \cdot \cos(t+1)$.
3. $y(t) = Ax(t)$.
4. $y(t) = \int_{-\infty}^{t+2} x(\tau)d\tau$.
5. $y(t) = Par\{x(t-1)\}$.

System Properties

Exmaples: Causility

- The system $y(t) = x(t) - x(t-1)$ is causal.
- The system $y(t) = 2x(t+3)$ is anticausal.
- The system $y(t) = x(t-1) - x(t+3)$ is .

Questions

6 (*)Study the causality of the following systems:

1. $y(t) = x(-t)$.
2. $y(t) = x(t) \cdot \cos(t+1)$.
3. $y(t) = Ax(t)$.
4. $y(t) = \int_{-\infty}^{t+2} x(\tau) d\tau$.
5. $y(t) = Par\{x(t-1)\}$.

System Properties

Exmaples: Causility

- The system $y(t) = x(t) - x(t-1)$ is causal.
- The system $y(t) = 2x(t+3)$ is anticausal.
- The system $y(t) = x(t-1) - x(t+3)$ is noncausal.

Questions

6 (*)Study the causality of the following systems:

1. $y(t) = x(-t)$.
2. $y(t) = x(t) \cdot \cos(t+1)$.
3. $y(t) = Ax(t)$.
4. $y(t) = \int_{-\infty}^{t+2} x(\tau) d\tau$.
5. $y(t) = Par\{x(t-1)\}$.

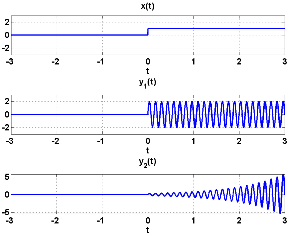
System properties

Stability

- A system is sadi to be **stable** when bounded inputs leads to bounded outputs, for any time, t. Mathematically, this property is expressed as (BIBO):

$$|x(t)| < K_x < \infty \Rightarrow |y(t)| < K_y < \infty$$

- A system is unstable whenever we are able to find a *specific* bounded input that leads to an unbounded output. Finding one such example enable sus to conclude that the given system is unstable.



System properties

Questions

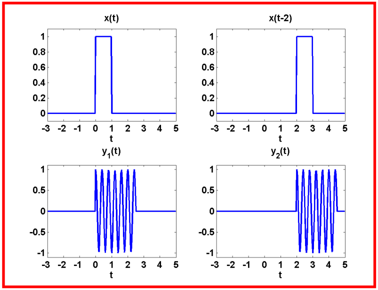
7 (*)Study the stability of the following systems:

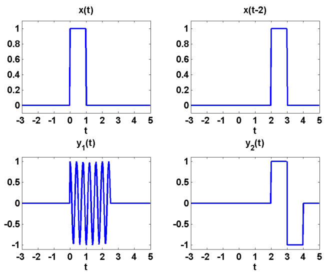
1. $y(t) = [x(t)]^2$.
2. Derivative system: $y(t) = \frac{dx(t)}{dt}$.
3. Integrator system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$.
4. $y(t) = t \cdot x(t)$.
5. $y(t) = x(-t)$
6. $y(t) = x(t-2) + 3x(t+2)$.
7. $y(t) = Impar\,(x(t))$.
8. $y(t) = e^{x(t)}$.

System Properties

Time Invariance (I)

- A system is **time invariant** if the behavior and characteristicas of the system are fixed over time.
- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.
- The system is said to be *time variant* otherwise.





System Properties

Time Invariance (II)

- There is a systematic way to study invariance:
  1. Let be $x_1(t)$ an arbitrary input, and let be $y_1(t)$ the output for this particular input.
  2. The output is shifted by a given t_0, $y_1(t-t_0)$.
  3. Then, consider a second input, $x_2(t)$, which is obtained by shifting $x_1(t)$ in time, $x_2(t) = x_1(t-t_0)$. The corresponding output is $y_2(t)$.
  4. We have to compare both outputs $y_2(t) \stackrel{?}{=} y_1(t-t_0)$, if the equality holds, then the system is time invariant.
- We can always use a counter-exmample to proof that the system is variant.

Questions

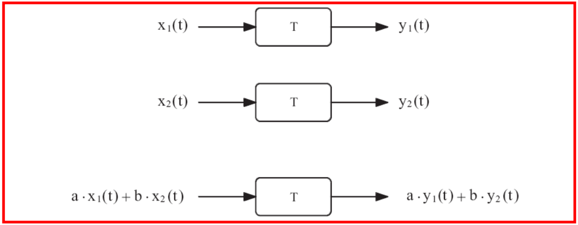
8 (*)Study the time invariance of the following systems.

1. $y(t) = \cos\left[x(t)\right]$.
2. $y(t) = t + x(t)$.
3. $y(t) = tx(t)$.
4. $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.
5. $y(t) = \frac{dx(t)}{dt}$.

System Properties

Linearity

- A systems is said to be **linear** when it possesse the property of superposition. The property of superposition has two properties: *additivity* and *scaling* or *homogeneity*:
  1. Additivity: the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
  2. Scaling: the response to $ax_1(t)$ is $ay_1(t)$ (whit $a \in \mathbb{C}$).
- The two properties can be combined. A system is linear when the response to $ax_1(t) + bx_2(t)$ is $ay_1(t) + by_2(t)$.
- Note, as a consequence, we can show that fo linear system an input which is zero for all time results in an output which is zero for all time.



System properties

Questions

9 (*)Study the linearity of the following systems:

1. $y(t) = t \cdot x(t)$.
2. $y(t) = x^2(t)$.
3. $y(t) = 2x(t) + 3$ (show using zero input property).

Comments on system properties

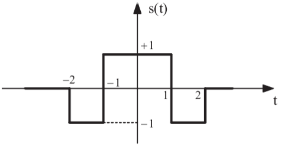
Study these properties at home

- Every memoryless system is causal
- The outuput of a linear system for a zero input is a zero output.
- If a system is time invariant, periodic inputs lead to periodic outputs.
- Let be a continous linear system. The system is causal if and only if, for an input which is zero up to a time t_0, leads to and output which is zero up to the same time t_0-
- A linear system is invertible if and only if the only input signal that leads to a zero output is the zero signal.

System properties

Questions

10 (*)Consider the following sytems:



$$y(t) = \begin{cases} 0: & t < 0 \\ x(t) + x(t-2): & t \geq 0 \end{cases}$$

$$y(t) = \begin{cases} 0: & x(t) < 0 \\ x(t) + x(t-2): & x(t) \geq 0 \end{cases}$$

Find the output signals when the input signal is $s(t)$. Study the properties of boths systems.

11 (*)Study the properties of the system $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

12 (*)Study the properties of the system for which we know the following outputs:

1. $x_1(t) = \delta(t) \rightarrow y_1(t) = 4u(t)$.
2. $x_2(t) = u(t-1) \rightarrow y_2(t) = t\sin(4t)u(t)$.
3. $x_3(t) = \delta(t-4) \rightarrow y_3(t) = 4\delta(t-2)$.
4. $x_4(t) = \delta(t) + \delta(t-4) \rightarrow y_4(t) = 4u(t)$.

Index

Linear Time-Invariant Systems

The system is linear and its "features" do not change with the time

Linear Time-Invariant Systems

- What if:
  1. It is possible to expresss any signal as a composition of simple signals. posible expresar una señal cualquiera en función de señales sencillas.
  2. It is possible to characterize a the output of a systems to those simple signals.

  Then, it will be very easy to compute the output of a system for every signal.
- Simple signals: unit impulse, unit step, pulse, sinc, sinusoides and exponentials.
- A set of important systems are those that are linear and time-invariant.

Linear Systems

- If we can decomopose any signal as a linear combination of simple shifted signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k s_k(t) = \sum_{k=-\infty}^{\infty} a_k s(t - kT)$$

- The output of the system for a input $s_k(t)$ is $v_k(t)$. If the system is linear, the output is:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k v_k(t)$$

Linear and Time-Invariant Systems

Linear and Time-Invariant Systems

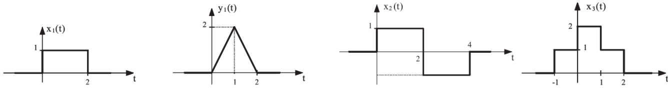
- But the previous expression is complex, since we need to know a bunch of outputs, one for echa shifted input signal, so we need a set of $v_k(t)$ for each k.
- If the system is time-invariant we only need to know the output fo the input $s(t) \Rightarrow v_0(t)$. Since the system is time-invariant the output for $s(t-kT)$ is $v_0(t-kT)$, and therefore:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k s_k(t) = \sum_{k=-\infty}^{\infty} a_k s(t-kT)$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k v_k(t) = \sum_{k=-\infty}^{\infty} a_k v_0(t-kT)$$

Question

13 (*)Consider the following *LTI* system, which response to the input $x_1(t)$ is the signal $y_1(t)$. Determine the response of the system for inputs $x_2(t)$ and $x_3(t)$.



A LTI system can be expressed with:
- convolution
- linear diff. equations with constant coefficients

Unit Impulse Response of LTI Systems

- We have seen that any continuous-time signal can be expressed as a infinte combination of unit impulses. Since the unit impulse has zero width, the sum should be compute using the integral:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- If $h_\tau(t)$ is the system response to the unit impulse at time instant τ, then the output fot the input $\delta(t)$ should be $h_0(t)$.
- It the system is LTI, then, for an input $\delta(t-\tau)$, the output should be $h_\tau(t) = h_0(t-\tau)$. The output $h_0(t)$, usually denoted simply as $h(t)$, is known as **Unit Impulse Response**.
- Therefor, the ouput of an LTI system can be obtained just as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- The previoues expression is known as the **Convolution Integral** between two signals.
- **If we know the unit impulse response of a LTIS, then we can compute the response for any signal. That is, the unit impulse response characterize completely the LTIS.**

Convolution

A LTI system can be expressed with:
- convolution
- linear diff. equations with constant coefficients

LTIS Convolution

- To characterize a LTIS, we compute its unit impulse response $h(t)$.
- To obtain the LTIS response for any particular signal $x(t)$, we use the convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

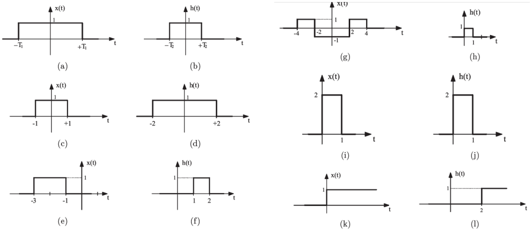
- So that, to get the output signal we only need to solve an integral.

Convolution procedure

- The procedure to compute the convolution is usually easier using graphical representations:
  1. Sketch $x(\tau)$.
  2. Mark the changes in the analytical expression of $x(\tau)$.
  3. For each relative position, starting at $t = -\infty$:
     1. Sketch $h(t-\tau)$.
     2. Mark the changes in the analytical expression of $h(t-\tau)$.
     3. Write down the integral and the correct limits regarding the different marks.
     4. Solve the integral.
     5. Establish the valid range for this expression.
  4. Put together the final solution.

Convolution

Questions



14 Compute the convolutions in the Figure, given by $y(t) = x(t) * h(t)$.

15 Convolve $x(t) = u(t-1)$ with $h(t) = e^{-t}u(t)$.

16 Convolve $x(t) = u(t) - u(t-1)$ with $h(t) = \delta(t-2)$.

17 Convolve $x(t) = u(t) - u(t-1)$ with $h(t) = \delta(t)$.

18 Convolve $x(t) = \Lambda(t)$ with $h(t) = \delta(t+1) - \delta(t-1)$.

19 Convolve $x(t) = e^{-\alpha t}u(t)$ with $h(t) = e^{-\beta t}u(t)$.

20 Convolve $x(t) = u(t+1) - u(t-2)$ with $h(t) = e^{-(t-2)}u(t-2)$.

21 Convolve $x(t) = 2u(t+1) - 2u(t-2)$ with $h(t) = sen(\pi t)u(t+3)$.

Index

Properties of the Convolution

Convolution and association of LTI Systems

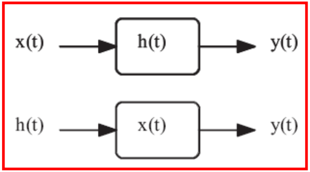
- Next we describe the properties of the convolution. Note that each of the following mathematical properties has a practial applicaiton of the systems.

Conmutative

- Changin the order of the signals does not change the result of the convolution:

$$x(t) * h(t) = h(t) * x(t)$$

- In terms of systems and signals, the output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI systems with input $h(t)$ and unit impulse response $x(t)$.



- *Proof:* $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau =_{(t-\tau=\sigma)} \cdots = h(t) * x(t)$

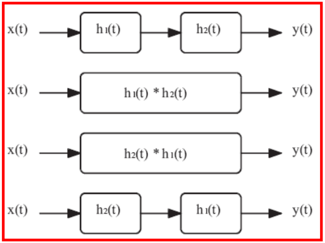
Properties of the Convolution

Associative

- El orden en que se realizan las convoluciones no altera el resultado:

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

- *Demostración:* ejercicio.



- When several LTI systems are interconnected in series, the global output signal is independent on the interconnection order.uando varios SLIT se conectan en serie, la señal de salida obtenida es independiente del orden en que se conecten los sistemas.
- The series interconnection of two LTI systems is equivalent to the single system with unit impulse response the convolution of the original unit impulse responses: $h_{eq} = h_1(t) * h_2(t)$.

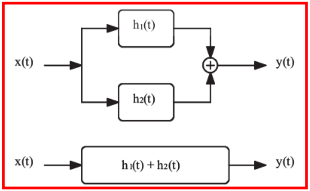
Properties of the Convolution

Distributive

- Convolution has the distributive property over addition:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

- *Proof:* TPC.

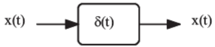


- Distributive property can be interpreted in systems language. LTI interconnected in parallel are equivalent a signle LTI systems with unit impulse response the sum of the original unite impulse responses. $h_{eq}(t) = h_1(t) + h_2(t)$.

Identity element of the convolution

- The unit impulse is the identity elemento of the convolution:

$$x(t) * \delta(t) = x(t)$$



Properties of the convolution

Convolution with a delayed impulse

- The convolution of a signal with a delayed unit impulse is the same original signal delayed the same amount of time:

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

IMP!!!!

- Proof:

$$y(t) = x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - t_0 - \tau)d\tau$$

The impulse is not zero at $t - t_0 - \tau = 0 \Rightarrow \tau = t - t_0$, therefore:

$$y(t) = \int_{-\infty}^{\infty} x(t - t_0)\delta(t - t_0 - \tau)d\tau = x(t - t_0)\int_{-\infty}^{\infty} \delta(t - t_0 - \tau)d\tau = x(t - t_0)$$

Question

22 (*)Compute $y(t) = x(t) * h(t)$ for $x(t) = u(t+2)$ and $h(t) = \delta(t-2) - \delta(t+2)$.

Properties of the convolution

Step response

- Let be an LTI suystems with unite impulse ($\delta(t)$) response $h(t)$.
- If the unit step ($u(t)$)response is $s(t)$, then:

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

  that is, the step response is the integral of the unit impulse response.
- Therefore, the impulse response is the derivative of the step response:

$$h(t) = \frac{ds(t)}{dt}$$

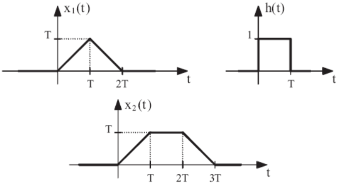
- The series interconnection of an LTI system ($h(t)$) with an **integrator** (differentiator) system has as impulse response the equivalent **integral** (derivative) of the $h(t)$.

Properties of the convolution

Questions

23 (*)Compute and sketch the following convolutions:

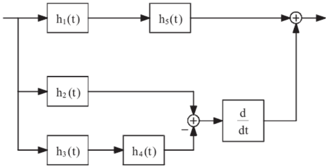
1. $h(t) * h(t)$.
2. $\frac{dx_1(t)}{dt} * h(t)$.
3. $\frac{dx_2(t)}{dt} * h(t)$.
4. $x_1(t) * h(t)$.
5. $x_2(t) * h(t)$.
6. $x_1(t) * \frac{dh(t)}{dt}$.
7. $x_2(t) * \frac{dh(t)}{dt}$.
8. $\frac{dh(t)}{dt} * \frac{dh(t)}{dt}$.



24 (*)Consider the interconnection of LTI systems given by the figure, wherw:

1. $h_1(t) = u(t) - u(t-3)$
2. $h_2(t) = h_3(t) = t \cdot u(t)$
3. $h_4(t) = \delta(t-1)$
4. $h_5(t) = h_1(-t)$

Compute and sketch the impulse response, $h_{eq}(t)$, of the equivalent system.



Properties of LTI Systems

Properties of LTI Systems

- The properties of the LTI systems are completely determined by its impulse response.
- If two LTI systems have the same impulse response, then they are the same system. This only holds for LTI systems.

Example: LTI systems and impulse response

- Let be an LTI system with $h(t) = \delta(t) - \delta(t-2)$. An alternativa representation can be obtained usign the input $x(t)$ and computing the output $y(t)$, that is:

$$y(t) = x(t) * h(t) = x(t) - x(t-2)$$

Although the equations are different (convolution and sum of delayed versions of the input) the LTI systems is the same.

Properties of LTI Systems

always memory,
except the case of
h(t)=A delta(t)

Memory in LTI Systems

- Let be an LTI system with $h(t)$. The output is given by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, the systems it memoryless iff , $h(\tau) = 0$ for all $\tau \neq 0$. That is, the only LTI systems memoryless are those with:

$$h(t) = A\delta(t)$$

Examples: LTI systems and memory

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ has memory.
- The LTI system with $h(t) = u(t) - u(t-1)$ has memory.

Properties of the LTI systems

we can also write the convolution in this way:

Causality for LTI Systems

- Let be an LTI systems with $h(t)$. The output is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, the impulse response of a causal LTI systems requieres to satisfy, $h(\tau) = 0$ for all $\tau < 0$, that is, causal LTI systems has the following type of impulse response:

$$h(t) = h(t)u(t)$$

Examples: LTI systems and causality

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ is causal.
- The LTI system with $h(t) = u(t) - u(t-1)$ is causal.
- The LTI system with $h(t) = u(t+1) - u(t)$ is anticausal.
- The LTI system with $h(t) = u(t+1) - u(t-1)$ is noncausal.

Properties of the LTI systes

Stability for LTI systems

- Let be an LTI systems with $h(t)$. The output is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, in order the system to be stable, bounded inputs should lead to bounded outputs.
- This can be expressed as:

$$|x(t)| \leq k_x \Rightarrow |y(t)| = |x(t) * h(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right| \leq$$

$$\leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq$$

$$\leq \int_{-\infty}^{\infty} k_x|h(\tau)|d\tau \leq k_x \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

integrable

Therefore, an LTI system is stable iff the impulse response is *absolutely integrabel*:

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau = k_h \leq \infty$$

Properties of the LTI systems

Examples: Stability and LTI systems

- The LTI system with $h(t) = \delta(t) - \delta(t-3)$ is stable.
- The LTI system with $h(t) = u(t) - u(t-1)$ is stable.
- The LTI system with $h(t) = u(t)$ is unstable.
- The LTI system with $h(t) = e^{-t}$ is unstable.
- The LTI system with $h(t) = \sin(t) * u(t)$ is unstable.

Invertibility of LTI Systems

- The inverse of the LTI system with $h(t)$ is the LTI with $h_i(t)$, that results in:

$$h(t) * h_i(t) = \delta(t)$$

- Obtain inverse of LTI systems is *hard task*, called *deconvolution*

Examples: LTI system and invertibility

- The inverse of $h(t) = \delta(t-3)$ is $h_i(t) = \delta(t+3)$.
- The inverse of $h(t) = u(t)$ is the differentiator.

Index

Bibliography

References for this topic

1. Chapters 1 y 2. *Signal and Systems*. A. V. Oppenheim, A.S. Willsky. Pearson Educación. 1997, 2^a edition.
2. Read sections 1.5, 1.6, 2.0, 2.2 y 2.3.

Index

Problems

Problem 1 (*)

Let be the continuous-time systems with input $x(t)$ and output $y(t)$, related by $y(t) = x\,(sen(t))$.

1. It is causal?
2. It is linear?

[Sol: (a) No. (b) Yes.]

Problem 2 (*)

Determine which properties (memory, time invariance, linear, causal, stable) have the following continuous-time systems :

1. $y(t) = x(t-2) + x(t+2)$.
2. $y(t) = x(t) \cdot cos(3t)$.
3. $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.
4. $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$
5. $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$
6. $y(t) = x(t/3)$.
7. $y(t) = \frac{dx(t)}{dt}$

[Sol: (1) Linear, stable. (2) memoryless, linear, causal, stable. (3) Linear. (4) Linear, causal, stable. (5) invariant, causal, stable. (6) Linear, stable. (7) invariant, linear.]

Problems

Problem 3 (*)

Comput and sketch the convolution of the following signals:

$$x(t) = \begin{cases} t+1, & 0 \leq t \leq 1 \\ 2-t, & 1 < t \leq 2 \\ 0, & \text{en otro caso} \end{cases} \qquad h(t) = \delta(t+2) + 2\delta(t+1)$$

Problem 4 (*)

Let be $x(t) = u(t-3) - u(t-5)$ y $h(t) = e^{-3t}u(t)$.

1. Compute $y(t) = x(t) * h(t)$.
2. Compute $g(t) = \left(\frac{dx(t)}{dt}\right) * h(t)$.
3. What is the relationship between $g(t)$ and $y(t)$?.

Problems

Problem 5 (*)

Determine whether each of the following systems is invertible. If it is noninvertible, find a counterexample, given by two different signals that lead to the same output.

1. $y(t) = x(t-4)$.
2. $y(t) = cos\,(x(t))$.
3. $y(t) = tx(t)$.
4. $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$.
5. $y(t) = x(t)x(t-1)$.
6. $y(t) = x(1-t)$.
7. $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$.
8. $y(t) = \frac{dx(t)}{dt}$.
9. $y(t) = x(2t)$.
10. $y(t) = \int_{-\infty}^{t} sin(\tau)x(\tau)d\tau$.

[Sol: (1) Invertible, $y(t) = x(t+4)$. (2) Noninvertible, $x_1(t) = x(t)$ y $x_2(t) = x(t) + 2\pi$. (3) Noninvertible, $x_1(t) = \delta(t)$ y $x_2(t) = 2\delta(t)$. (4) Invertible, $y(t) = dx(t)/dt$. (5) Noninvertible, $x_1(t) = \delta(t)$ y $x_2(t) = \delta(t-2)$. (6) Invertible. (7) Invertible, $y(t) = x(t) + dx(t)/dt$. (8) Noninvertible. (9) Invertible, $y(t) = x(t/2)$. (10) Noninvertible.]

Problems

Problem 6

Considere an LTI system with its output related with the input as:

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2) d\tau$$

1. What is the impulse response $h(t)$?
2. What is the output when the input is $x(t) = u(t+1) - u(t-2)$?

[Sol: (1) $h(t) = e^{-(t-2)} u(t-2)$s. (2) $y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 \leq t \leq 4 \\ e^{-(t-4)} - e^{-(t-1)}, & t > 4 \end{cases}$]

Problem 7

Answer the following questions

1. Considere an time-invariant systems with input $x(t)$ and output $y(t)$. Show that if $x(t)$ is periodic with period T, then, the output $y(t)$ is also periodic with period T.
2. Give an example of an time-invariant systems with an input una señal de entrada $x(t)$ nonperiodic that leads to an output $y(t)$ that is periodic.

[Sol: (2) $x(t) = t, y(t) = sen(x(t))$]

Problems

Problem 8 $(^*)$

Answer to the following questions:

1. Show that causality in a linear system is equivalent to: *For any instant time* t_0*, and any* $x(t)$*, such as* $x(t) = 0$ *para* $t < t_0$*, the output* $y(t)$ *is also zero for* $t < t_0$.
2. Find a nonlinear system that holds the previous property but being noncausal.
3. Find a nonlinear causal system that does not hold the previous property.

[Sol: (2)E.g. $y(t) = x(t)x(t+1)$. (3)E.g. $y(t) = x(t) + 1$.]

Problem 10

Let be $x(t) = u(t) - u(t-1)$, and $h(t) = x(t/\alpha)$, where $0 < \alpha \leq 1$.

1. Compute and sketch $y(t) = x(t) * h(t)$.
2. If $dy(t)/dt$ has three discontinuities, what is the value of α?

[Sol: (1) $y(t) = \begin{cases} t, & 0 \leq t < \alpha \\ \alpha, & \alpha \leq t < 1 \\ 1 + \alpha - t, & 1 \leq t < (1+\alpha) \\ 0, & \text{resto} \end{cases}$. (2) $\alpha = 1$.]

Problems

Problem 9

9. Para cada uno de los siguientes pares de formas de ondas, use la integral de convolución para encontrar la respuesta $y(t)$ a la entrada $x(t)$ del sistema LTI cuya respuesta al impulso es $h(t)$.

a) $\left.\begin{matrix} x(t) = e^{-\alpha t} \cdot u(t) \\ h(t) = e^{-\beta t} \cdot u(t) \end{matrix}\right\}$ para $\alpha \neq \beta$, y para $\alpha = \beta$

b) $x(t) = u(t) - 2 \cdot u(t-2) + u(t-5)$
$h(t) = e^{2t} \cdot u(1-t)$

c) $x(t)$ y $h(t)$ son como se muestra en el panel (a).

d) $x(t)$ y $h(t)$ son como se muestra en el panel (b).

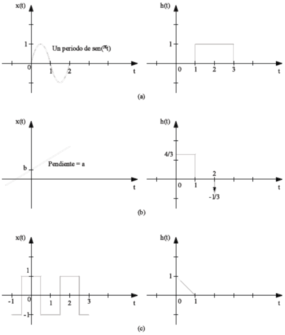
e) $x(t)$ y $h(t)$ son como se muestra en el panel (c).

[Sol.: (a) $y(t) = \begin{cases} e^{-\alpha t} \cdot t \cdot u(t) & : \ \alpha = \beta \\ \dfrac{e^{-\alpha t}}{\beta - \alpha} \cdot \left(e^{(\beta-\alpha)t} - 1\right) u(t) & : \ \alpha \neq \beta \end{cases}$;

(b) $y(t) = \begin{cases} \frac{1}{2} \cdot \left(e^{2t} - 2 \cdot e^{2\cdot(t-2)} + e^{2\cdot(t-5)}\right) & : \ t \leq 1 \\ \frac{1}{2} \cdot \left(e^{2} - 2 \cdot e^{2\cdot(t-2)} + e^{2\cdot(t-5)}\right) & : \ 1 \leq t \leq 3 \\ \frac{1}{2} \cdot \left(e^{2\cdot(t-5)} - e^{2}\right) & : \ 3 \leq t \leq 6 \\ 0 & : \ t > 6 \end{cases}$;

(c) $y(t) = \begin{cases} 0 & : \ t \leq 1 \\ \frac{1}{\pi} \cdot [1 - \cos(\pi \cdot (t-1))] & : \ 1 \leq t \leq 3 \\ \frac{1}{\pi} \cdot [\cos(\pi \cdot (t-3)) - 1] & : \ 3 \leq t \leq 5 \\ 0 & : \ t > 5 \end{cases}$;

(d) $y(t) = x(t)$; (e) $y(t) = \begin{cases} -t^2 + t + \frac{1}{4} & : \ -\frac{1}{2} < t < \frac{1}{2} \\ t^2 - 3t + \frac{7}{4} & : \ \frac{1}{2} < t < \frac{3}{2} \end{cases}$, periodo "2"]



Problems

Problem 11 $(*)$

Determine whether each of the following statements concerning LTIsystems is true or false. Justify your answers.

1. If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable.
2. The inverse of a causal LTI system is always causal.
3. If $|h(t)| \leq K$ for each t, where K if a given number, then the LTI systems with impulse response $h(t)$ is stable.
4. Si un SLIT tiene una respuesta al impulso $h(t)$ de duración finita, el sistema es stable.
5. If an LTI systems is causal, then is stable
6. The cascade of a noncausal LTI system with a causal one is necessarily noncausal.
7. A continuous-time LTI system is stable if and only if its step response $s(t)$ Un SLIT es stable si y sólo si su respuesta al escalón $s(t)$ es absolutamente integrab is absolutely integrable, that is, iff
$$\int_{-\infty}^{\infty} |s(t)|dt < \infty$$
.
8. A discrete-time LTI system is causal if and only if its step response $s(t)$ is zero for $t < 0$.

[Sol: (1) T. (2) F. (3) F. (4) T. (5) F. (6) F. (7) F. (8) T.]

Problems

Problem 12

Let be an LTI systems with $h(t) = \delta(t - T_1) - \delta(t + T_1)$, an the input

$$x(t) = \begin{cases} T_0 + t, & -2T_0 \leq t < -T_0 \\ |t|, & -T_0 \leq t < T_0 \\ 0, & \text{resto} \end{cases}$$

where $T_0 > 0$. Sketch the signals $x(t)$ y $h(t)$, and compute, graphically the output of the system. Consider the following cases $T_1 = 2T_0$ y $T_1 = T_0/2$.

Problem 13

Let be the real signals $x(t) = e^{-t}u(t)$, $y(t) = e^{t}u(-t)$, y $z(t) = e^{-3t}u(t)$, compute the following convolutions:

1. $r(t) = x(t) * x(t)$.
2. $v(t) = y(t) * z(t)$.

Problems

Problem 14

For each of the following pair of signals, calculate the convolution to find the output $y(t)$ given the input $x(t)$ of an LTI system with impulse response $h(t)$.

1. $x(t) = e^{-3t}u(t)$, with $h(t) = u(t-1)$.
2. $x(t) = u(t) - 2u(t-2) + u(t+5)$, with $h(t) = e^{2t}u(1-t)$.
3. $h(t) = u(t) - u(t-1)$, with $\begin{cases} e^{t}, & t < 0 \\ e^{5t} - 2e^{-t}, & t \geq 0 \end{cases}$