Topic 1- part 1 - “Signals”

Discrete Time Systems (DTS)

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In this slides, WE WILL SEE:

1.1 Definition of signals, examples and classification

1.2 Basic operations with signals in continuous time (CT), and important signals in CT and main properties

1.3 Basic operations with signals in discrete time (DT), and important signals in DT and main properties

1.1 Definition of signals, examples and classification

1.1.1 Signals: definitions and classification

Main concept: signals

- What is a signal? numbers/data varying with time and/or space….
- Examples: time series, images etc.
- Mathematically:

$$x(t), \quad x[n] \qquad t \in \mathbb{R}$$

$$y(t), \quad y[n] \qquad n \in \mathbb{Z}$$

$$z(t), \quad z[n] \qquad n = \ldots -3, -2, -1, 0, 1, 2, 3\ldots$$

- It is a function - one dimensional, x(t) - or bidimensional x(t1,t2) (e.g., images)

Main concept: signals

- Generally, we have *real* signals

$$x(t), x[n] \in \mathbb{R}$$

Examples:

$$x(t) = \sin(t)$$
$$x(t) = t$$

- or *complex* signals

$$x(t), x[n] \in \mathbb{C}$$

Examples:

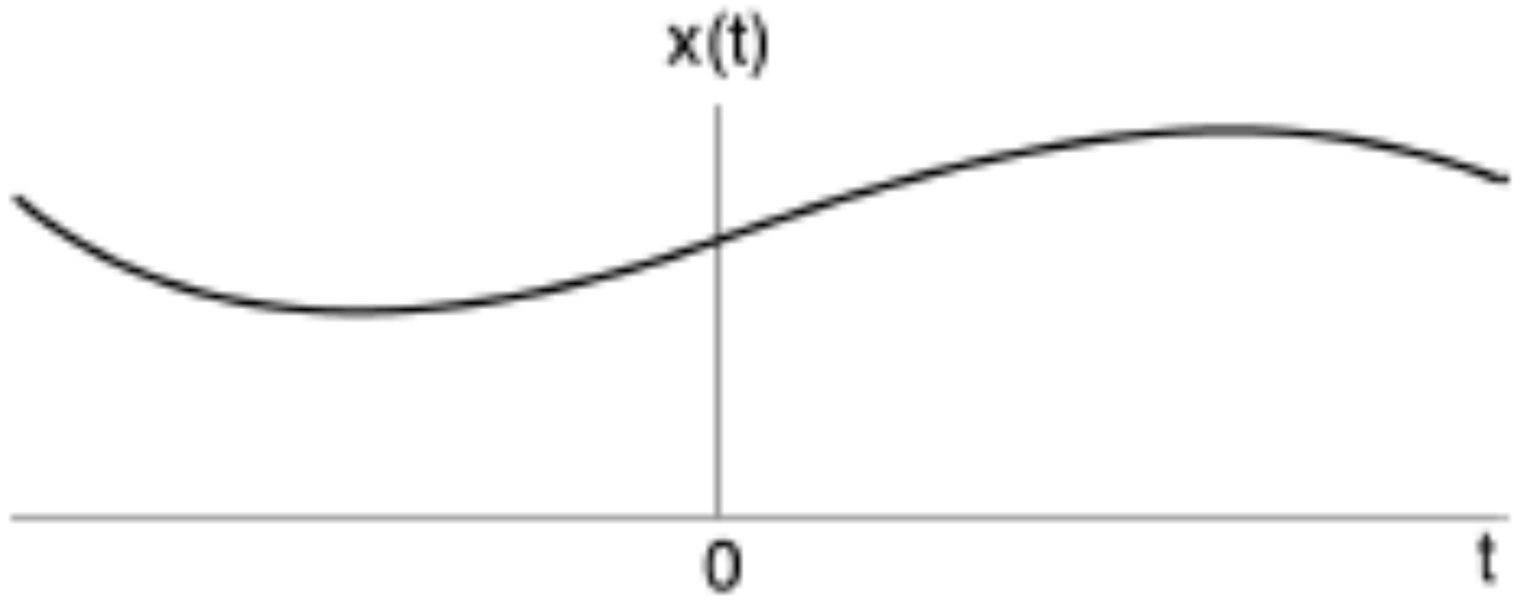
$$x(t) = e^{-jt}$$

$$x(t) = j\sin(t)$$

Signals

Recall of signals and systems in continuous time

- ¿What is a signal?
  - Is a ``mathematical model” (a function) which represents a variable of interests, that changes with the time.
  - Examples of signals: radio, volts, temperature, ...



signals and systems in continuous time (CT): where *t takes continuous values*

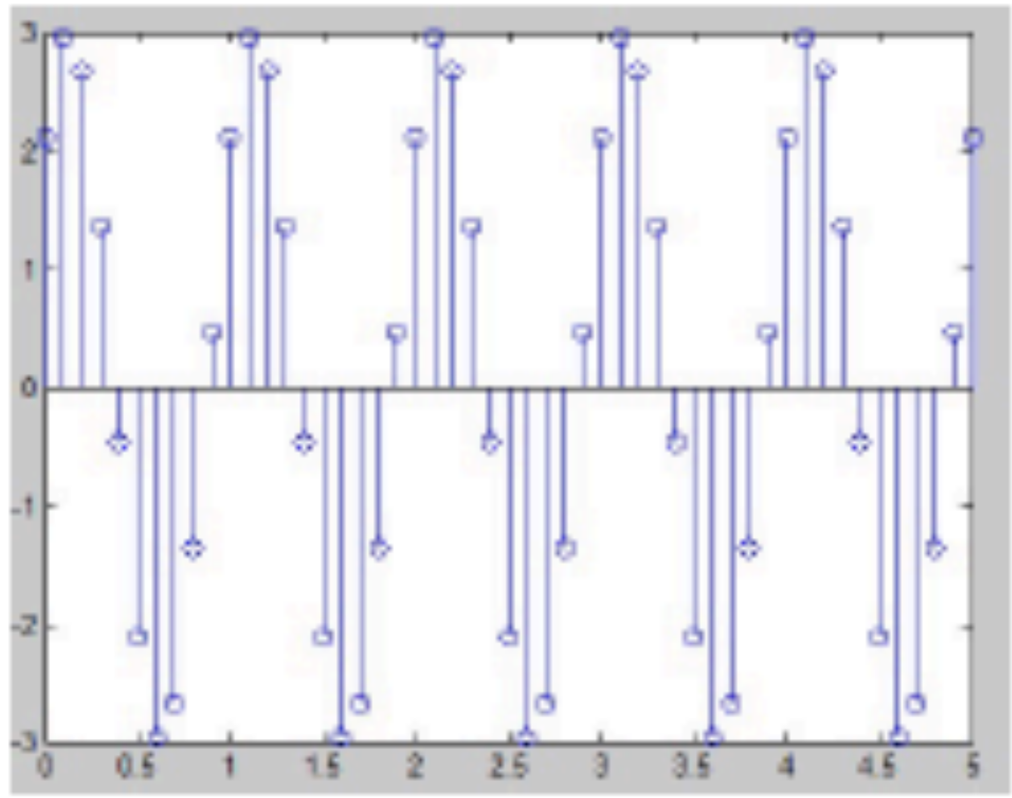
$t \in \mathbb{R}$

- One-dimensional (temperature in a place) vs. Multidimensional (for instance, an image)

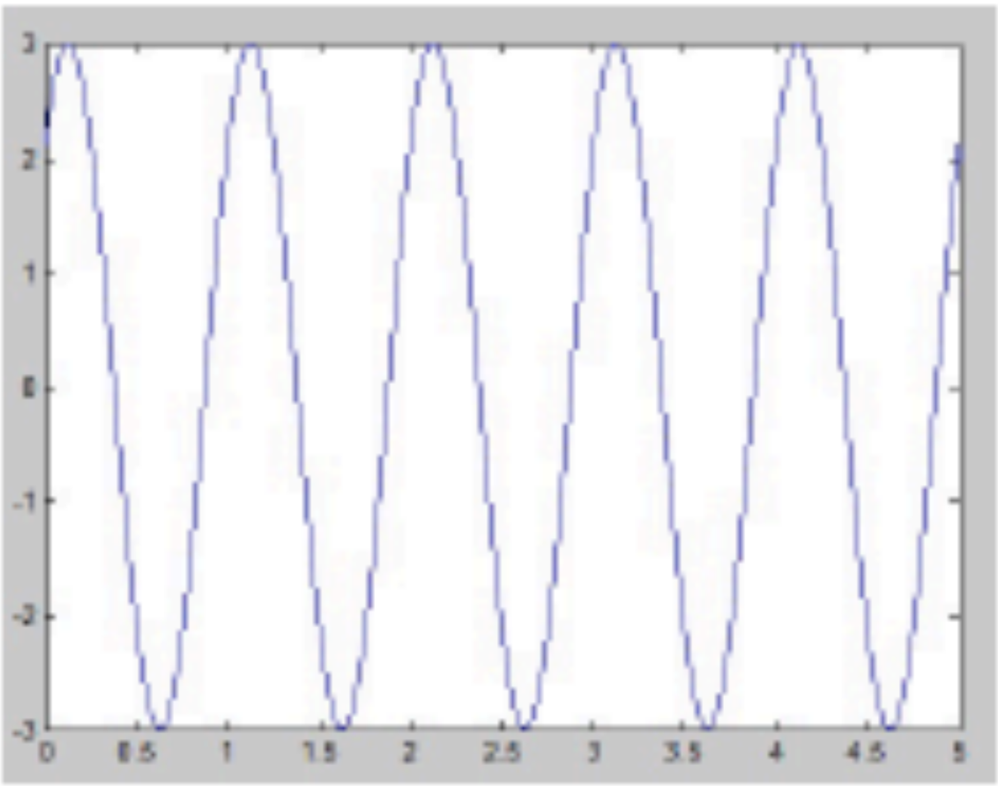
Signals

- Continuous signals vs. discrete signals
  - Continuous: defined for any real values.
    - Example: voice.
  - Discrete: defined for only for certain time values.
    - Example: final prize of stocks (in a stock market), every day.

$n = \ldots -3, -2, -1, 0, 1, 2, 3\ldots$



Discrete Signal



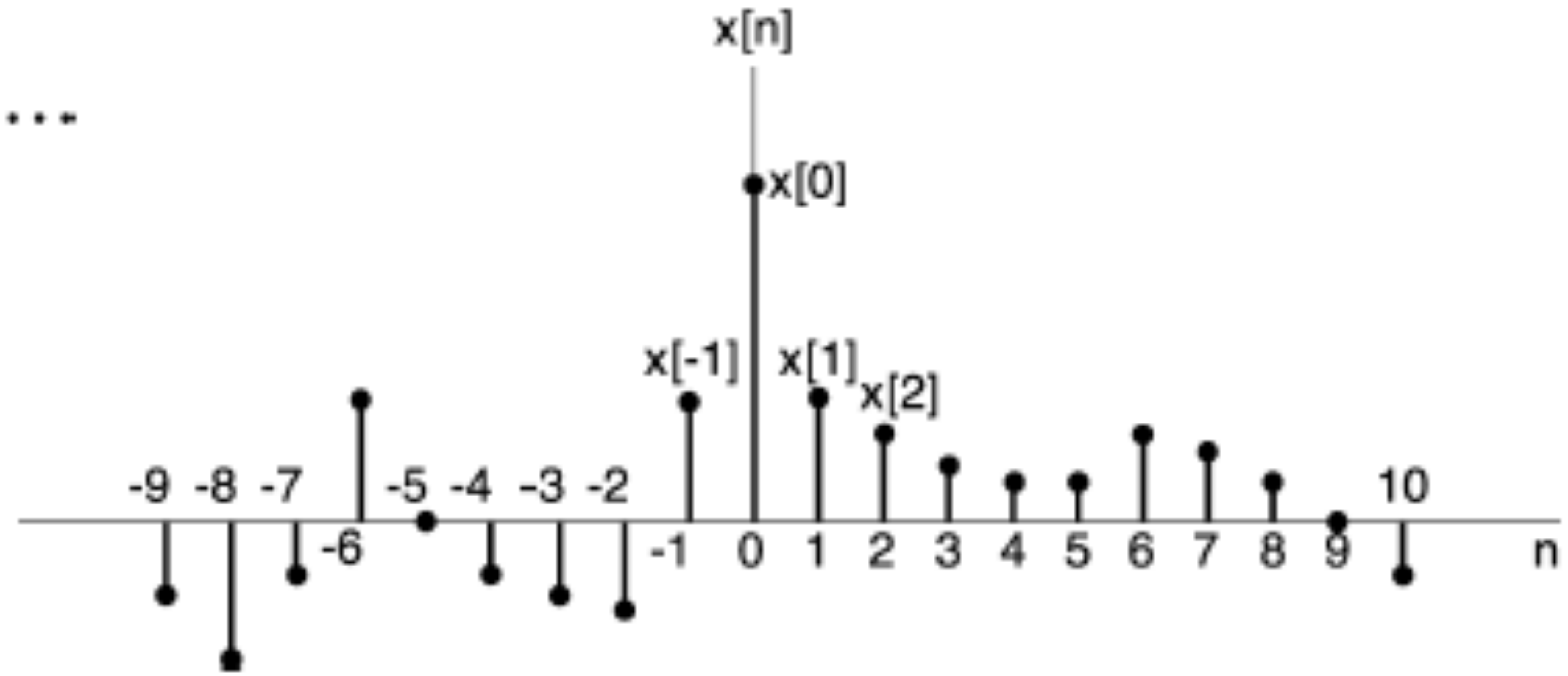
Continuous Signal

REMARK: here we just look the x-axis…

DISCRETE Signals

➢ A discrete signal is a *sequence of real numbers* and is denoted as:

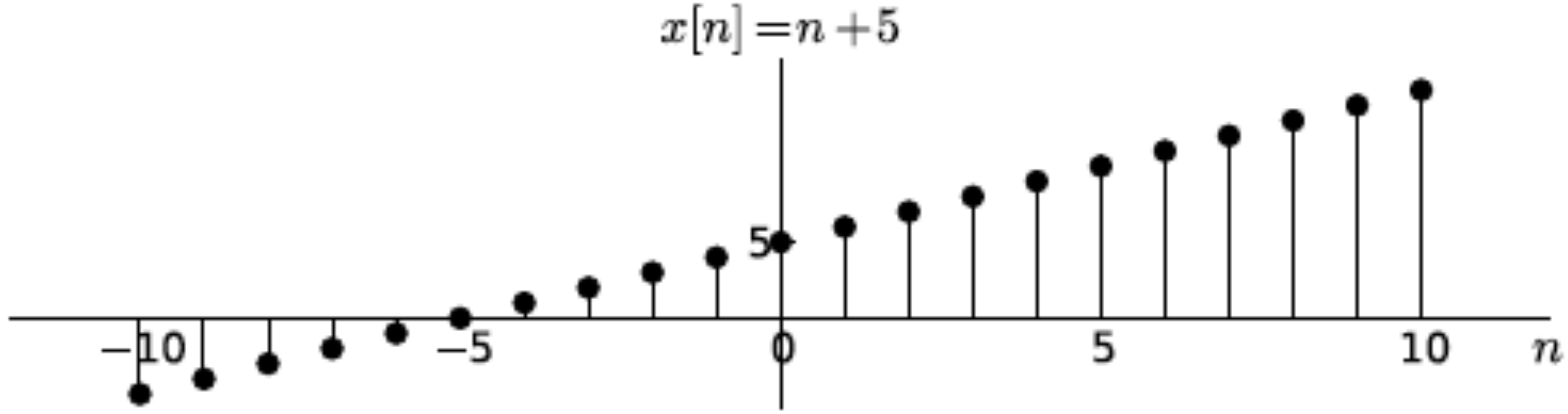
$$x[n], y[n], z[n]...$$

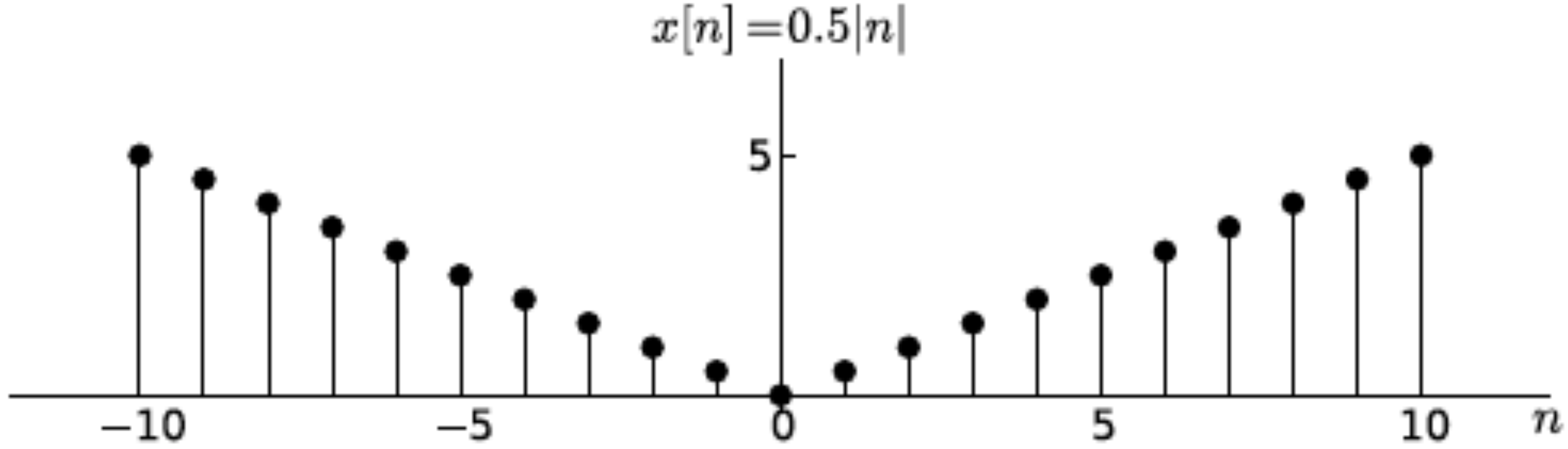


➢ In a Discrete Signal:

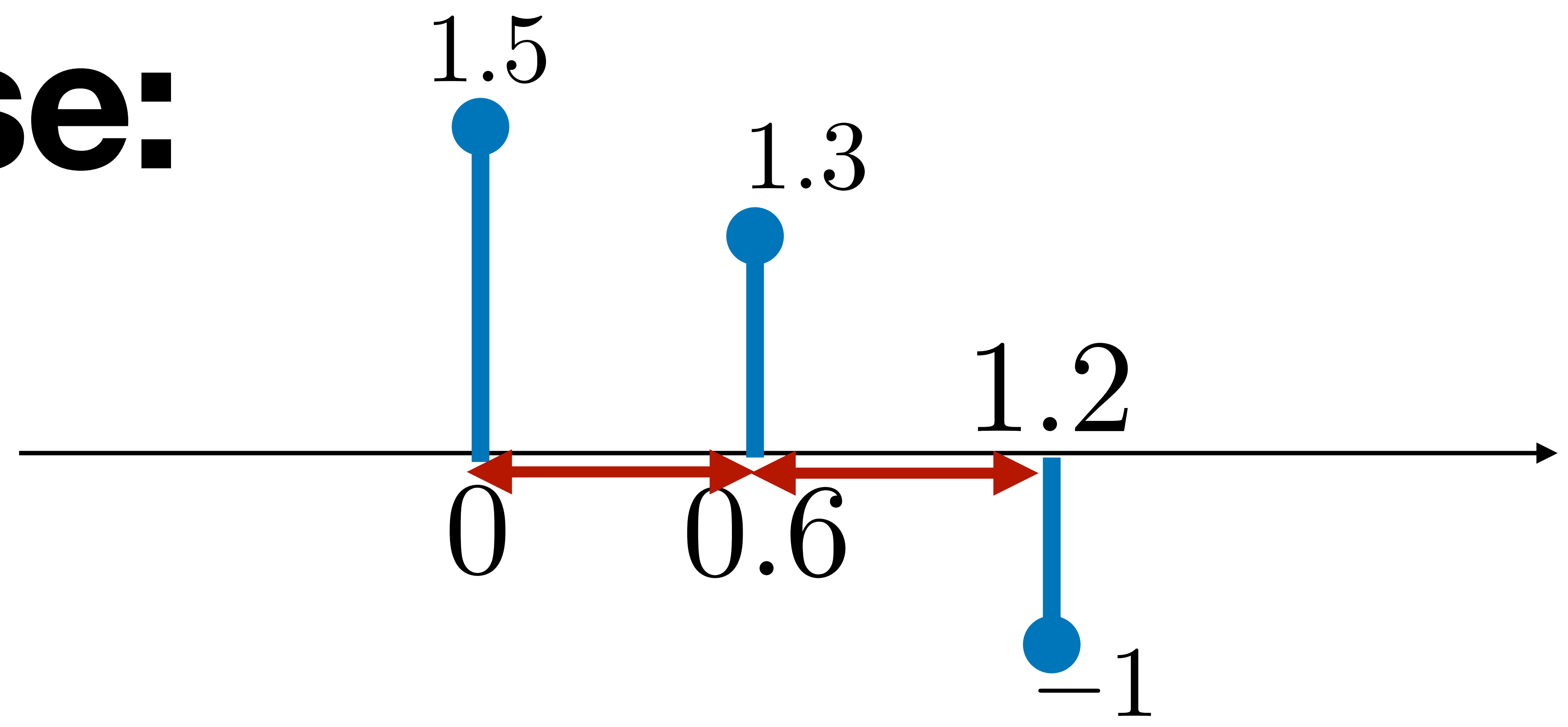
- The independent variable (*n*) takes <u>integer</u> values, i.e., discrete
- The dependent variable (*x*, *y*, *z*…) takes real values, i.e., continouos

DISCRETE Signals: examples

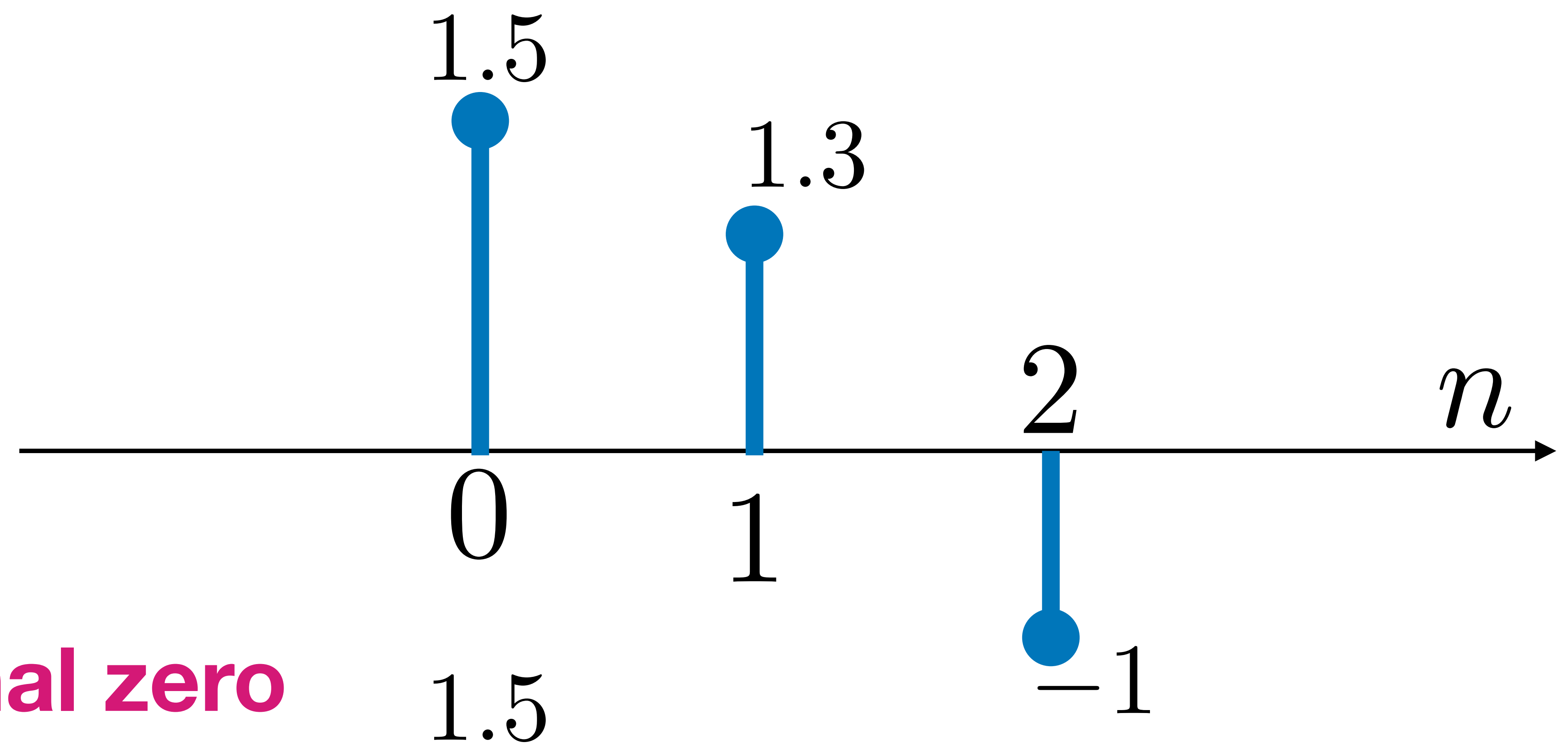




Special case:

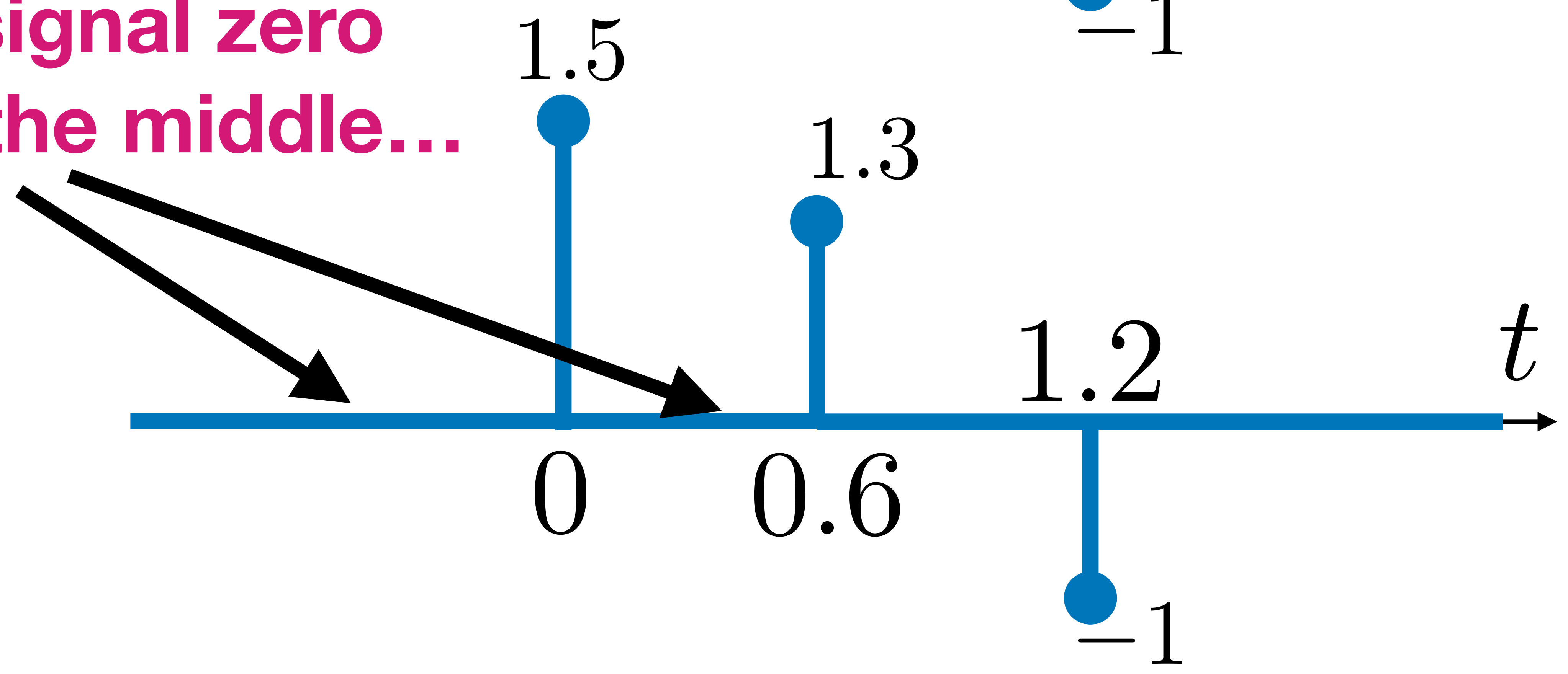


Is a continuous or a discrete signals?



Each step represents a jump of 0.6 in a continuos line…

Considering the signal zero in these regions in the middle…



Signals

Signals

continuous

discrete

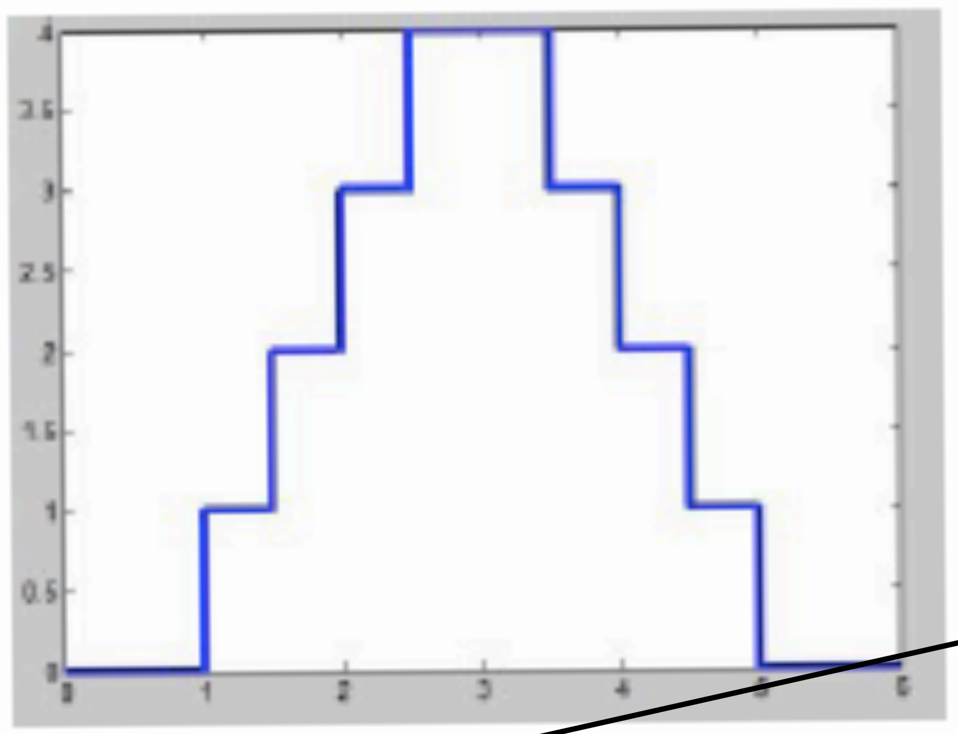
(<u>mainly</u> in this course)

We look the x-axis !!!

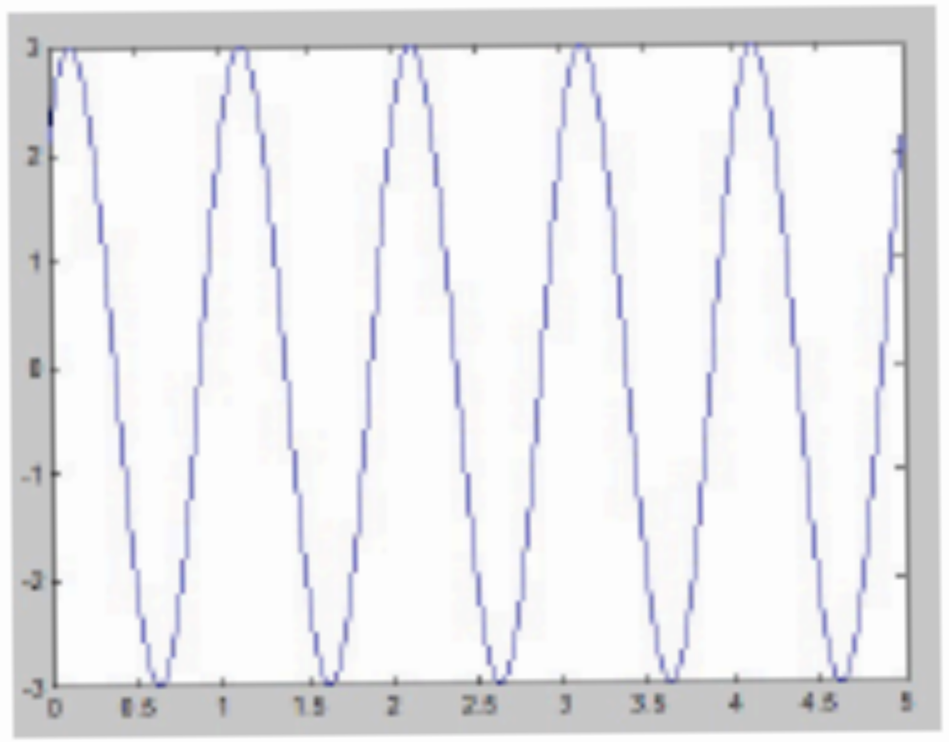
Signals

➤ Digital Signals vs. Analog Signals

- Digital Signals: take only certain values (a finite number of values, in general) within an interval of time.
- Analog Signals: can take a (infinite) number of continuous values in a (bounded or unbounded) interval.



Señal digital

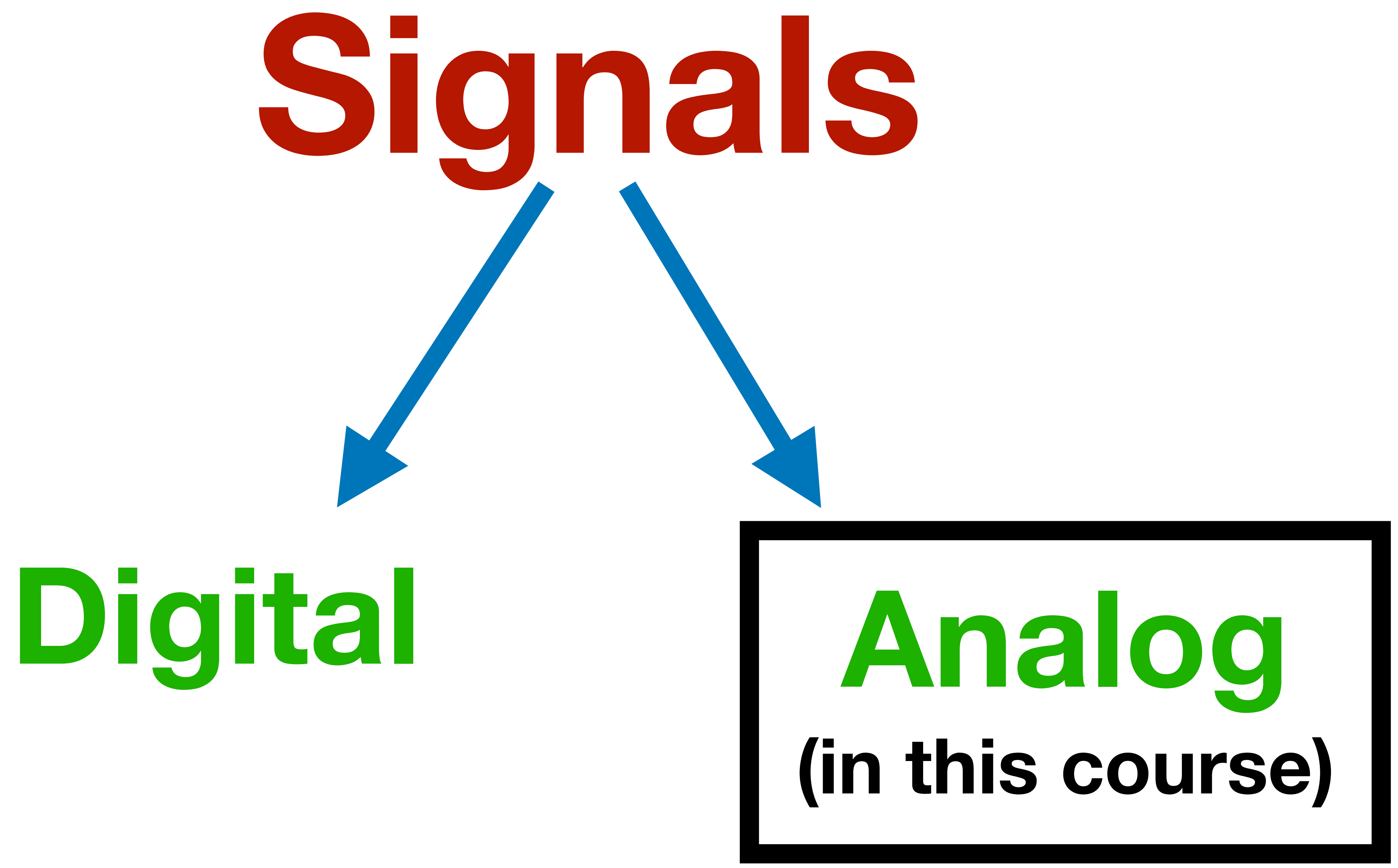


Señal analógica

(Both)
continuous signals

We look the y-axis !!!

Signals

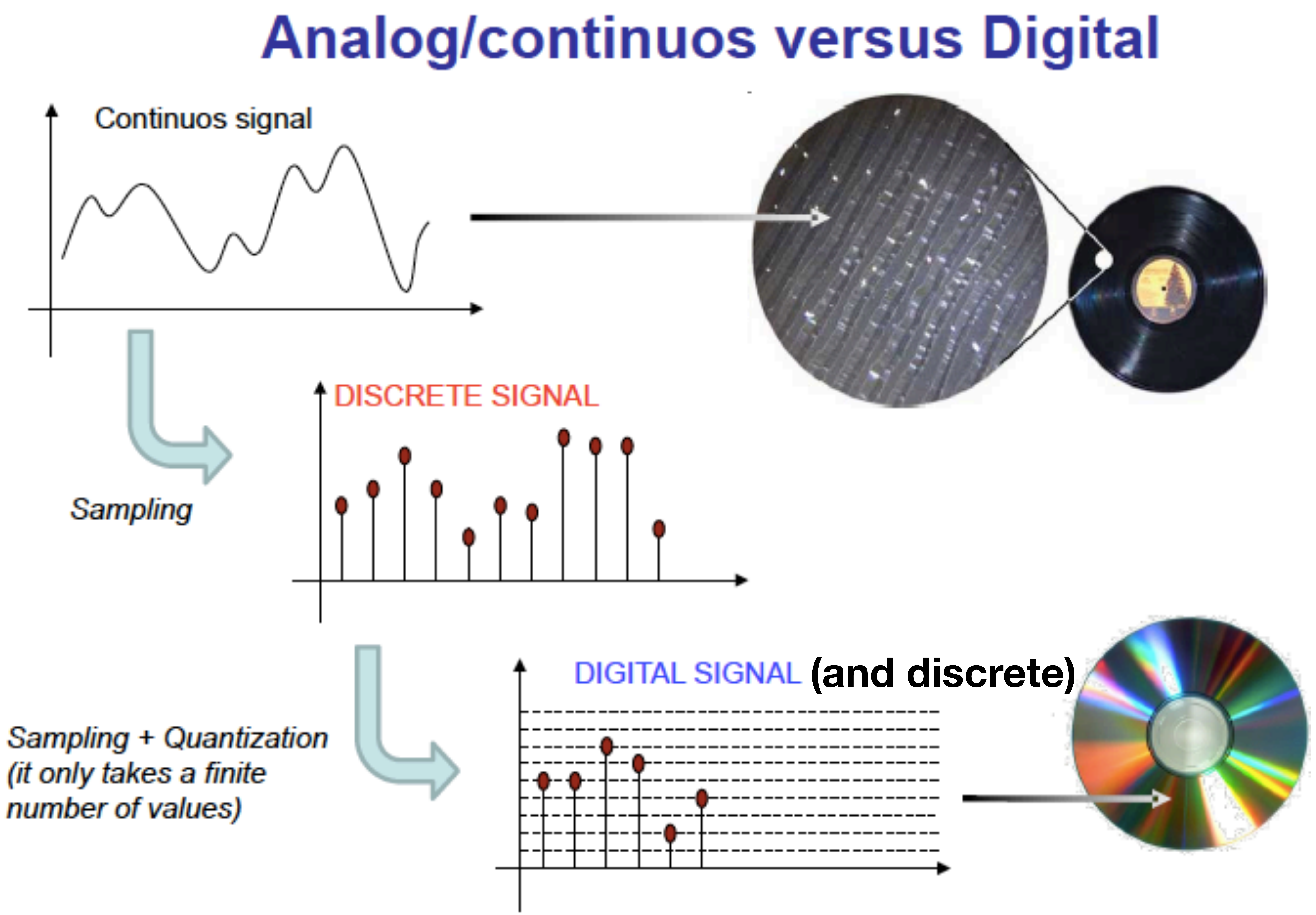


We look the y-axis !!!

Signals

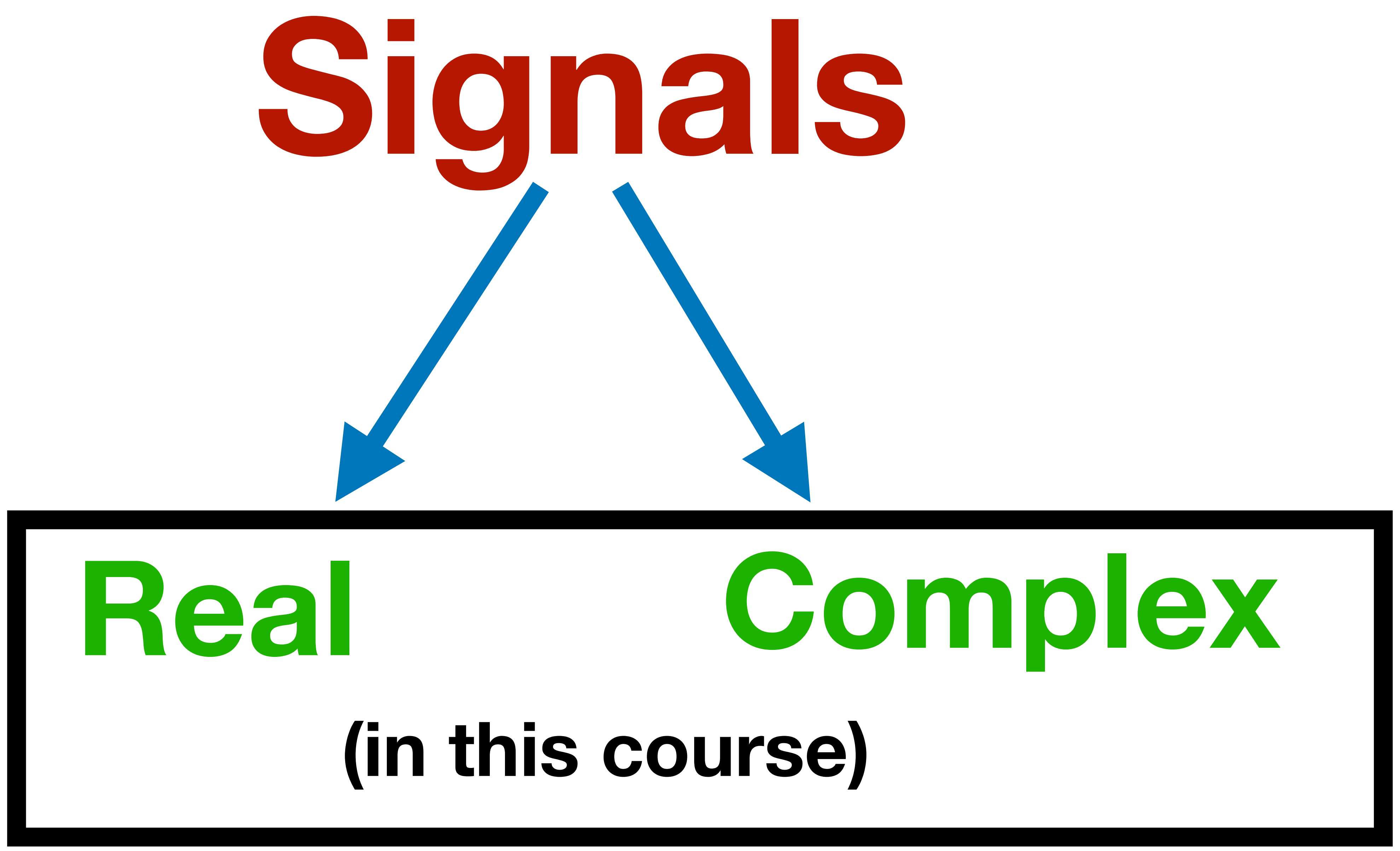
Can you plot a *discrete* and *digital* signal ?

Signals


Analog/continuos versus Digital
Continuos signal
DISCRETE SIGNAL
Sampling
DIGITAL SIGNAL (and discrete)
Sampling + Quantization
(it only takes a finite
number of values)

Signals

...and we have already saw:



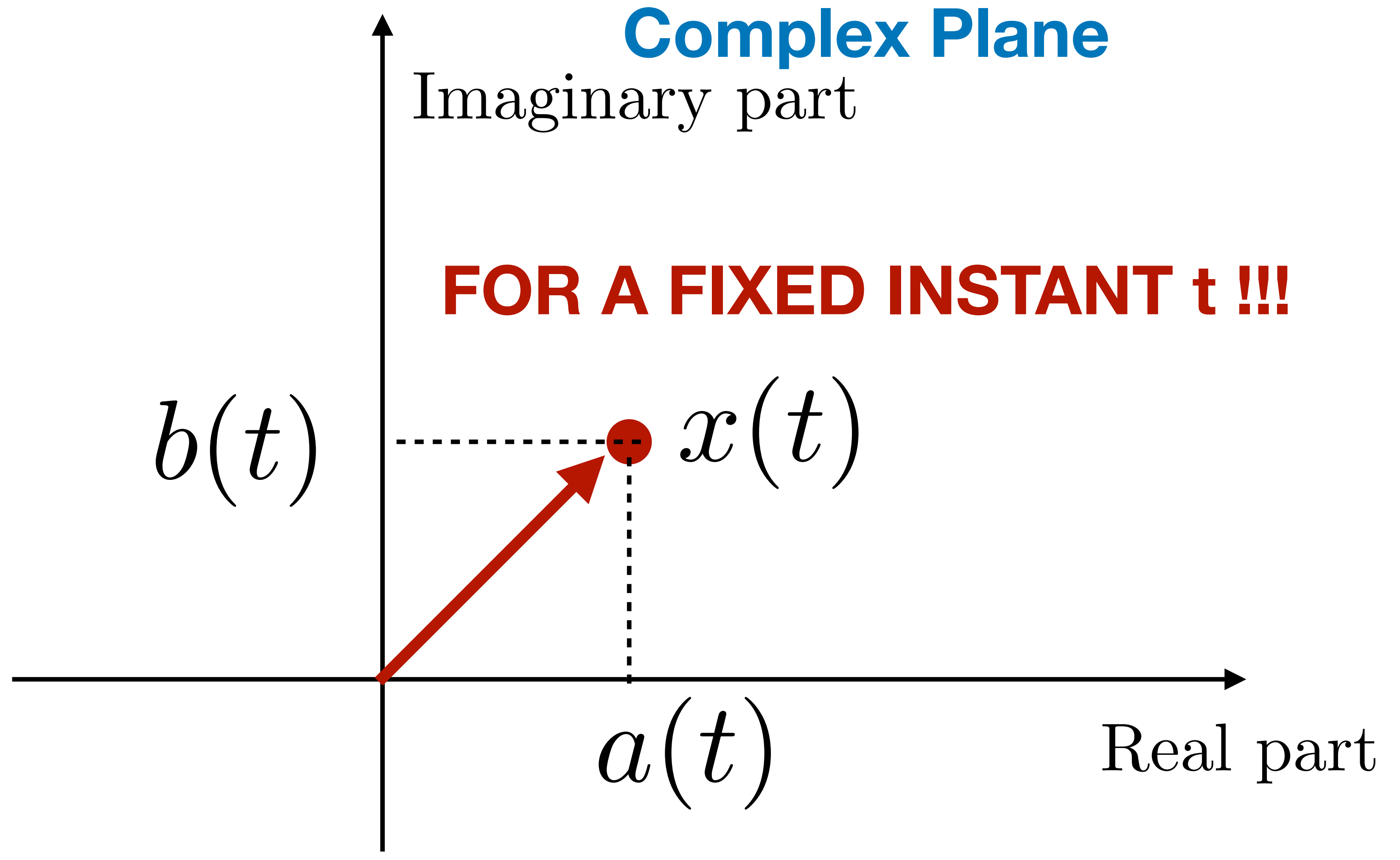
We look the y-axis !!!

Complex signals

$$x(t) = a(t) + jb(t)$$

$$\text{Re}\{x(t)\} = a(t)$$

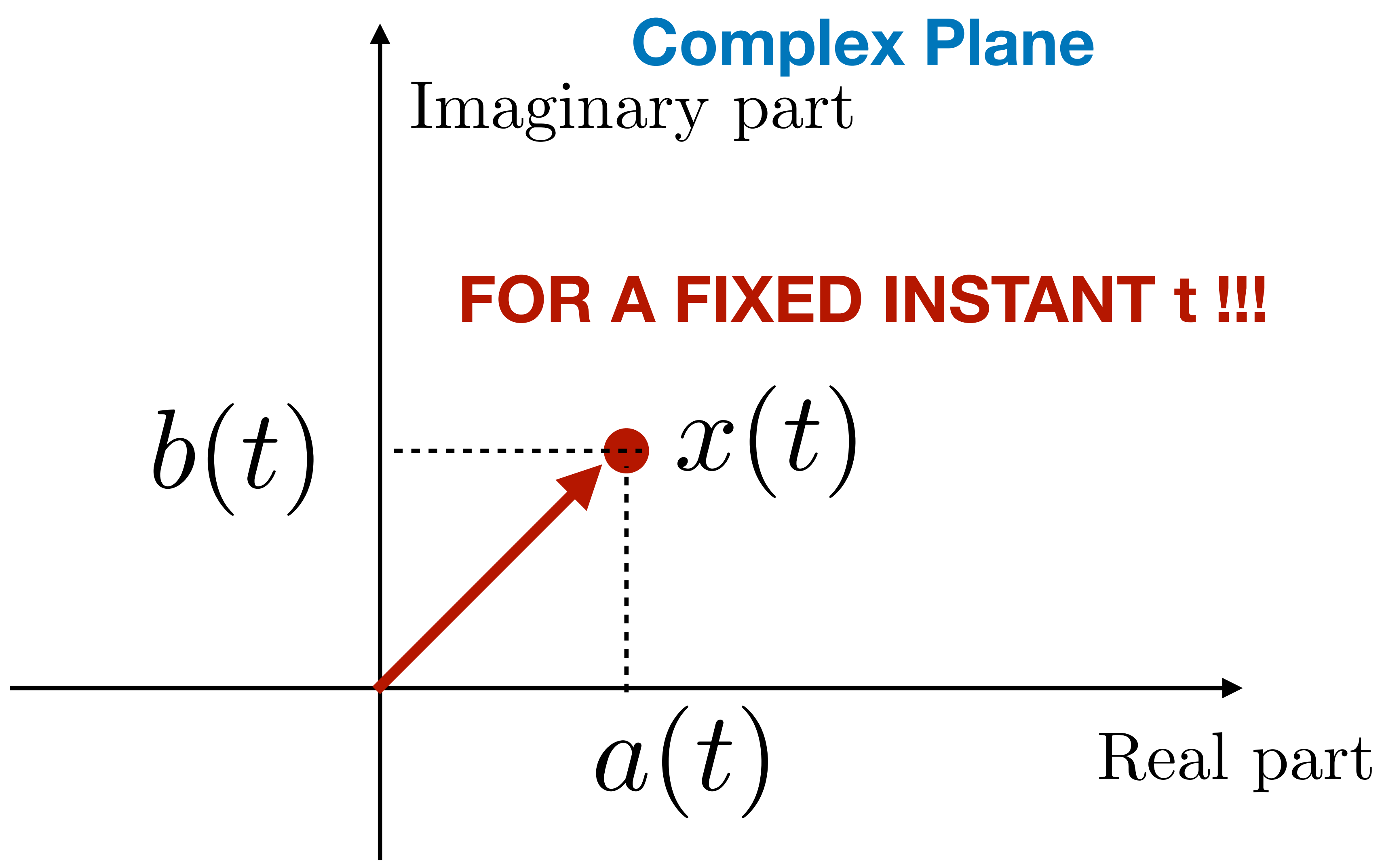
$$\text{Im}\{x(t)\} = b(t)$$

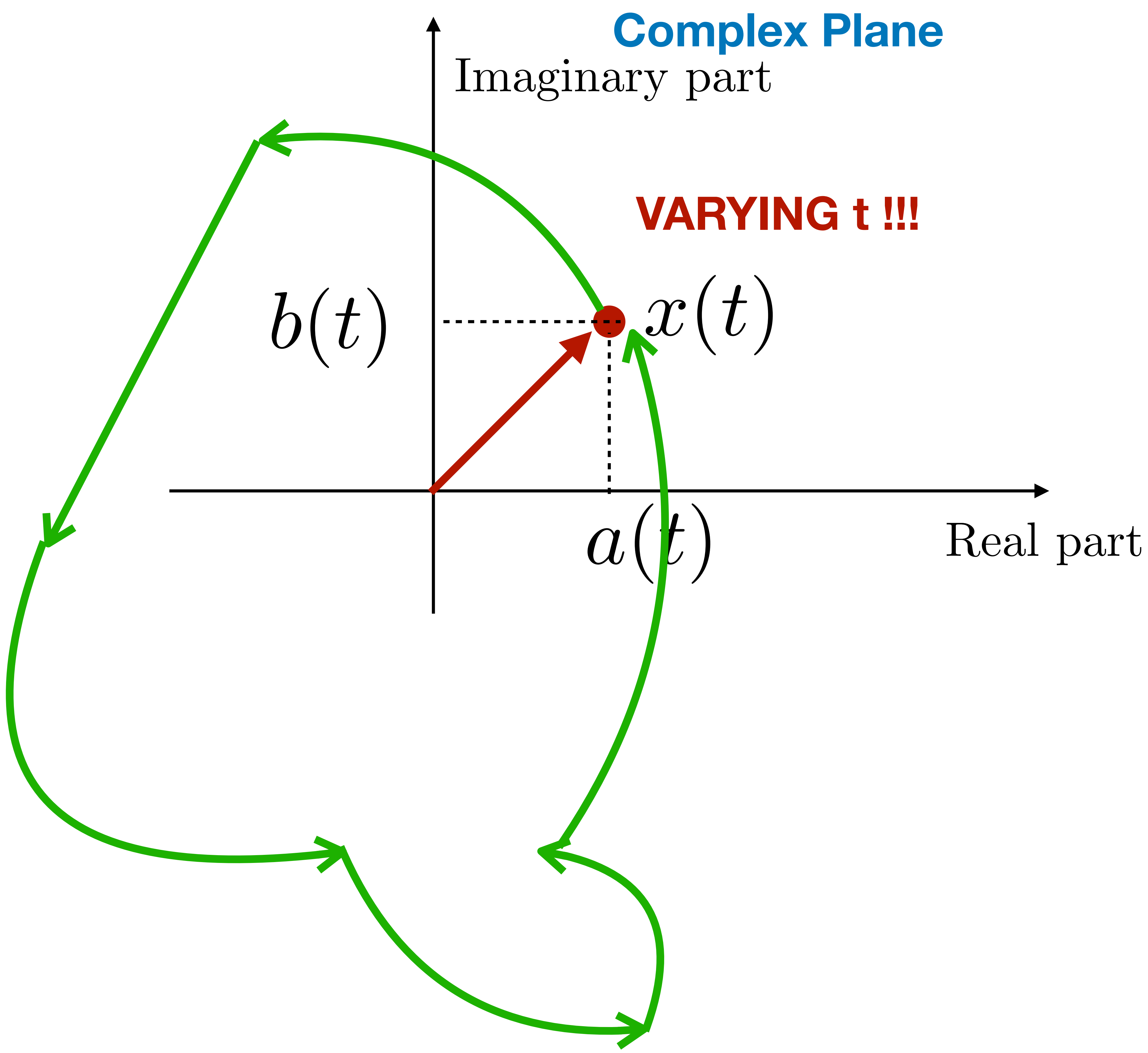


$$\text{Module}\{x(t)\} = |x(t)|^2 = a(t)^2 + b(t)^2$$

$$\text{phase}\{x(t)\} = \arctan \frac{b(t)}{a(t)}$$

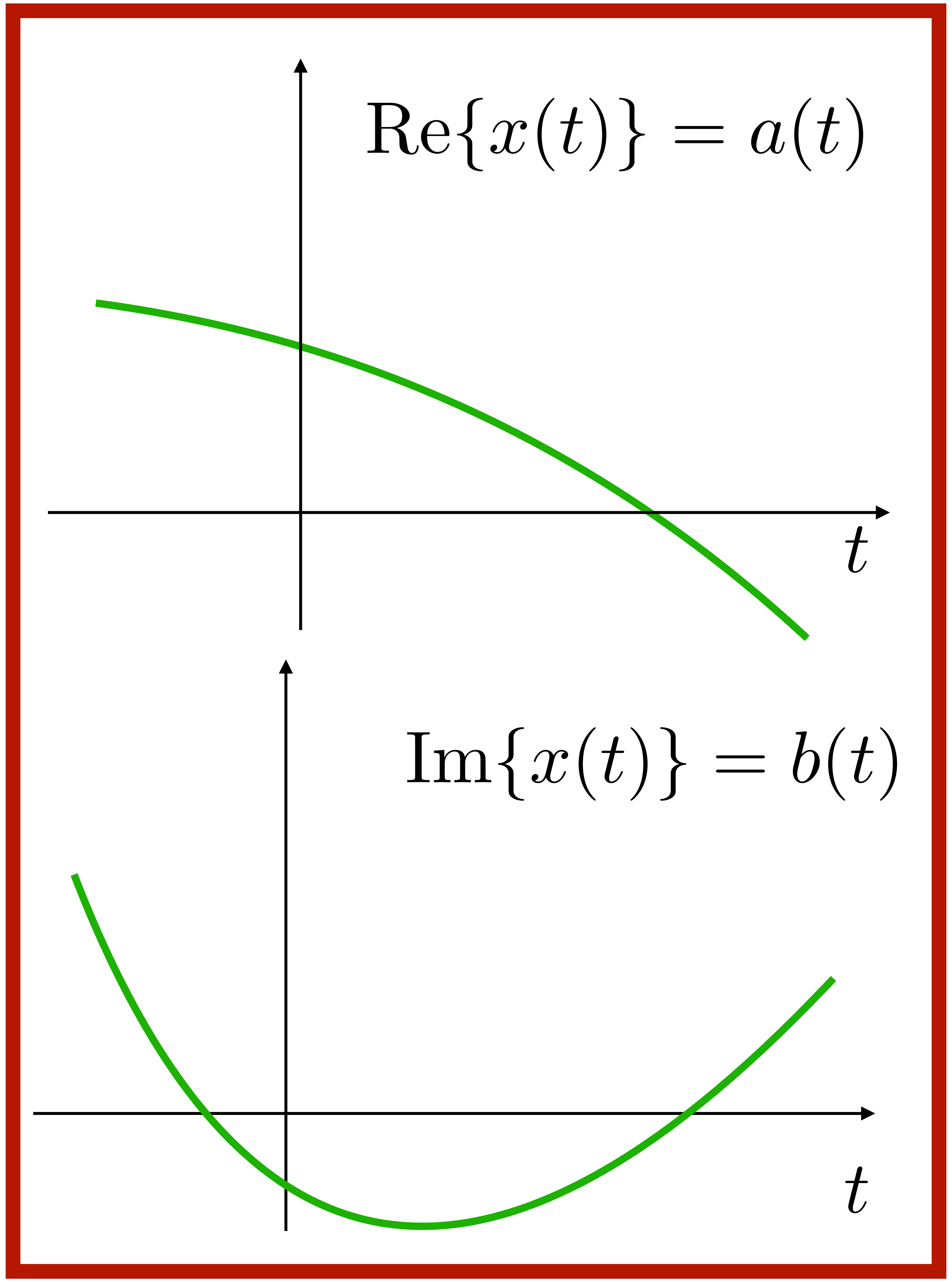
Way to plot a complex signal

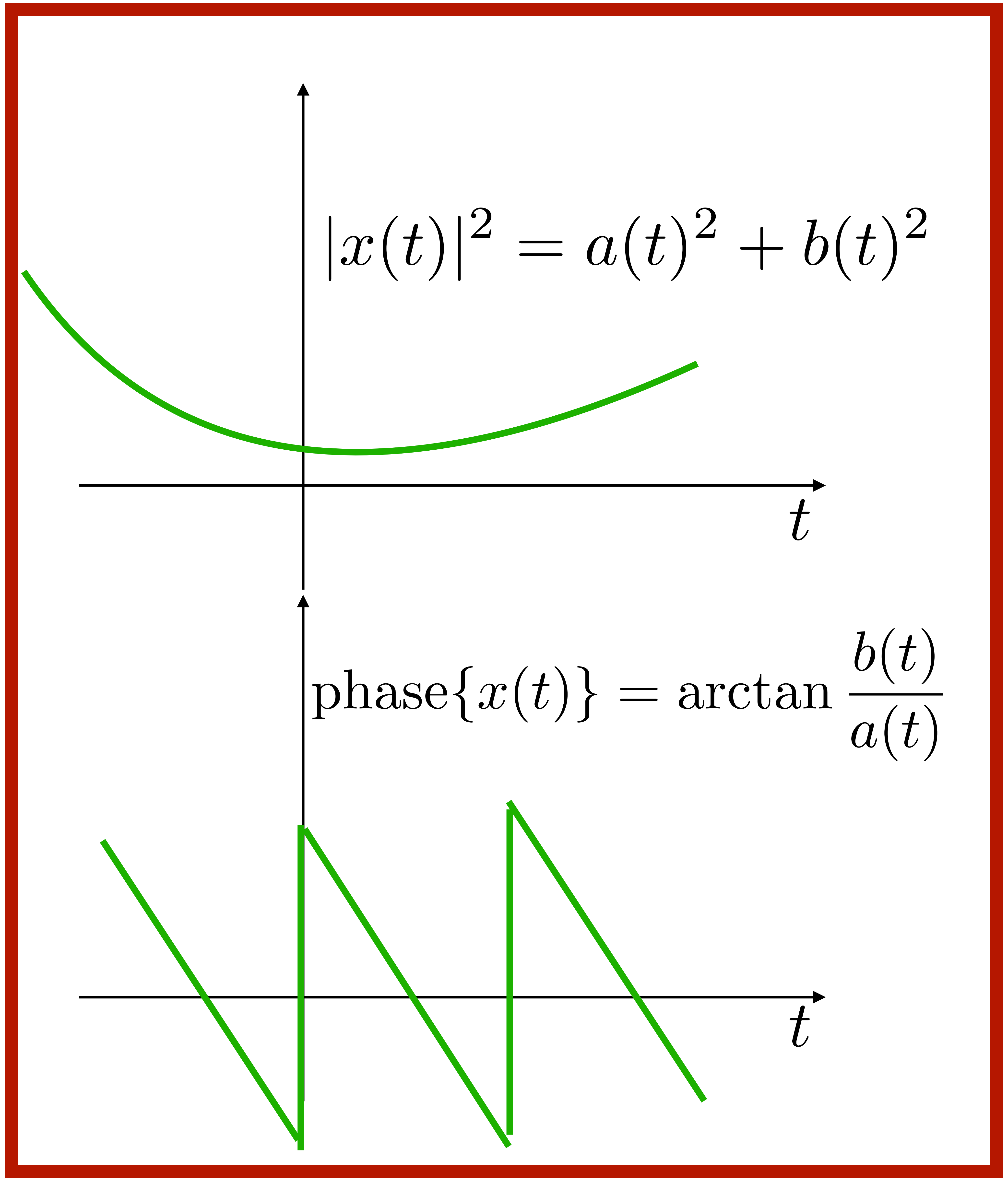




IN THIS CASE IS A PERIODIC SIGNAL: why ?

Way to plot a complex signal





Complex signals: conjugate, real and im. parts

- Complex conjugate of a signal:

$$x^*(t) = \Re\{x(t)\} - j\Im\{x(t)\} = a(t) - jb(t)$$

- Real and imaginary parts can be obtained as:

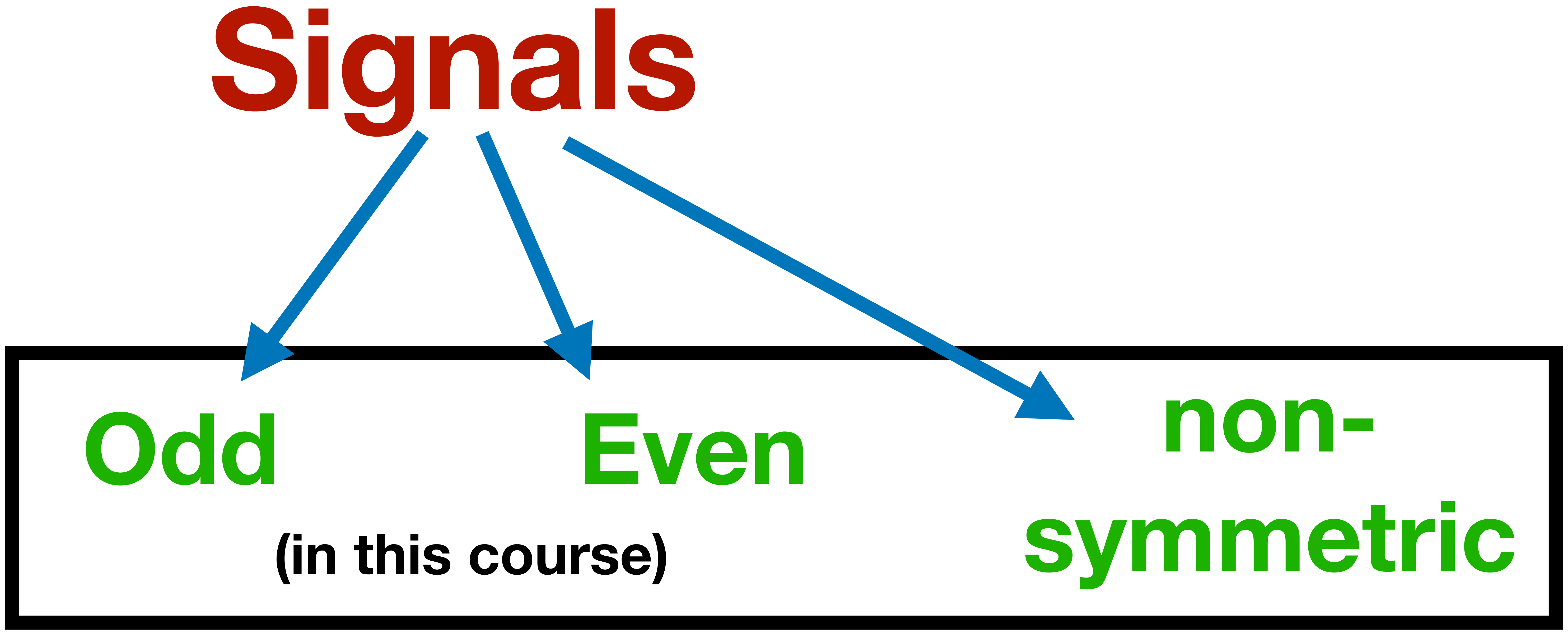
$$\Re\{x(t)\} = \frac{1}{2}\left[x(t) + x^*(t)\right]; \quad \Im\{x(t)\} = \frac{1}{2j}\left[x(t) - x^*(t)\right]$$

- The magnitude and argument can be obtained as:

MODULE $|x(t)|^2 = x(t) \cdot x^*(t) = (\Re\{x(t)\})^2 + (\Im\{x(t)\})^2$

$$\angle\{x(t)\} = \operatorname{arc\,tg} \frac{\Im\{x(t)\}}{\Re\{x(t)\}}$$

Signals



Examples (in continuos time)

- Real and Complex signals
  - Complex signal

$$x(t) = x_r(t) + jx_i(t)$$

Example of a complex signal: $y(t) = e^{j0.3t} = cos(0.3t) + jsin(0.3t)$

Example of a real signal: $x(t) = cos(0.25t)$

- Odd and even signals: both real signal such that

$$x_e(t) = x_e(-t)$$

$$x_o(t) = -x_o(-t)$$

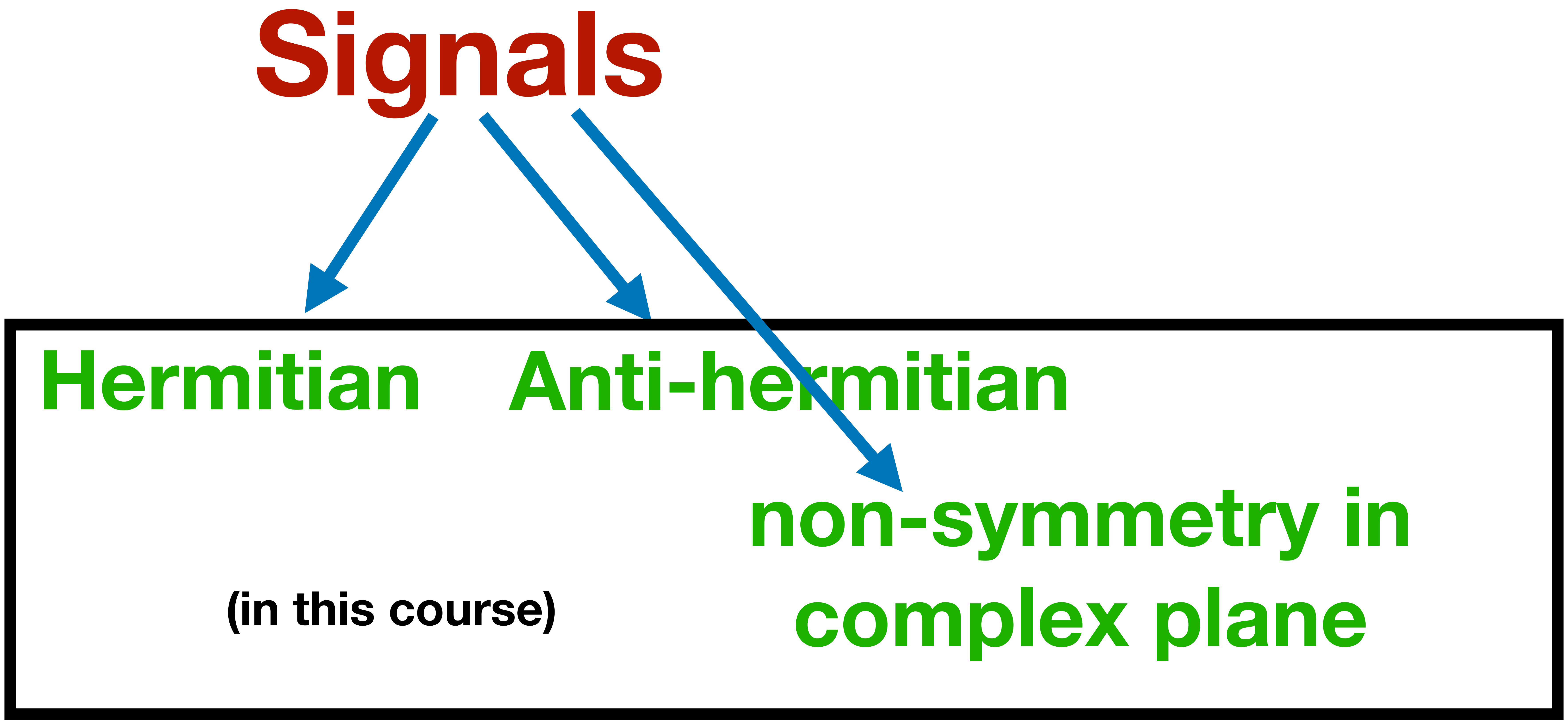
- Generally, we can write

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Signals


Signals
Hermitian
Anti-hermitian
non-symmetry in complex plane
(in this course)

Hermitian/anti-hermitian signals (cont. time)

- Hermitian and anti-hermitian signals:
  - Hermitian signals (if real, then is an even signal):

$$x(t) = x^*(-t)\ \forall t$$

  - anti-hermitian signals (if real, then is an odd signal):

$$x(t) = -x^*(-t) \forall t$$

Hermitian/anti-hermitian signals (cont. time)

- Every complex signal has two components: a **hermitian part** and an **antihermitian part**, that is,

$$x(t) = x_h(t) + x_a(t)$$

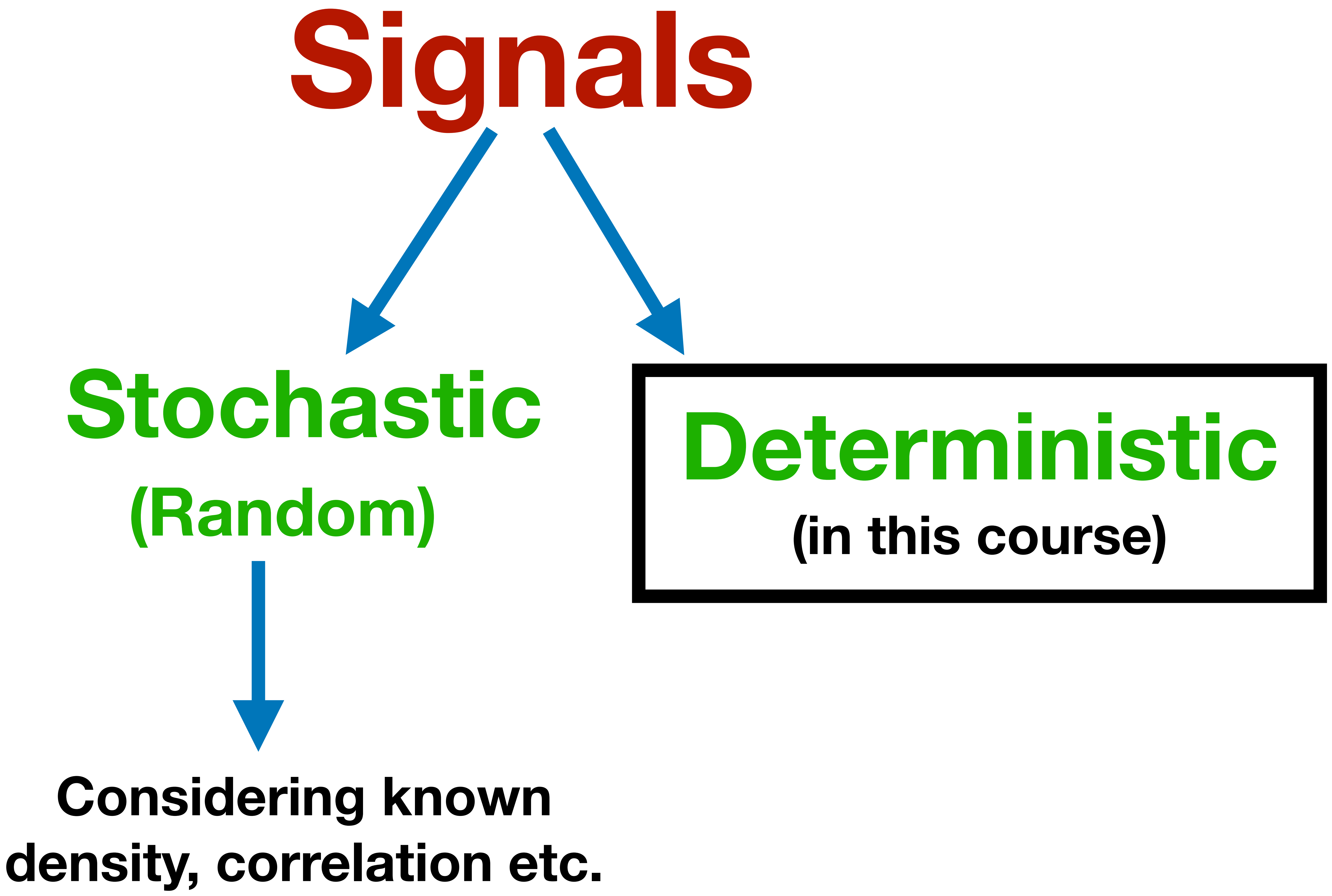
- Hermitian and antihermitian components can be computed as:

$$x_h(t) = \frac{1}{2}[x(t) + x^*(-t)]; \quad x_a(t) = \frac{1}{2}[x(t) - x^*(-t)]$$

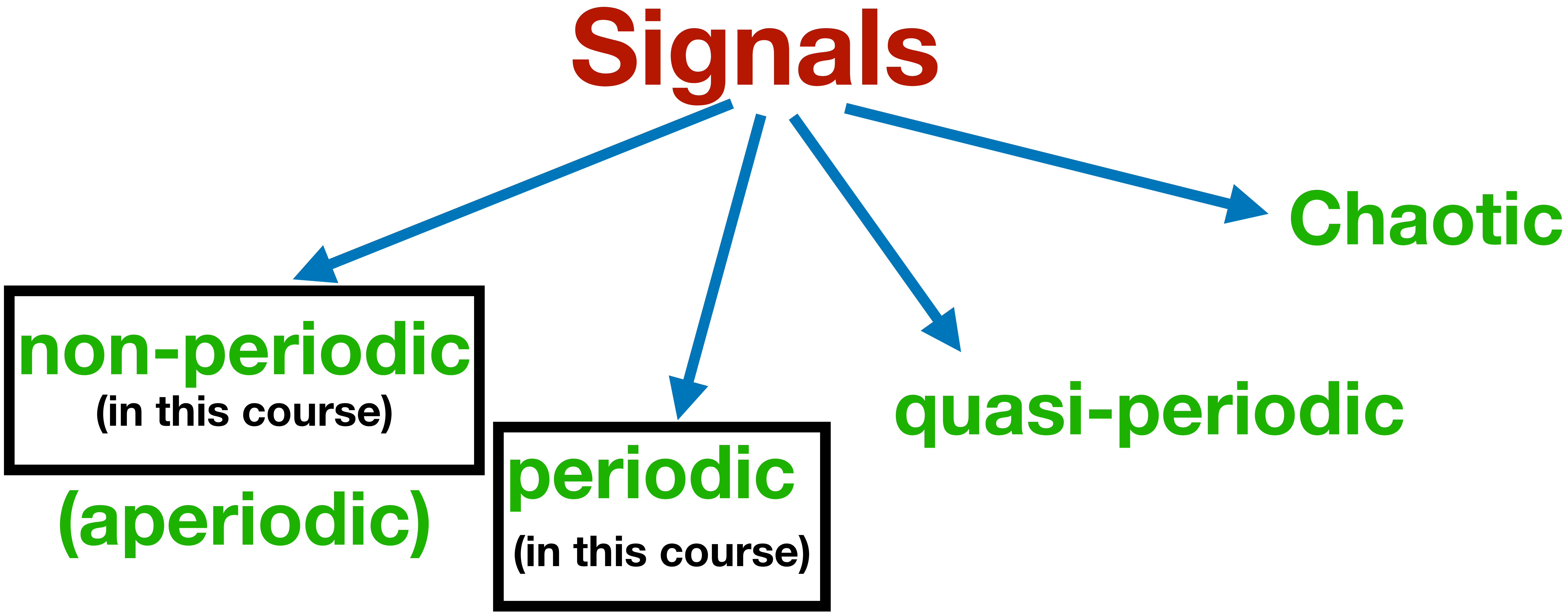
Signals

- Deterministic Signals vs. Stochastic Signals
  - Stochastic Signals: contain randomness
  - the definitions, formulas and the treatment change

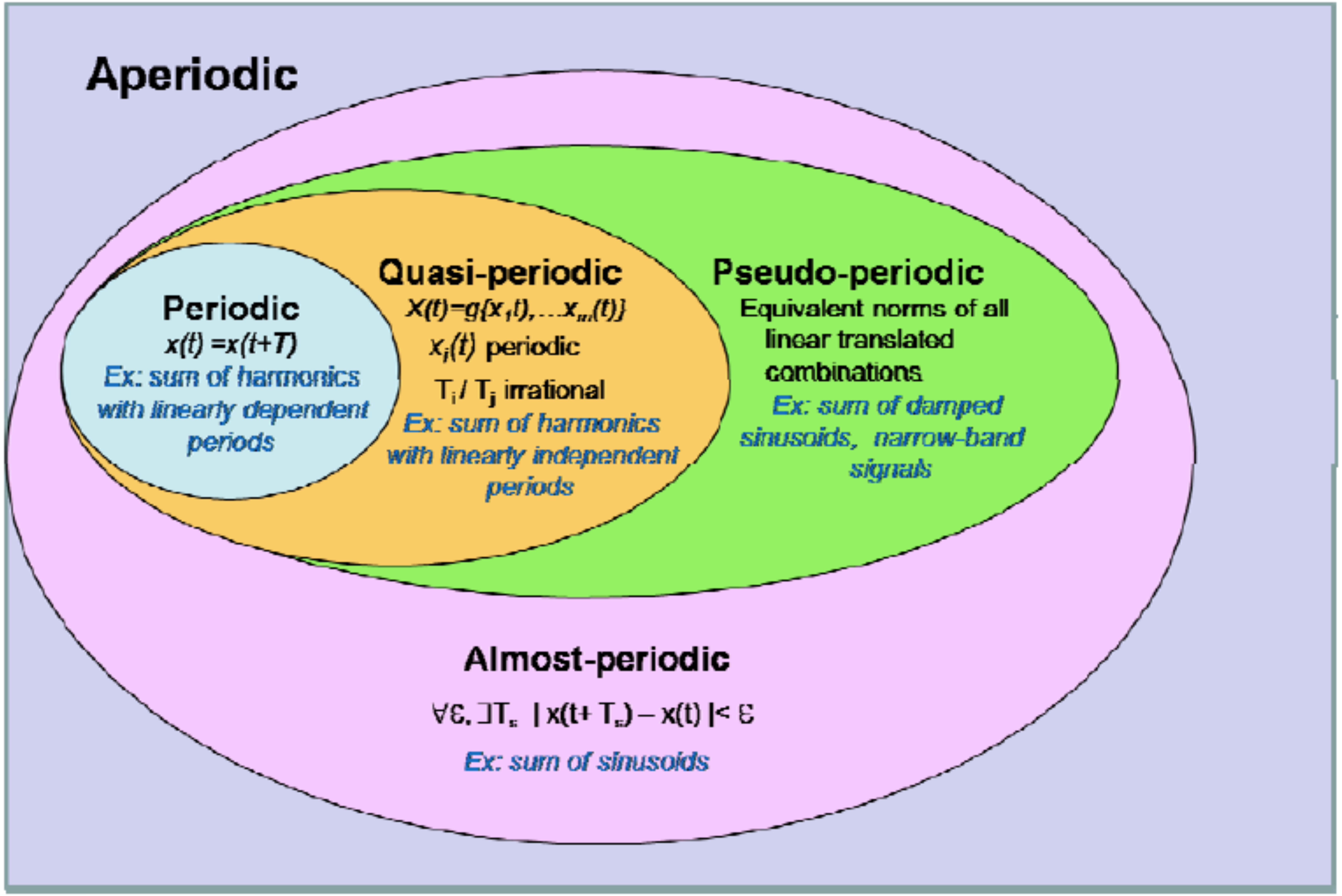
Signals


Signals
Stochastic
(Random)
Considering known density, correlation etc.
Deterministic
(in this course)

Signals


Signals
non-periodic
(in this course)
(aperiodic)
periodic
(in this course)
quasi-periodic
Chaotic

Summary - periodicity

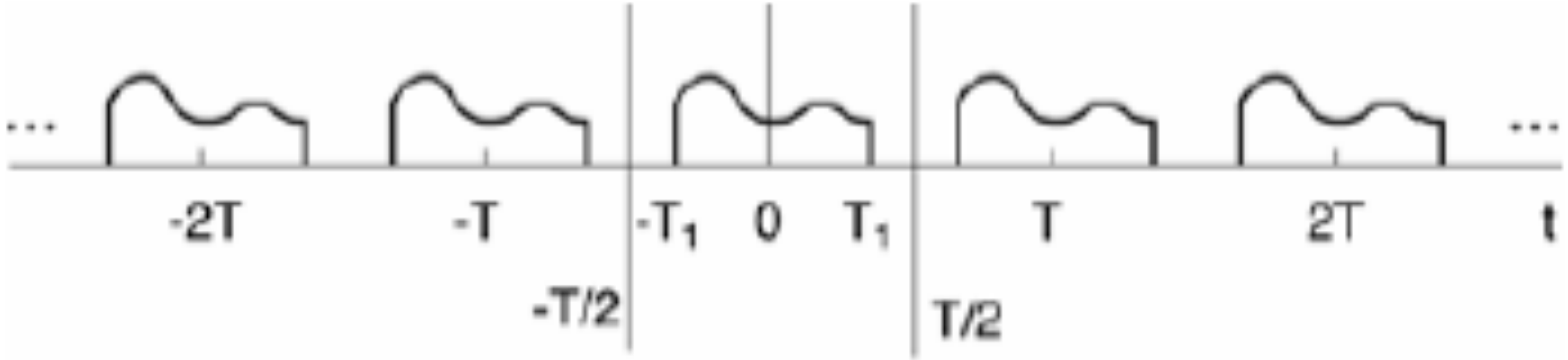


Periodic, quasi-periodic, almost and pseudo-periodic signals.

Periodic signals (continuous time)

- Periodic signals:

$$CT: \ \exists\, T > 0 \mid x(t) = x(t+T), \ \forall t$$



T is the *fundamental period*; note that the signal is repeating each 2T, 3T, …

Question: if we sum different periodic signals (with different period), the obtained signal is periodic? And if we multiply different periodic signals?

T is a real positive number

We will come back to this point… in another class

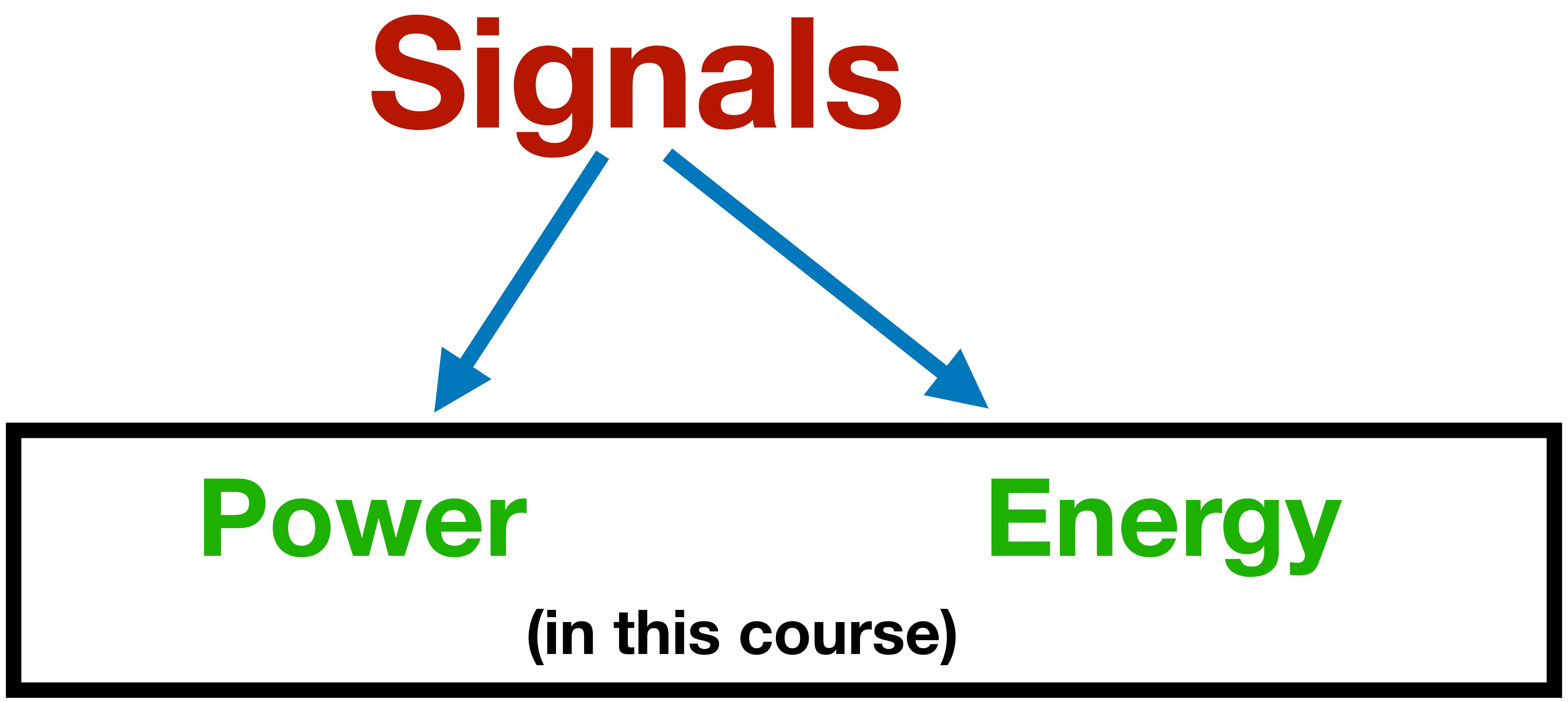
We will come back to periodicity in aonther class, since the discrete case is quite more complicated.

Periodic signals (continuous time)

Fundamental period

- If $x(t)$ is periodic with period T, it is also periodic with periods $2T, 3T, \ldots$.
- We call *fundamental period*, T_0, to the smallest value of T for which the equation $x(t) = x(t+T)$ holds.

Signals

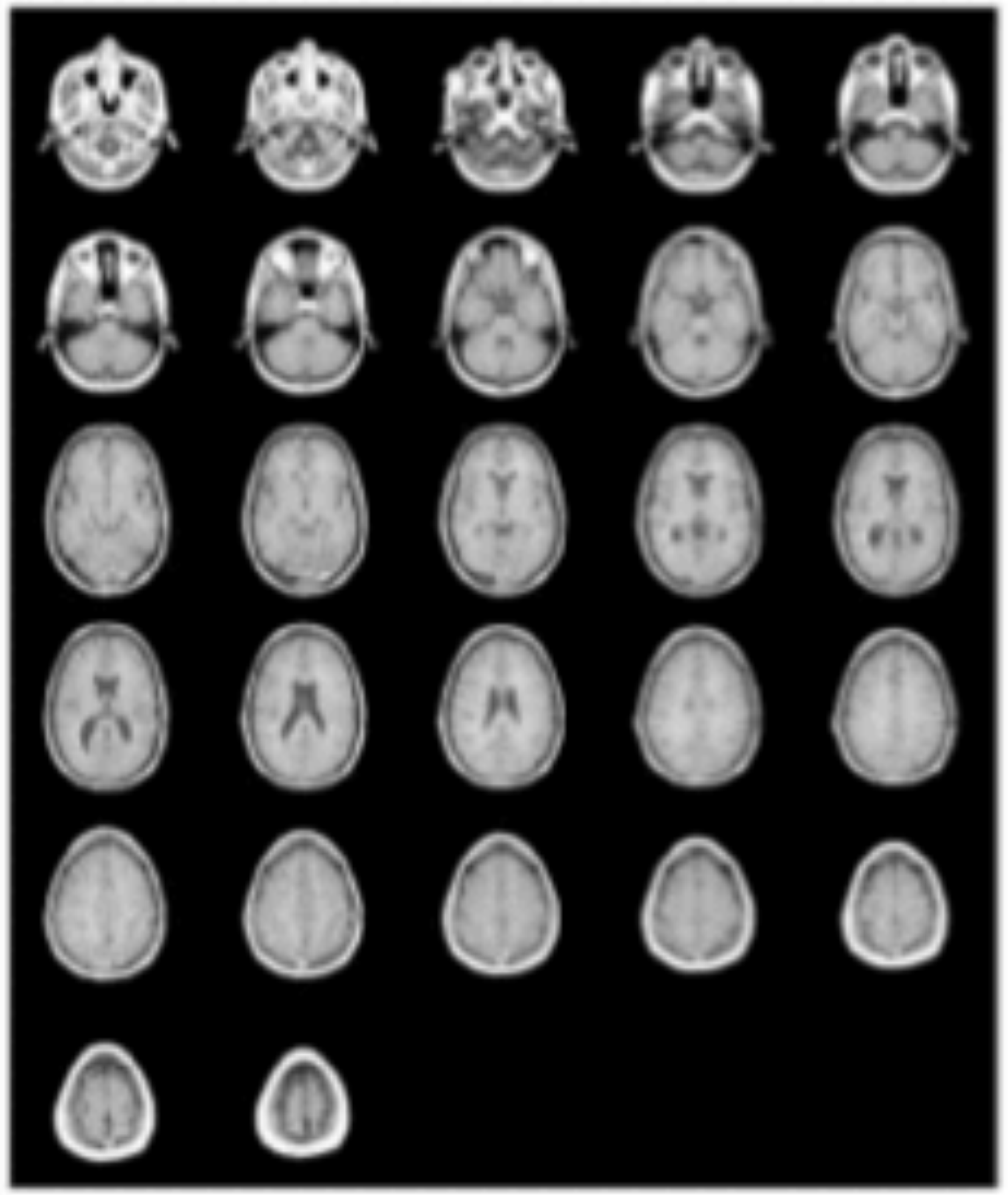


- Recall:
  - There are *power signals* and *energy signals*:
    - Finite Energy → then the power is zero → energy signal
    - Finite Power → then the energy is infinite → power signal
    - Some signals are neither energy nor power signals.
    - For a periodic signal: if the energy in one period is finite, then it is a power signal

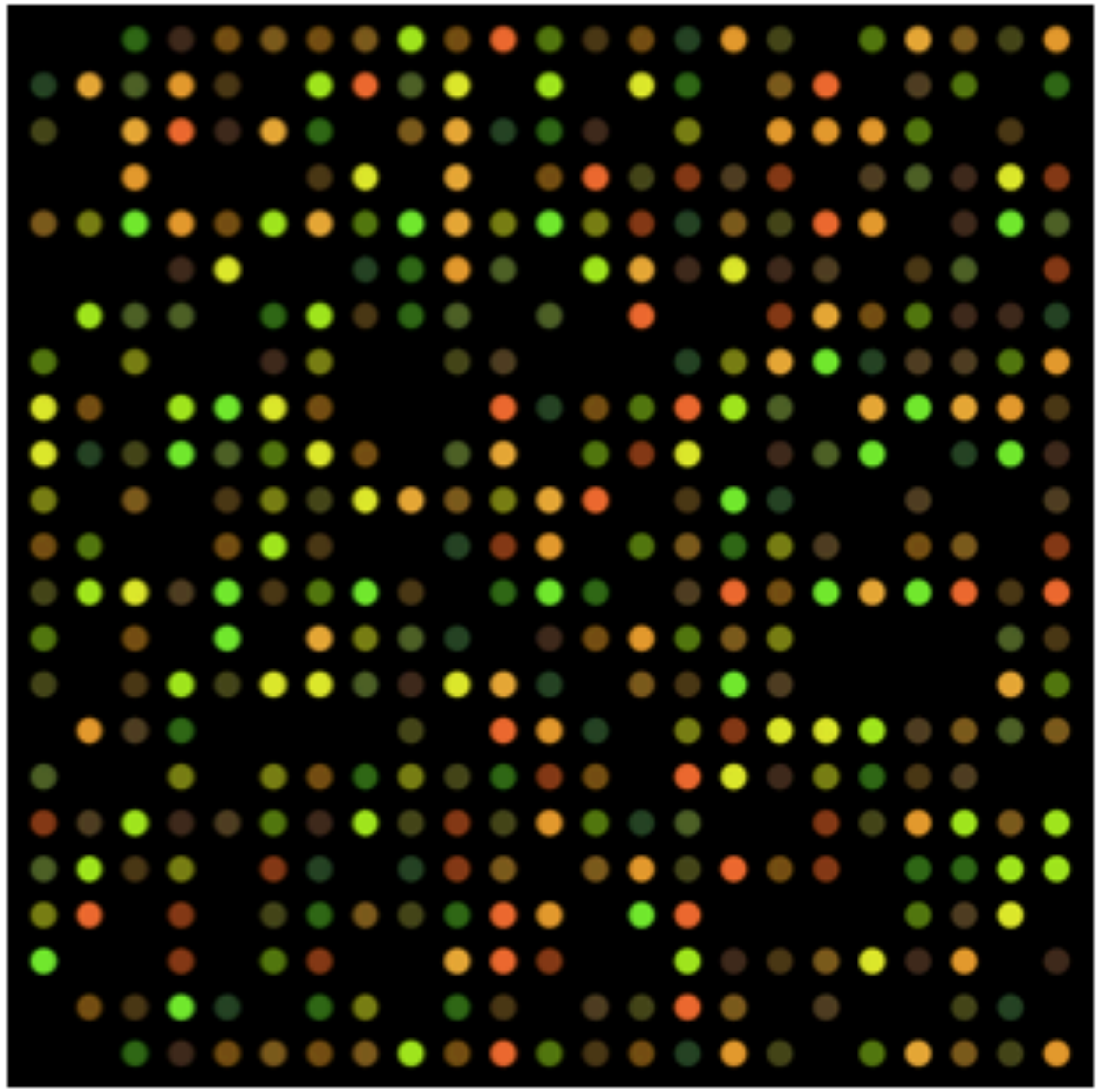
1.1.2 (Motivational) Examples of signals

"Motivational" examples

Medical Image: CAT

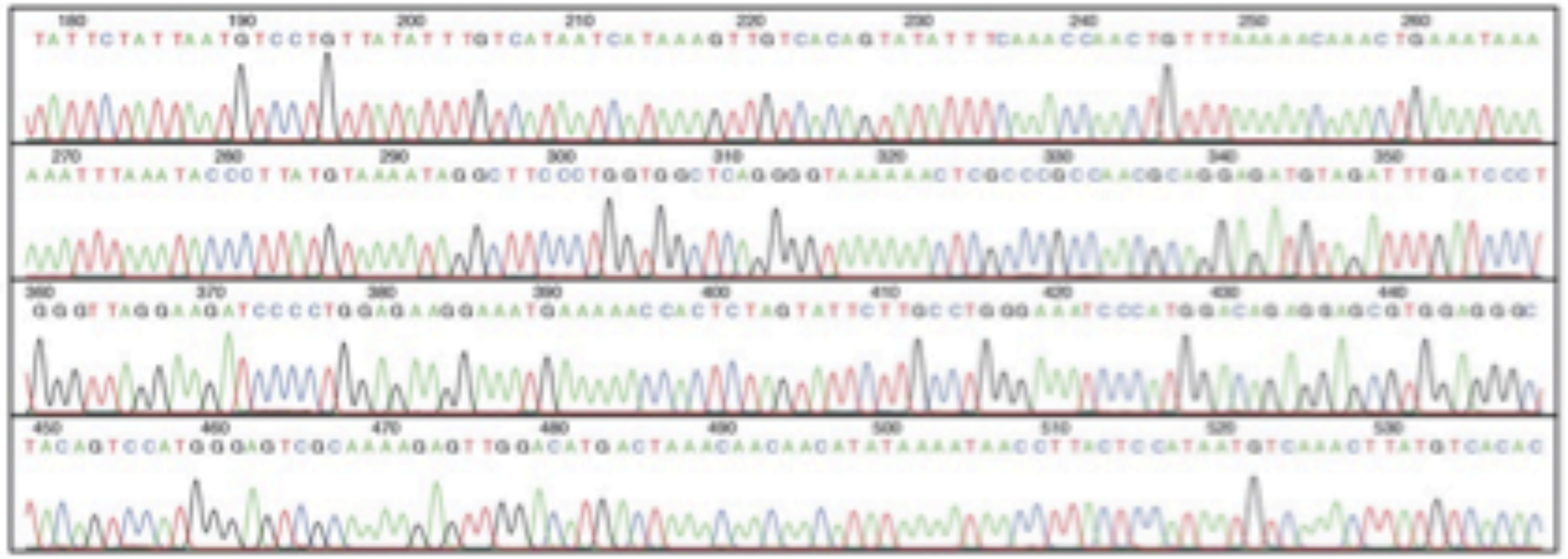


ADN: Microarray



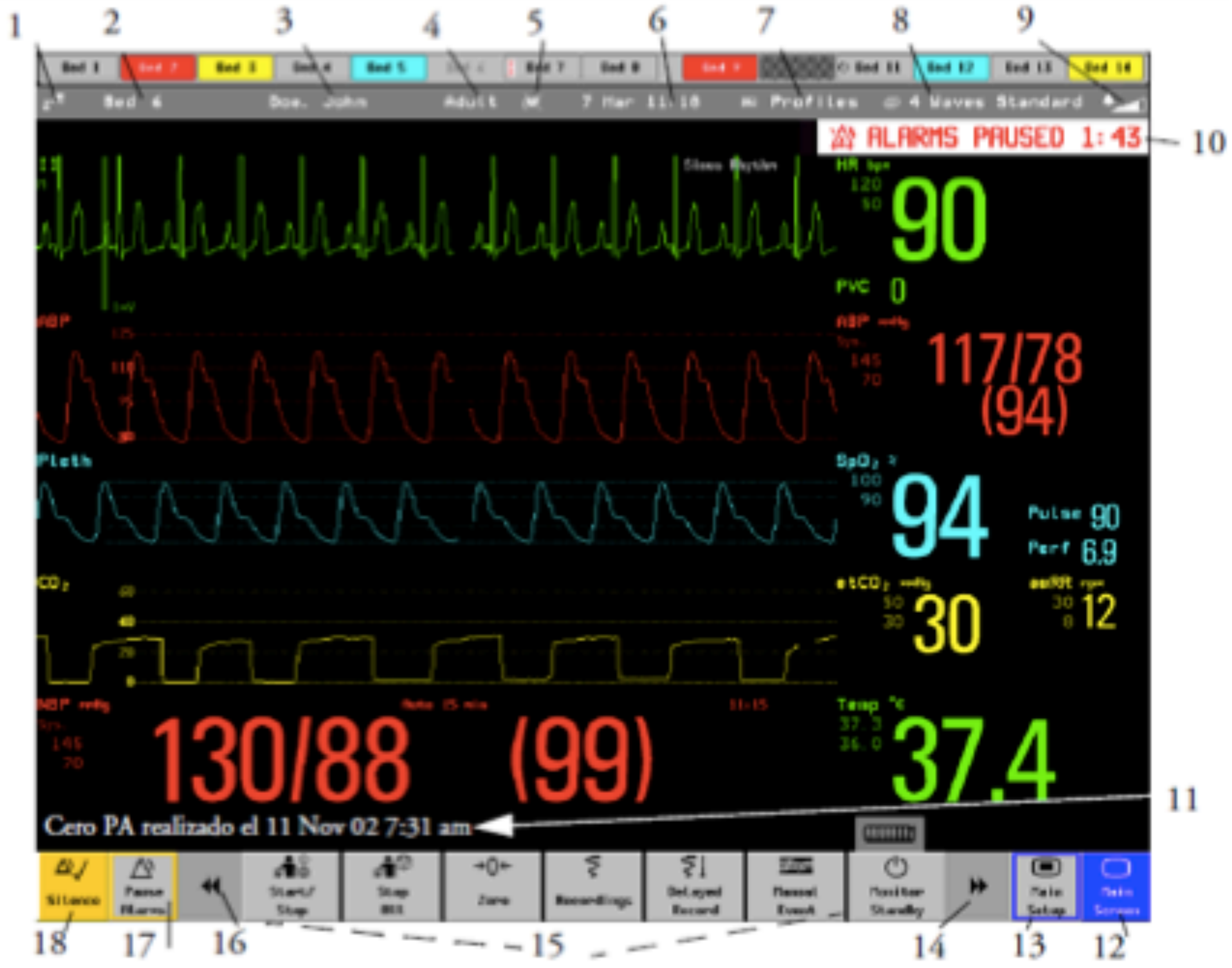
“Motivational” examples

DNA sequence

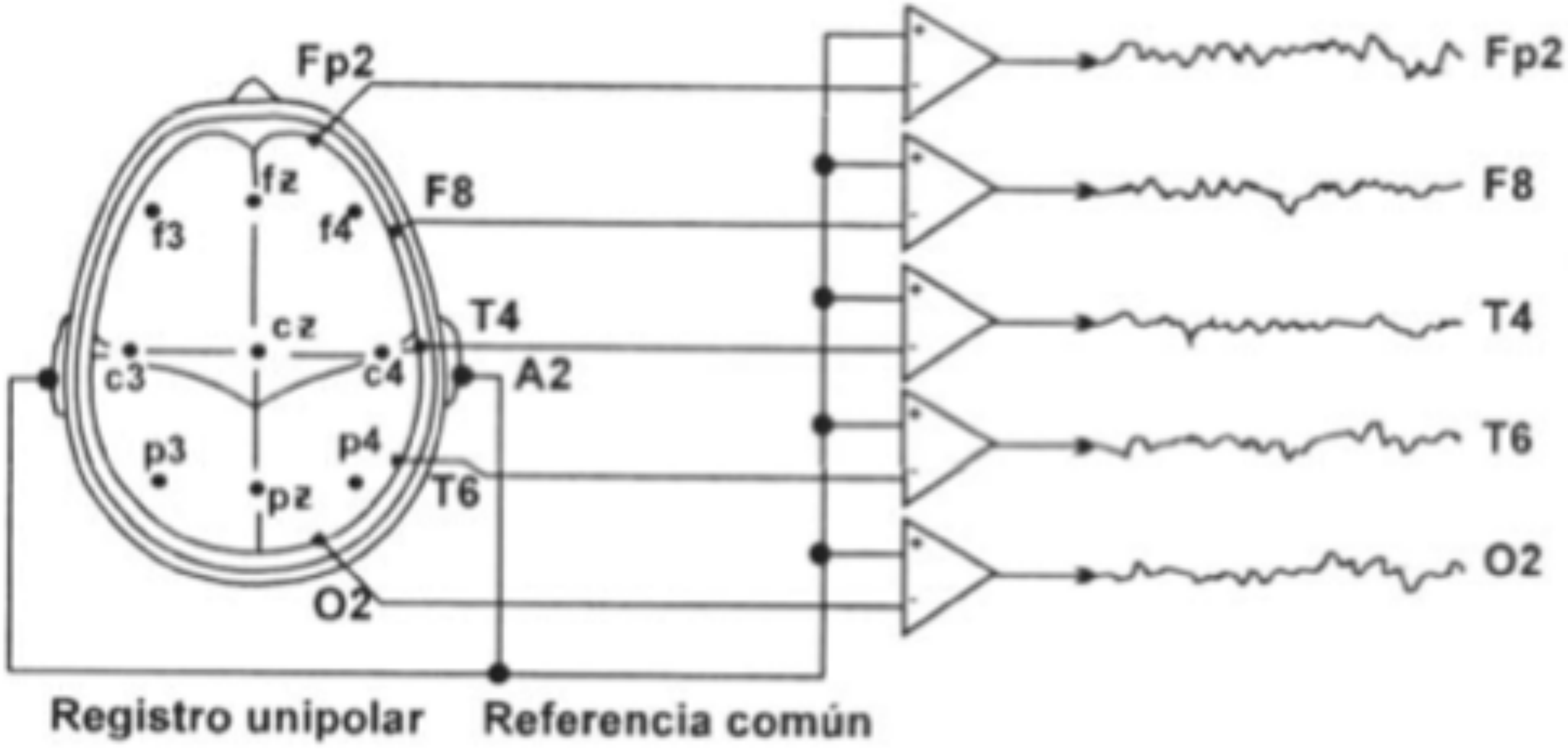


Secuencias de ADN obtenidas de una máquina de secuenciación automática

Intensive care monitor

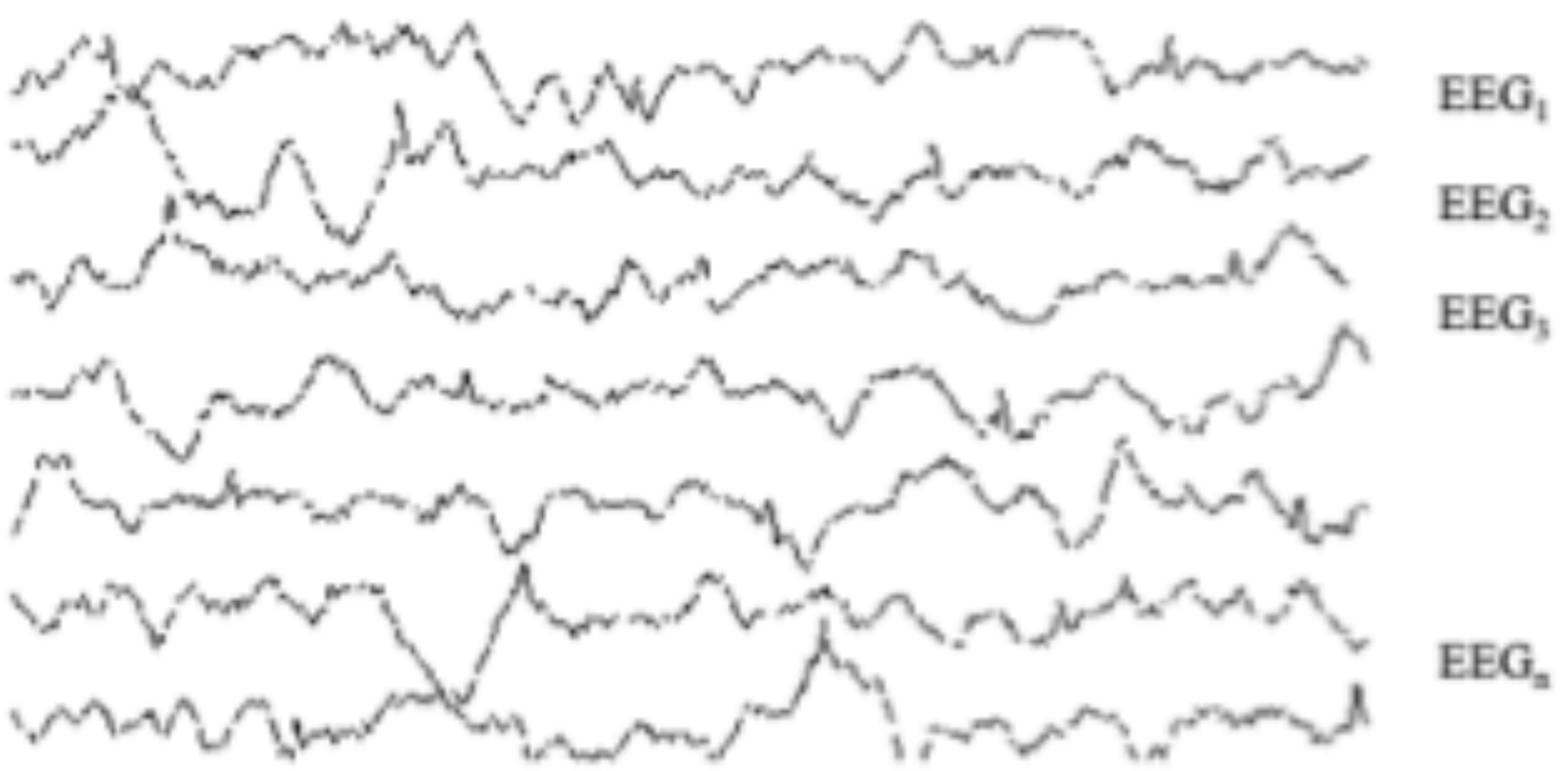


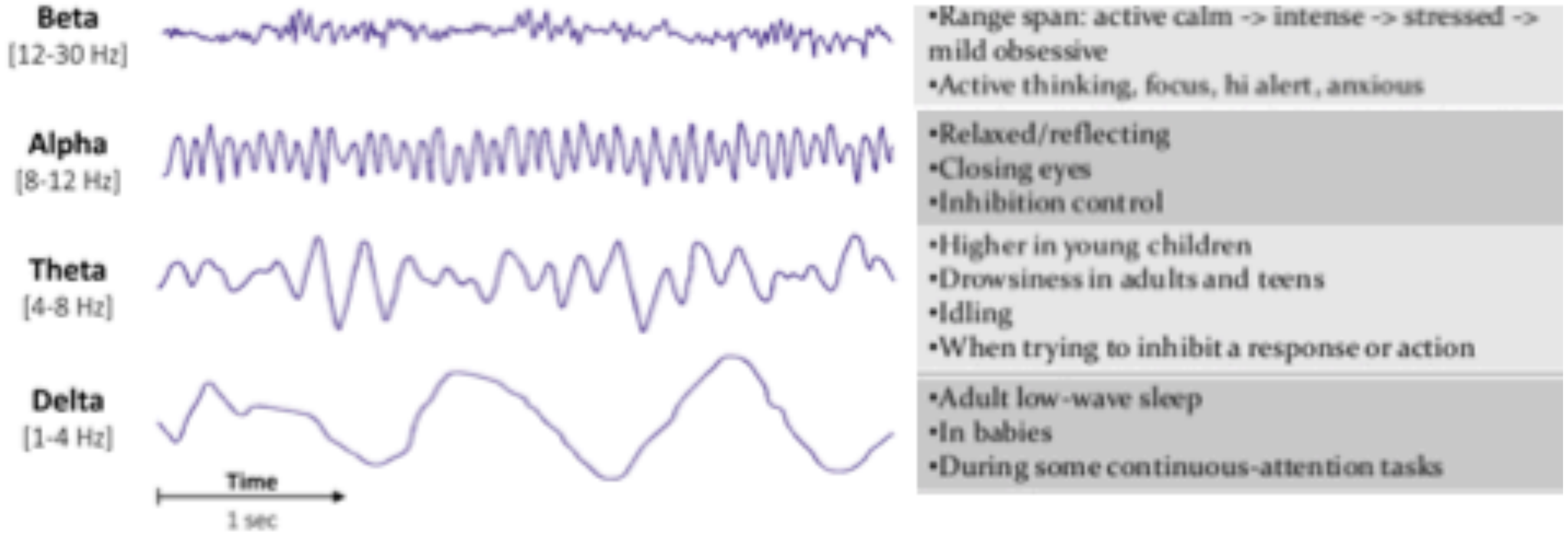
“Motivational” examples



“Motivational” examples

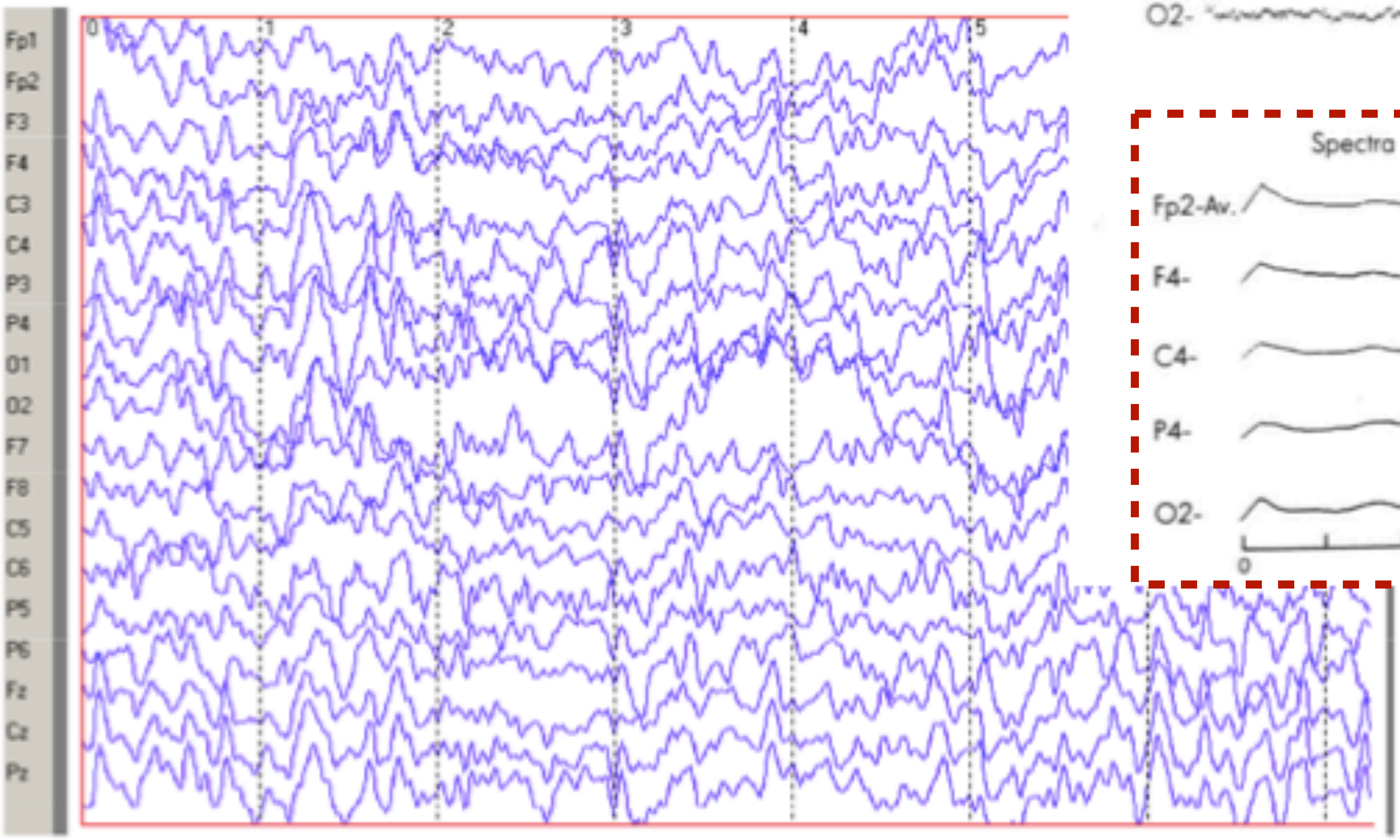
EEG: children 8-12 years, healthy in REM

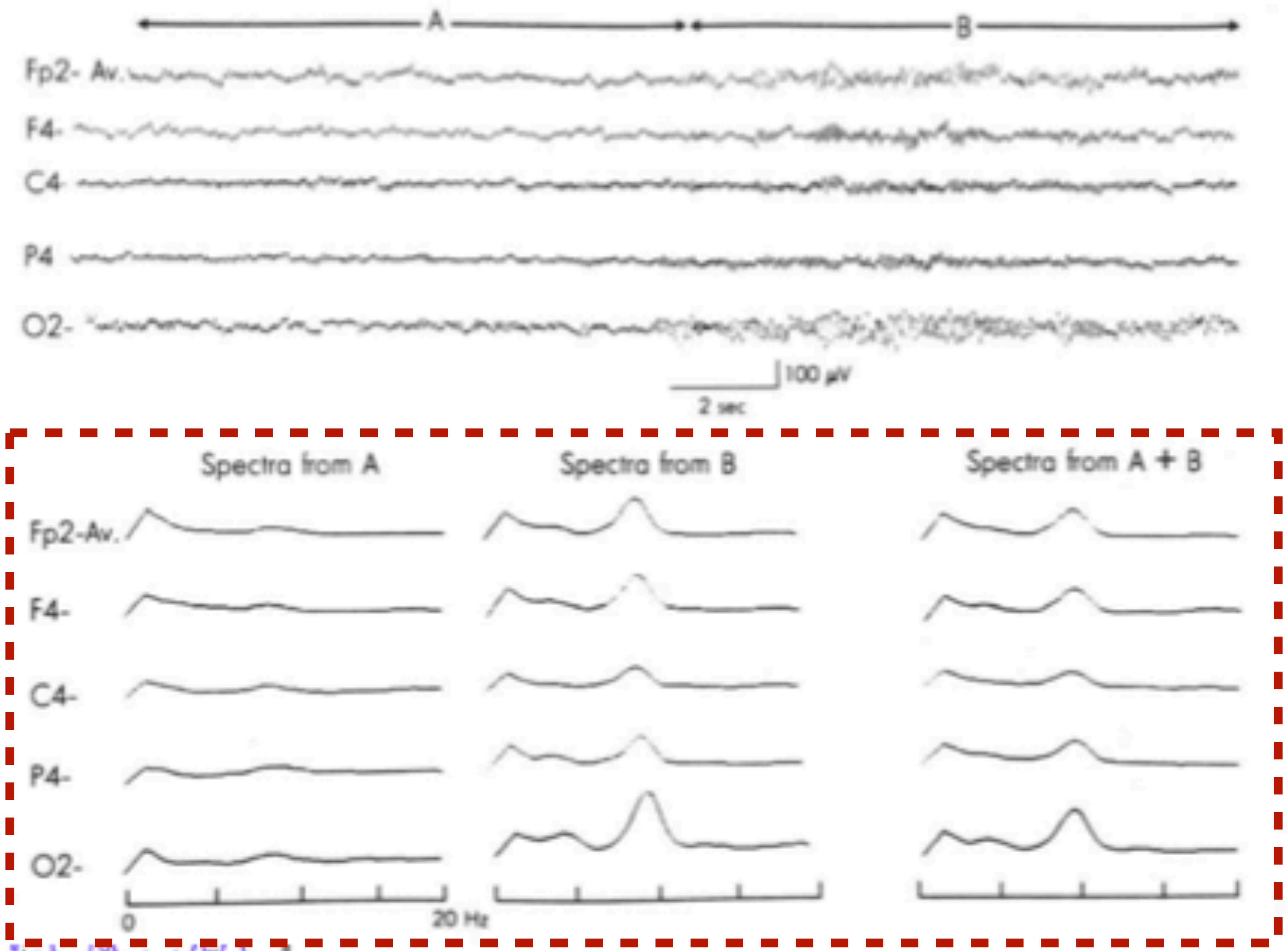




Amplitude between 20-100 uV
Frequency between 0-30 Hz.

“Motivational” examples





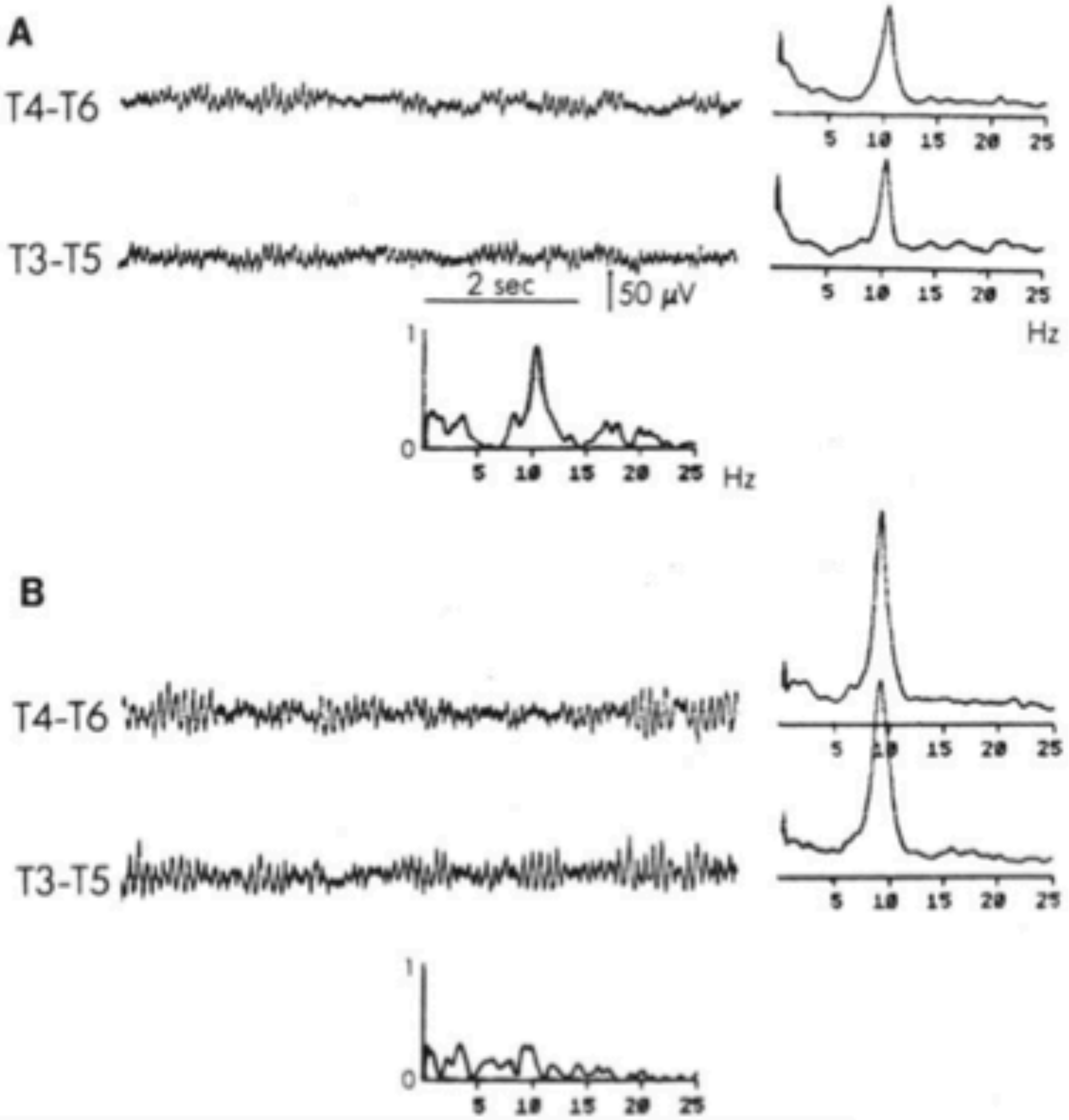
Spectra!

Frequency response

“Motivational” examples

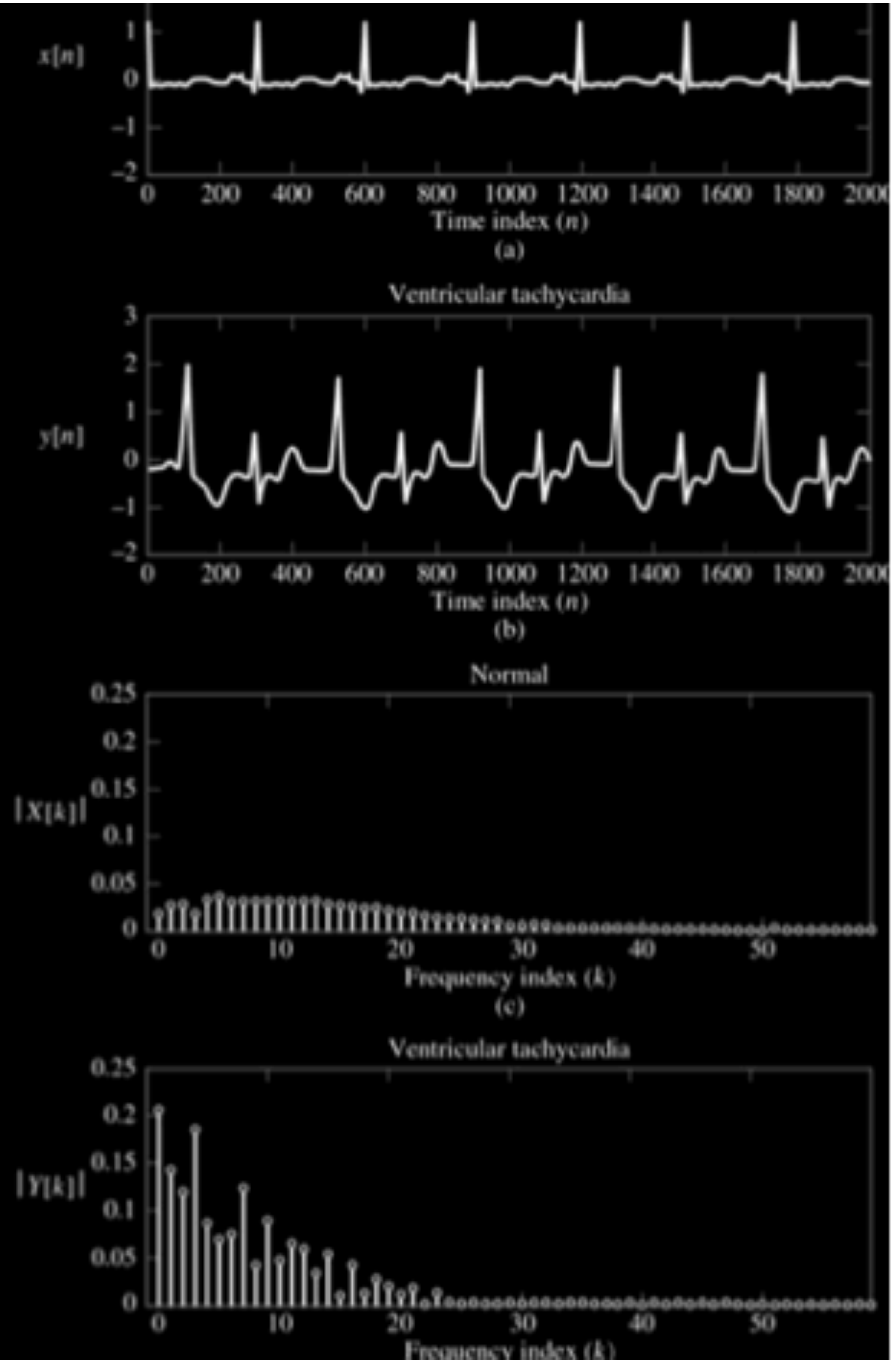
$$Cxy = \frac{|Pxy(f)|}{(Pxx(f)Pyy(f))}$$

Measures the synchronization between different brain cortex regions

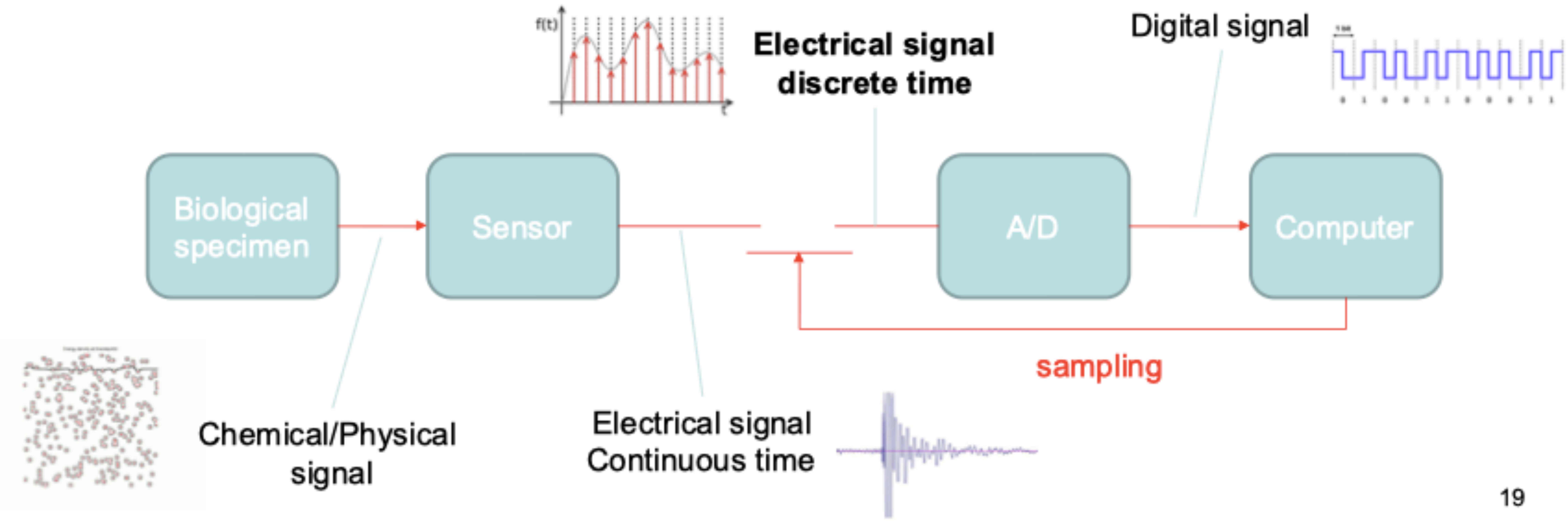


“Motivational” examples

Figure 3.15 (p. 214)
Electrocardiograms for two different heartbeats and the first 60 coefficients of their magnitude spectra.
(a) Normal heartbeat.
(b) Ventricular tachycardia.
(c) Magnitude spectrum for the normal heartbeat.
(d) Magnitude spectrum for ventricular tachycardia.



"Motivational" examples: in this course…



1.2 Basic operations with signals in cont. time, and important signals in cont. time and main properties

1.2.1 Basic operations with signals in CT

Operations with signals

- What can we do?
  - Any mathematical operation.

- Examples:
  - Level/amplitude change: $\Longrightarrow Ax(t)$
  - Translation: $\Longrightarrow x(t - t_0)$
  - Time inversion: $\Longrightarrow x(-t)$
  - Change of scale: $\Longrightarrow x(at)$
  - Derivation $\Rightarrow \dfrac{dx(t)}{dt}$
  - integration $\Longrightarrow \int_0^t x(\tau)d\tau$

Recall on the blackboard and in your mind…

Operations with signals

Suggested strategies: build a table !!

t	$Ax(t), x(t-t_0), x(at)$, etc.
0 →	correspondent value
-1 →	correspondent value
1.5 →	correspondent value
2 →	correspondent value
....	....

...and then make a plot !!!

1.2.2 Important signals in CT

Specific relevant signals in cont. time

Complex exponential (just imaginary part) in continuos time:

$$\text{Euler Formula: } e^{jw_o t} = cos(w_o t) + jsin(w_o t)$$

Specific relevant signals in cont. time

- Complex Exponential (CE)

$$x(t) = ce^{st} = re^{j\theta}e^{\sigma t}e^{jw_0 t} = re^{\sigma t}e^{j(w_0 t+\theta)}$$

real

Amplitude

complex

$$c = re^{j\theta}$$
$$s = \sigma + jw_0 \qquad w_0 = 2\pi f_0$$

Specific relevant signals in cont. time

Complex exponential in continuos time:

in general, we will consider the simpler formula

$$x(t) = e^{st} = e^{\sigma t} e^{j w_0 t}$$

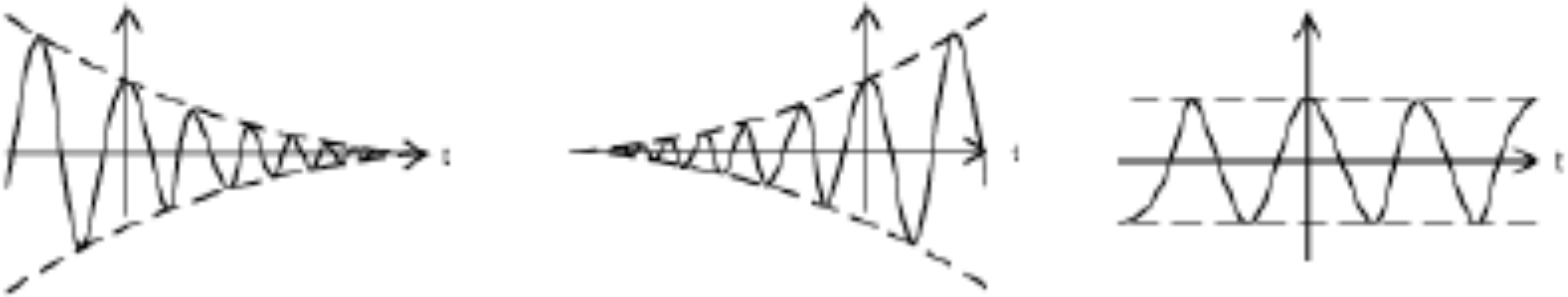
where:

$$s = \sigma + j\omega_0$$

Specific relevant signals in cont. time

EXTREMELY IMPORTANT SLIDE:

$$x(t) = e^{\sigma t}\cos(\omega_0 t)$$



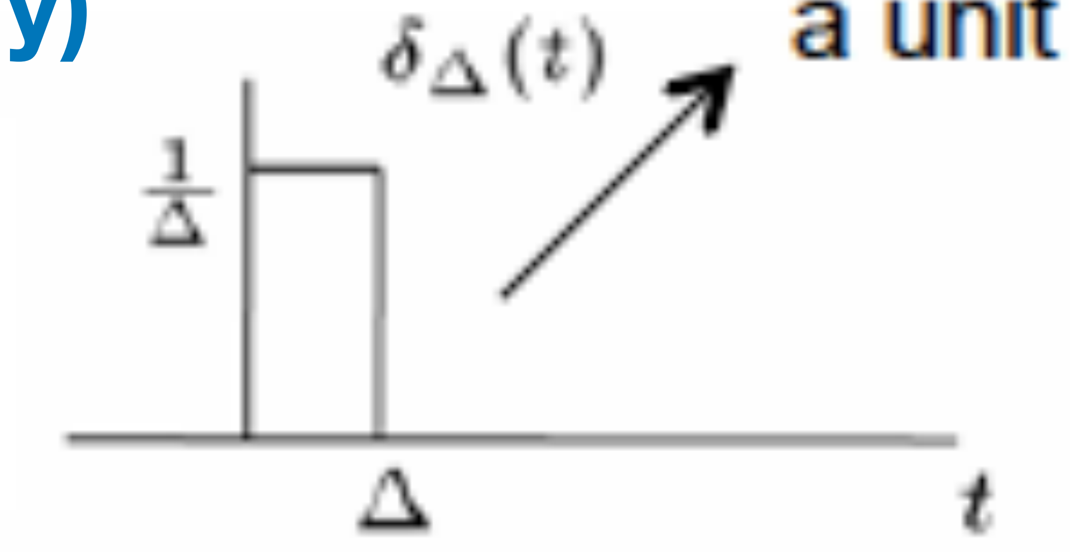
σ<0 σ>0 σ=0

Dirac delta or impulse function

- Dirac delta or impulse: $\delta(t)$

Possible definition
(no so "good" mathematically)

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



Note it has a unit area!

  - Properties:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

  - *Note that the Dirac delta in t=0 diverges (takes the values "infinite"); it is not a stirctly function, it is a generalized function (or a distribution)*

Properties of the Dirac delta

Properties of the unit impulse

1. The area under the function is 1:

$$\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$$

2. Scaling property:

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

3. Even property

$$\delta(-t) = \delta(t)$$

4. Sampling property

$$x(t)\delta(t) = x(0)\delta(t)$$

5. Sampling property (ii)

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

6. Sampling property (iii)

$$x(t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t_0 - \tau) d\tau$$

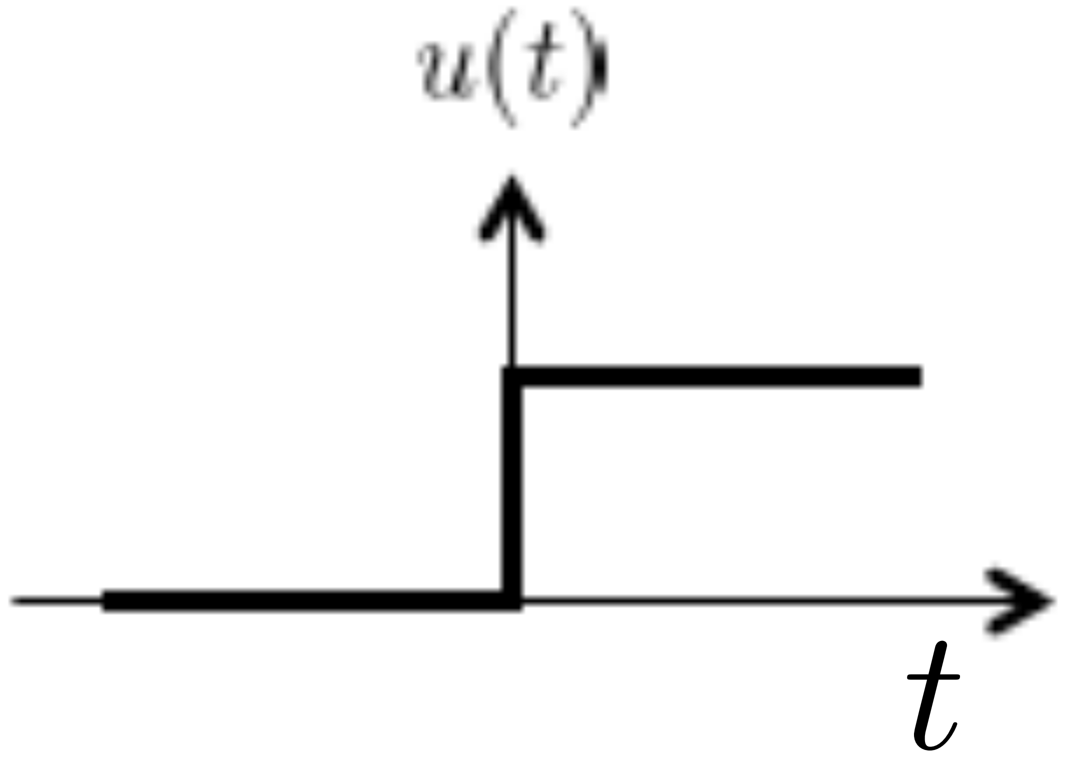
7. Therefore, any continuous-time signal can be decompose as a (infinte) linear combination of shifted and scaled unit impulses

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau) d\tau$$

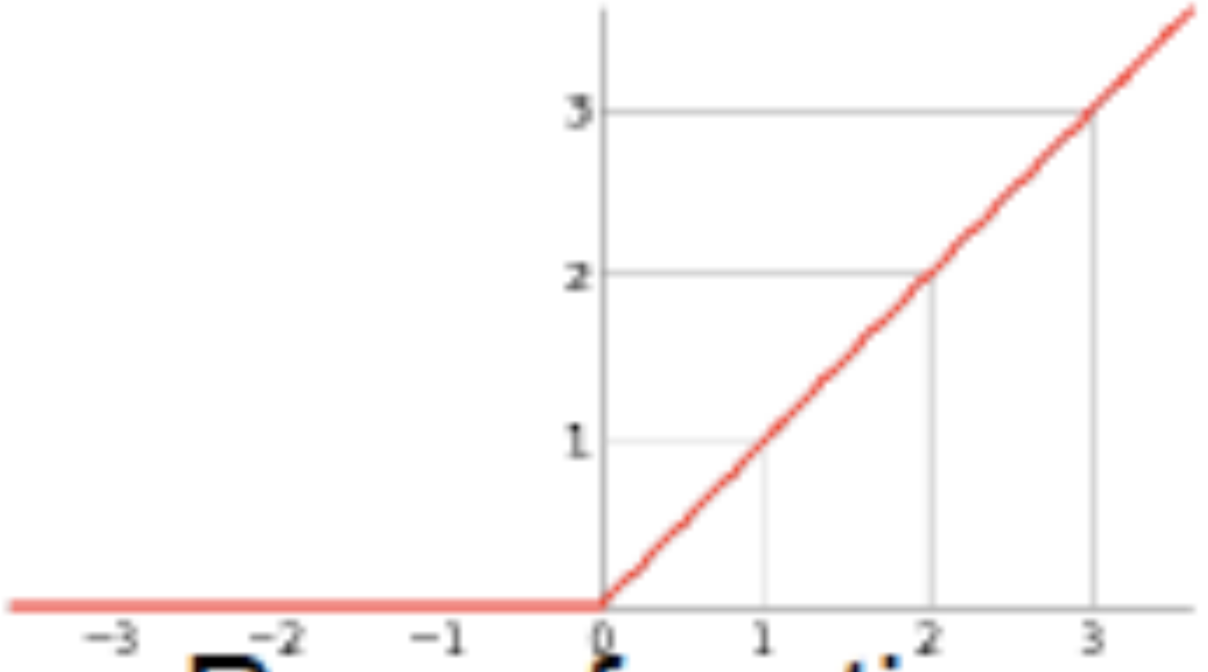
Heaviside function - step function

Heaviside step function $u(t)$:

$$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$



$$u(t) = \int_{-\infty}^{t} \delta(x) dx$$



Ramp function

$$u(t) = \frac{d}{dt} \max\{t, 0\}$$

It can be seen as the derivative of the ramp function

Heaviside function - step function

The Dirac delta can be seen as the derivative of the step function:

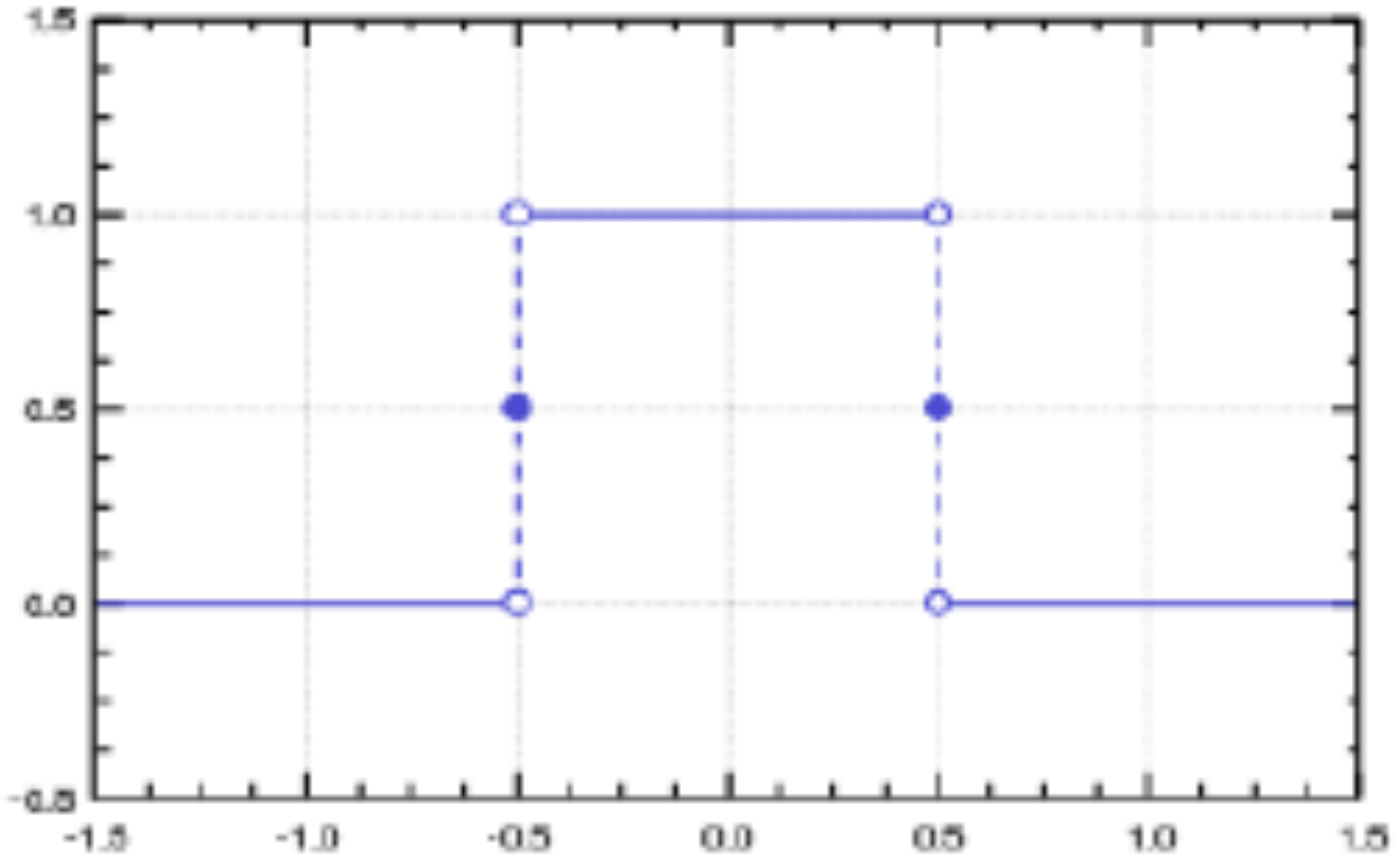
$$\delta(t) = \frac{du(t)}{dt}$$

Mathematically, we need the distribution theory….

Rectangular function

- Unit Rectangle: $p(t)$

$$p(t) = \begin{cases} 1, & \text{si } |t| < 1/2 \\ 0, & \text{si } |t| > 1/2 \end{cases}$$



  - Without unit area: $p_T(t) = p(t/T)$

$$p_T(t) = \begin{cases} 1, & \text{si } |t| < T/2 \\ 0, & \text{si } |t| > T/2 \end{cases}$$

Sinc function

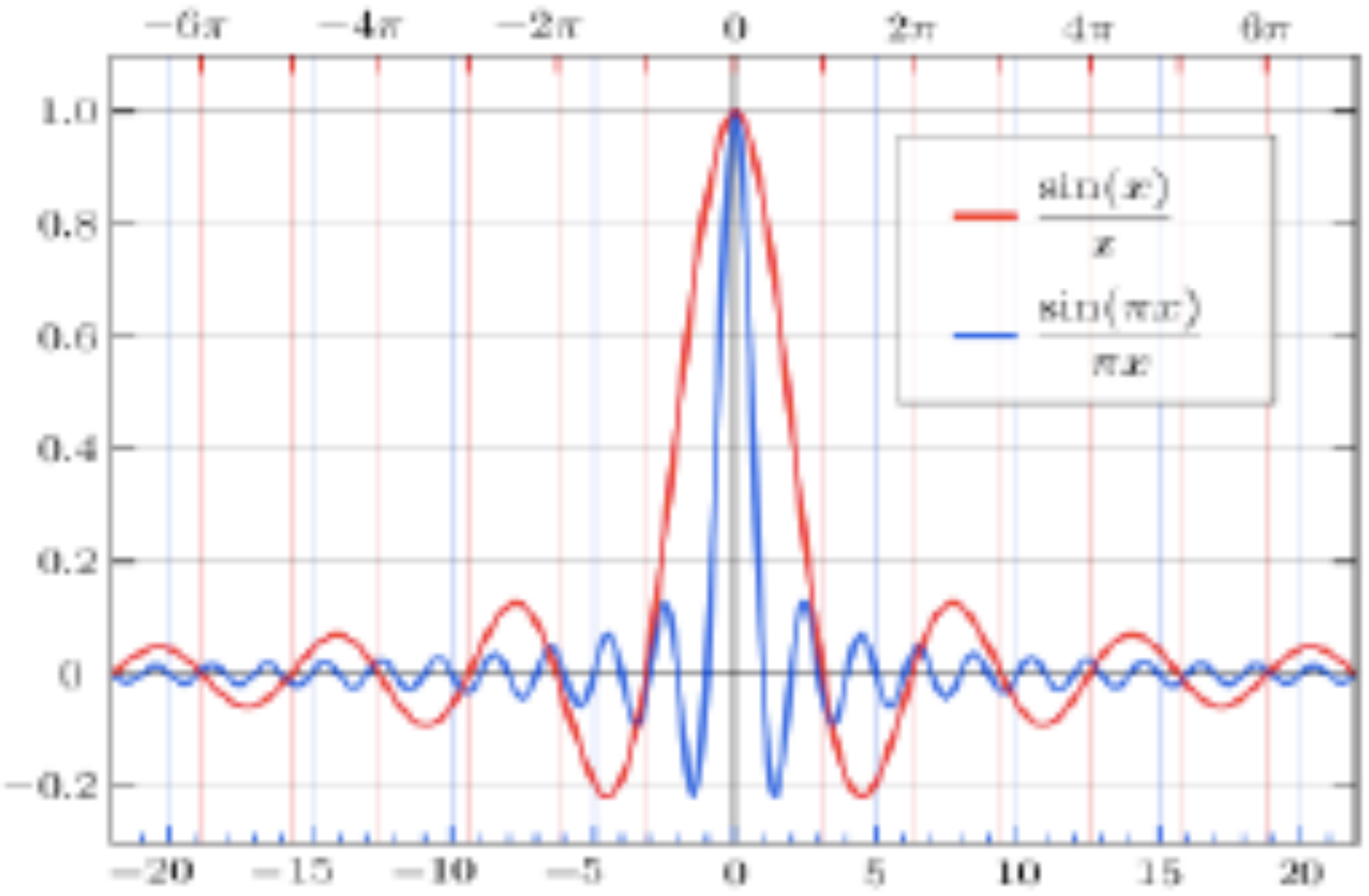
- Sinc function: $sinc(t)$

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

  - Without unit area:

$$sinc_T(t) = sinc(t/T) = \frac{sinc(\pi t/T)}{\pi t/T}$$

* IMPORTANT REMARK: The zeros are at multiples of T !!!



Zeros at the multiples of 1 (or T) !!!!

1.2.3 Some properties in CT

Main properties (signals in cont. time)

- Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For non-periodic signals

- Mean Power:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

→ Energy in a unit of time

Main properties (signals in cont. time)

For a periodic signal:
POWER = mean energy in an period To

$$P = \frac{1}{T_0} \int_{<T_0>} |x(t)|^2 dt$$

Main properties (signals in cont. time)

- Recall:
  - There are *power signals* and *energy signals*:
    - Finite Energy → then the power is zero → energy signal
    - Finite Power → then the energy is infinite → power signal
    - Some signals are neither energy nor power signals.
    - For a periodic signal: if the energy in one period is finite, then it is a power signal

1.3 Basic operations with signals in discrete time, and important signals in discrete time and main properties

1.3.1 Basic operations with signals in DT

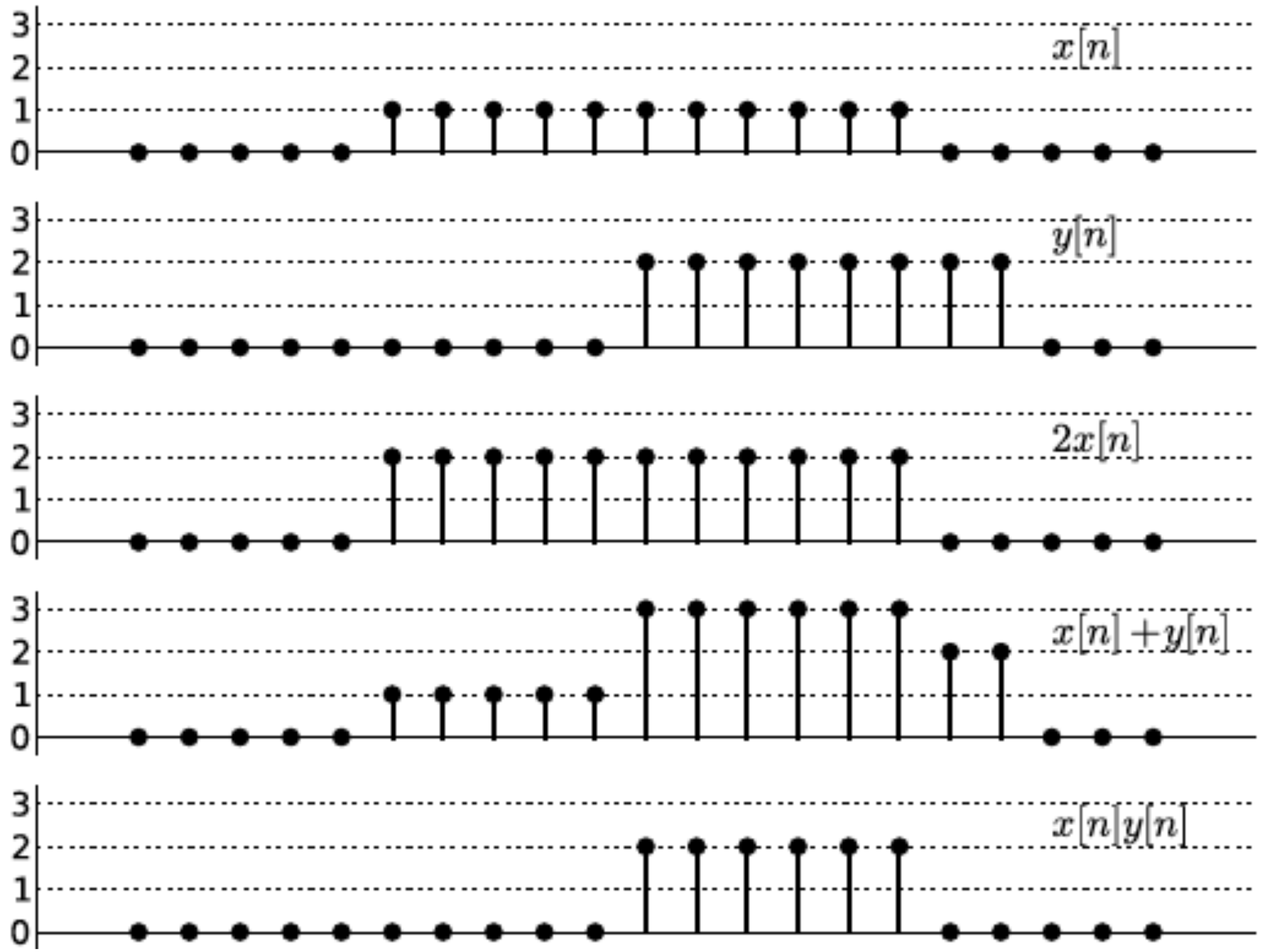
Operations with signals in DT

Basic operations about the dependent variable

- Change of scale of y[n]: $y[n] = K \cdot x[n]$
- Sum: $y[n] = x_1[n] + x_2[n]$
- Product: $y[n] = x_1[n] \cdot x_2[n]$

Operations with signals in DT

Basic operations about the dependent variable



Operations with signals in DT

Basic operations about the independent variable

- Translation/movement:

$$y[n] = x[n + n_0] \rightarrow \begin{cases} n_0 < 0 \rightarrow \text{To the right} \\ n_0 > 0 \rightarrow \text{To the left} \end{cases}$$

  - The value n_0 must be an integer

- Symetric signal with respt to the the y-axis:

$$y[n] = x[-n]$$

- Change of scale: (rational values in DT)

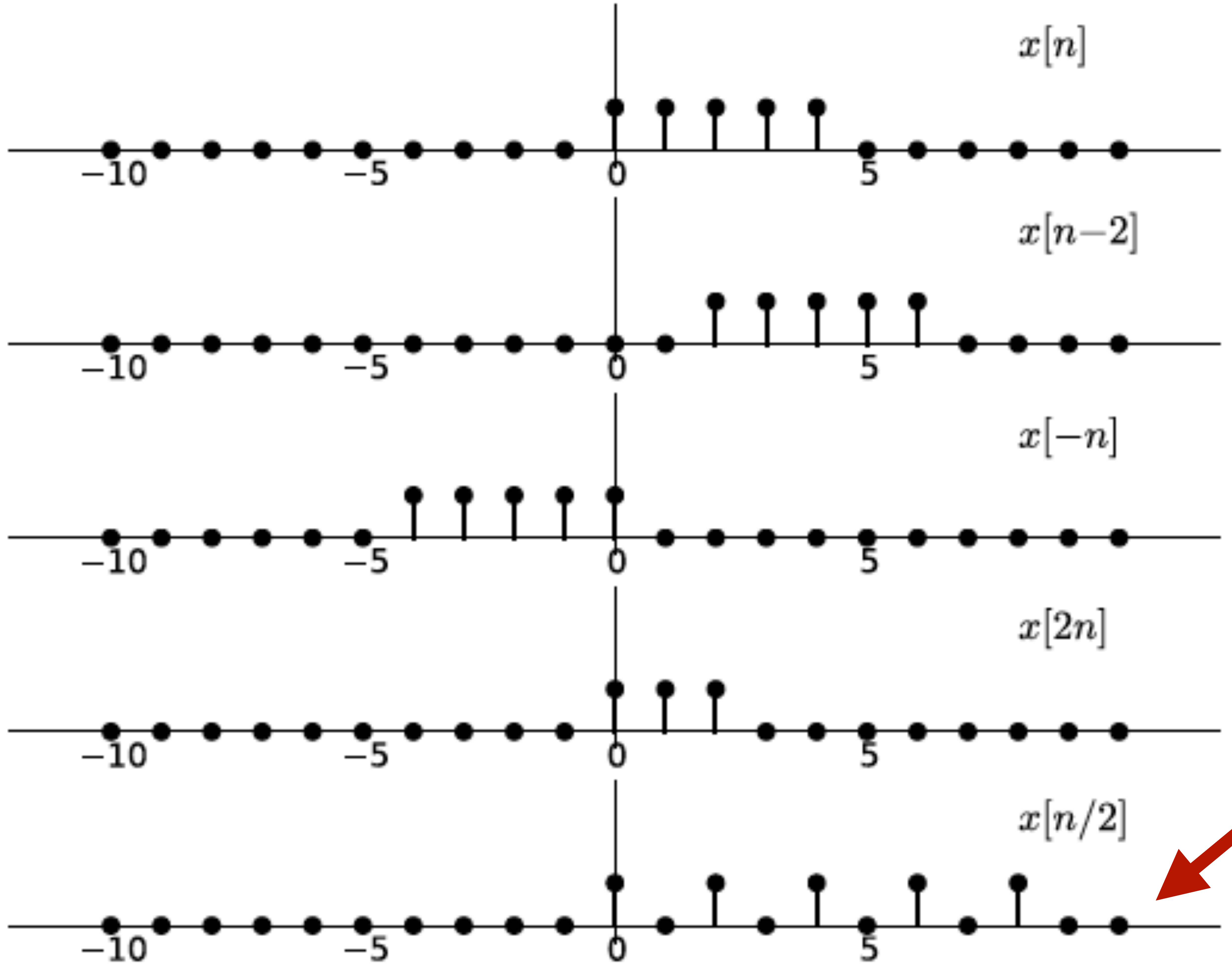
WARNING:
be careful, in my opinion, the change of scale is NOT well-defined in DT !!!

$$y[n] = x[an] \begin{cases} \text{expansion} & 0 \leq a < 1 \\ \text{contraction} & a > 1 \end{cases}$$

See other slides….

Operations with signals in DT

Basic operations about the independent variable



WARNING:
we are “inventing” samples !!

Operations with signals in DT

- IMPORTANT: en DT, the scale change produces the following consequences:

  - During **compression**, we **lose samples**
  - During **expansion**, we have to **add new samples** (typycally zeros)
  - REMARK: in the topic "SAMPLING", we will see how to do it without having any problems/issues

Operations with signals in DT

Difference and sum

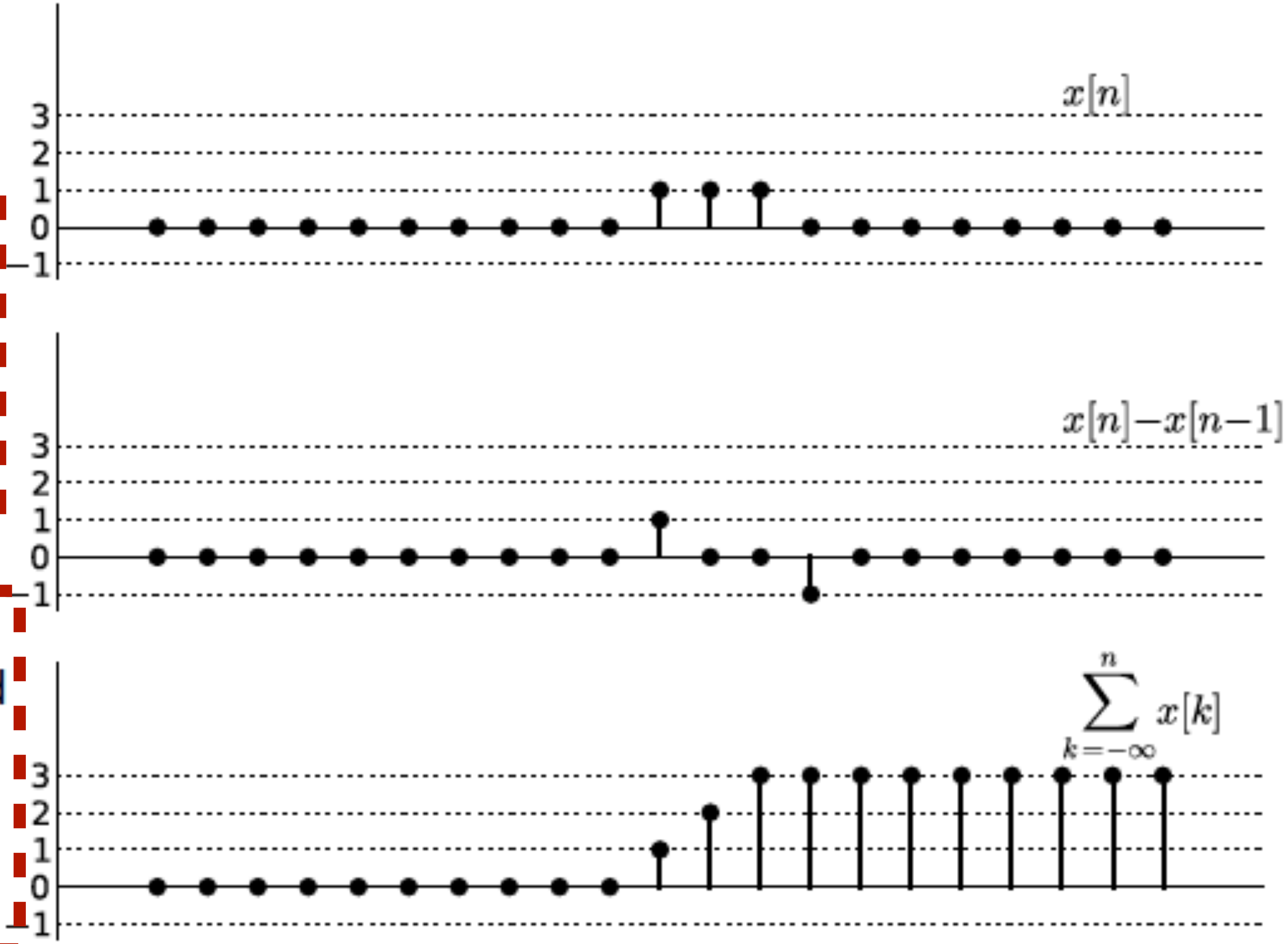
Original Signal:

Difference (related to derivative in CT):

$$y[n] = x[n] - x[n-1]$$

Sum (also called "accumulator", related to the integral in CT):

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

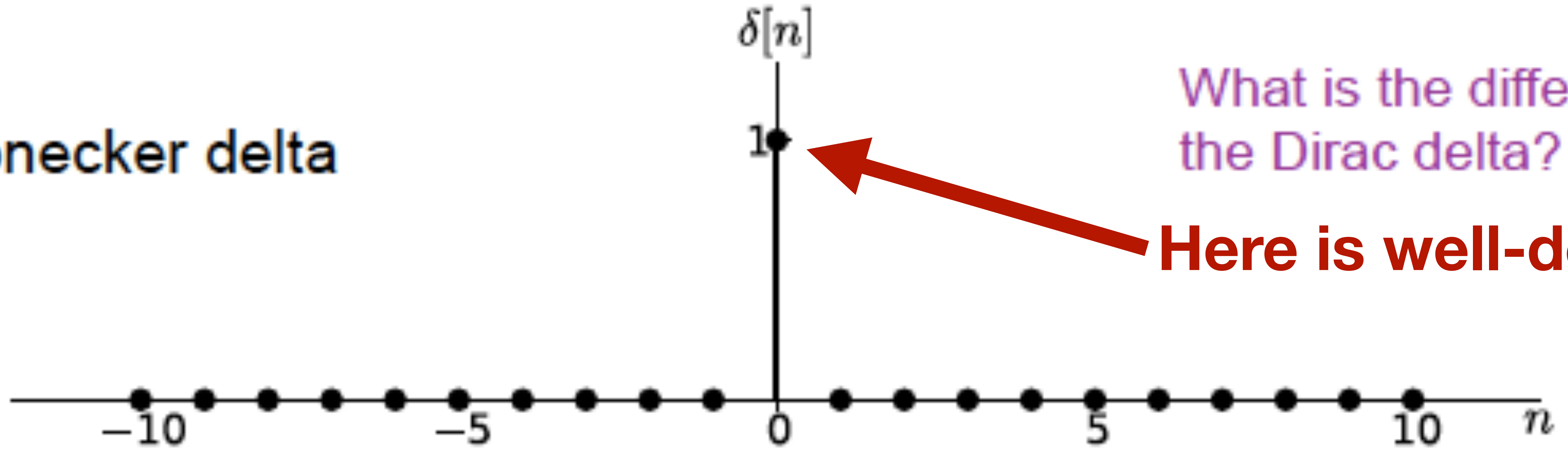


1.3.2 Important signals in DT

Important signals

Basic signals: delta and step functions

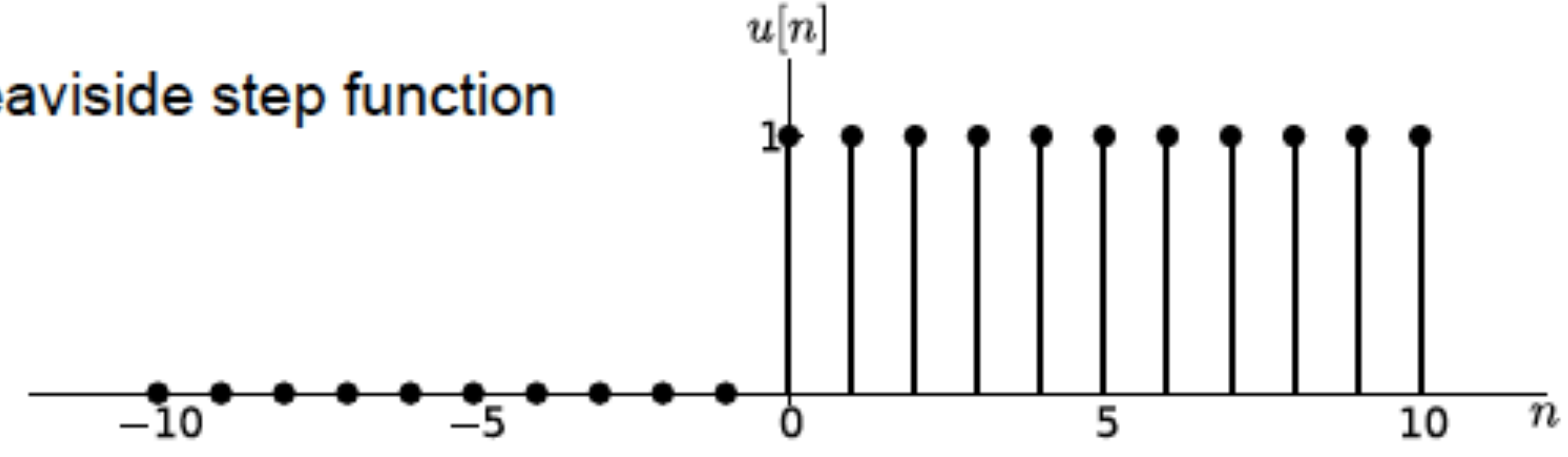
Kronecker delta



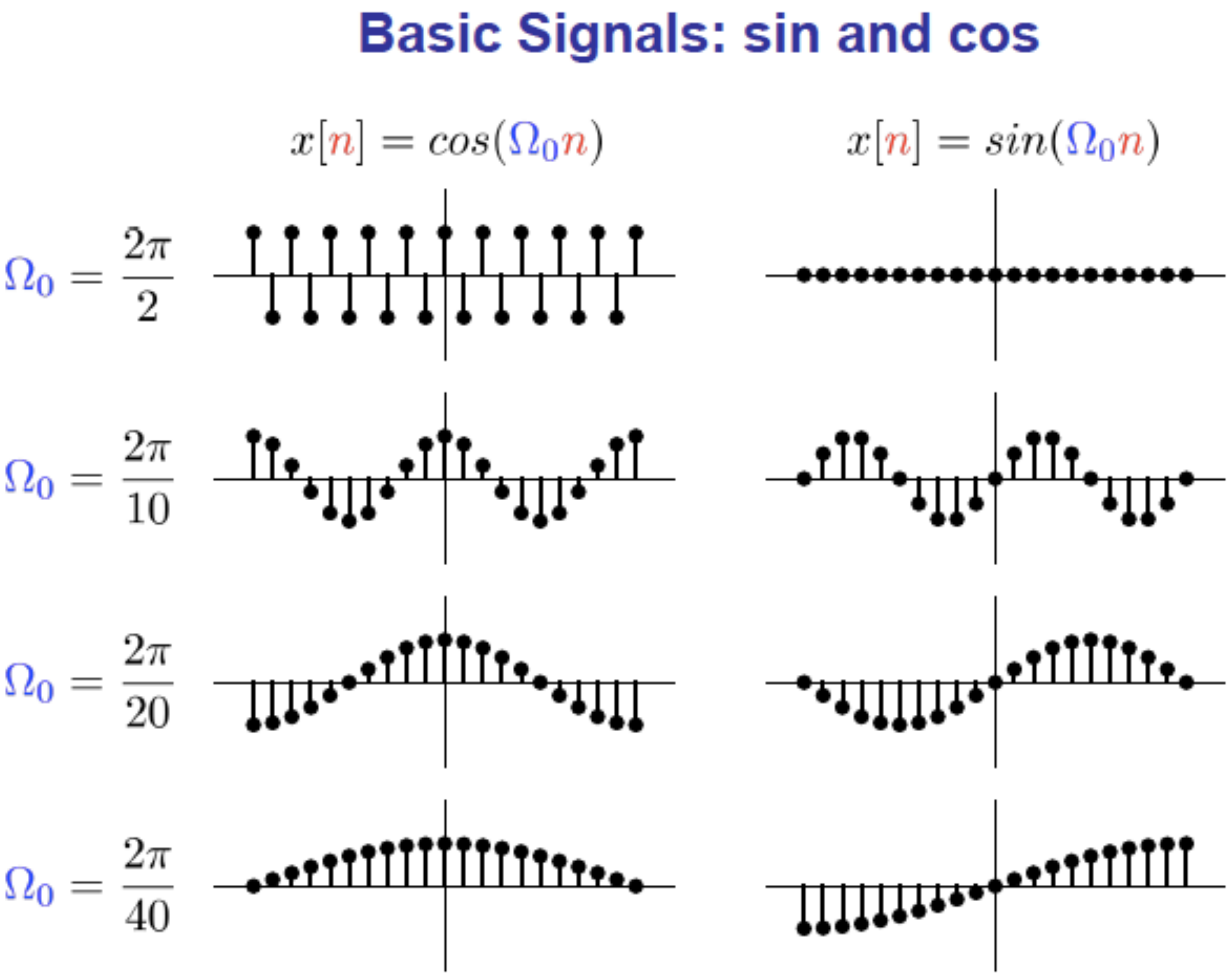
What is the difference with the Dirac delta?

Here is well-defined !!!

Heaviside step function



Important signals



WARNING:
IN DISCRETE TIME
cosine AND sine
ARE VERY "STRANGE"
AND COMPLICATED FUNCTIONS.

Cosine and sine: some properties in DT

Basic Signals: sin and cos

- Properties of the sinusoidal signals in DT:
  - Two sinusoidal signals with angular frequency of

$$\Omega_0, \quad \Omega_1 = \Omega_0 + 2\pi$$

are identical.

just true in discrete time

WARNING:
IN DISCRETE TIME
cosine AND sine
ARE VERY “STRANGE”
AND COMPLICATED FUNCTIONS.

  - They are periodic if and ony if the angular frequency can be expressed as:

$$\Omega_0 = 2\pi \frac{m}{N}$$

$\cos(3n) \Rightarrow$ NON-PERIODIC !!!!!!

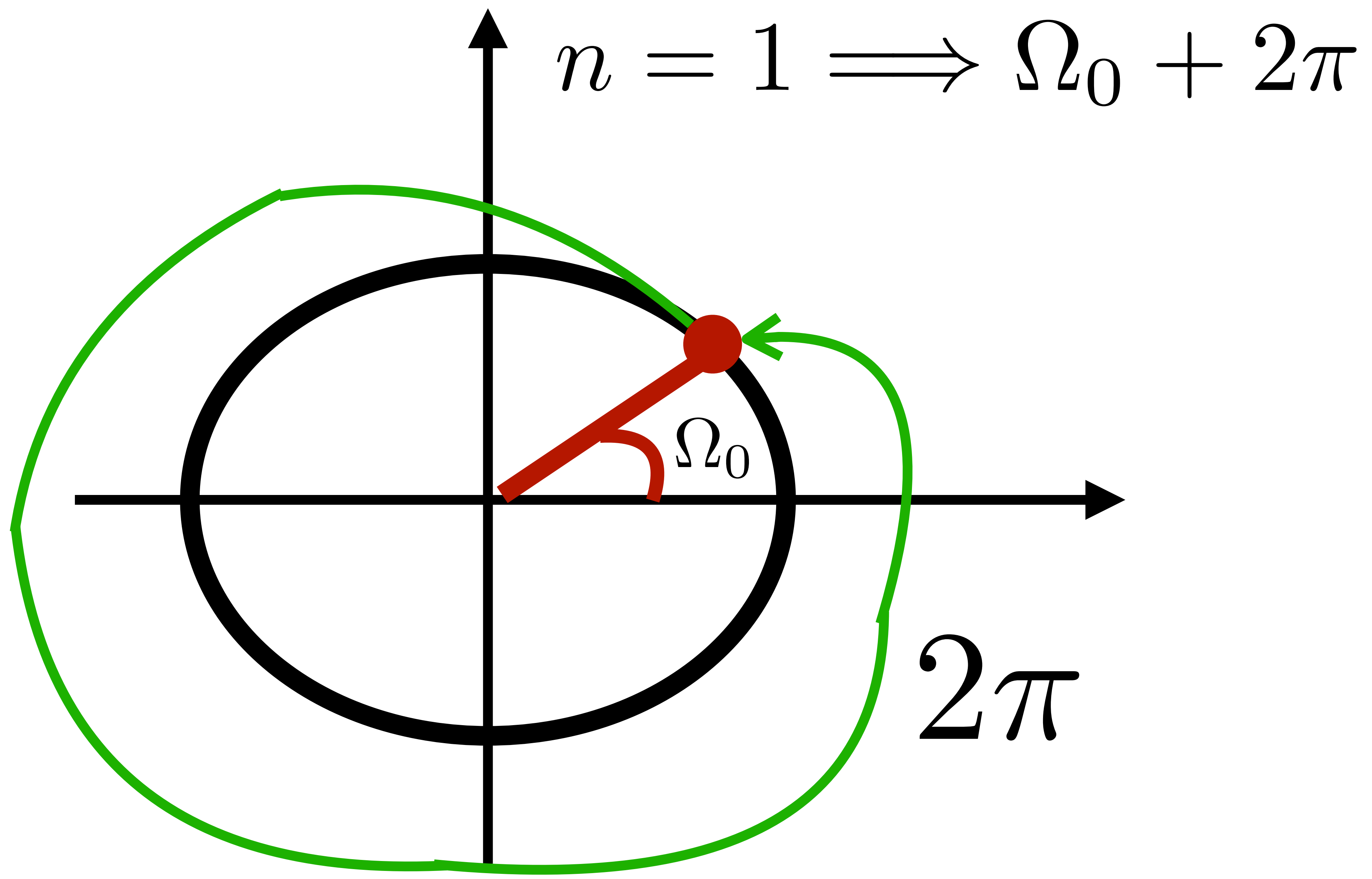
We will come back on this point in another class

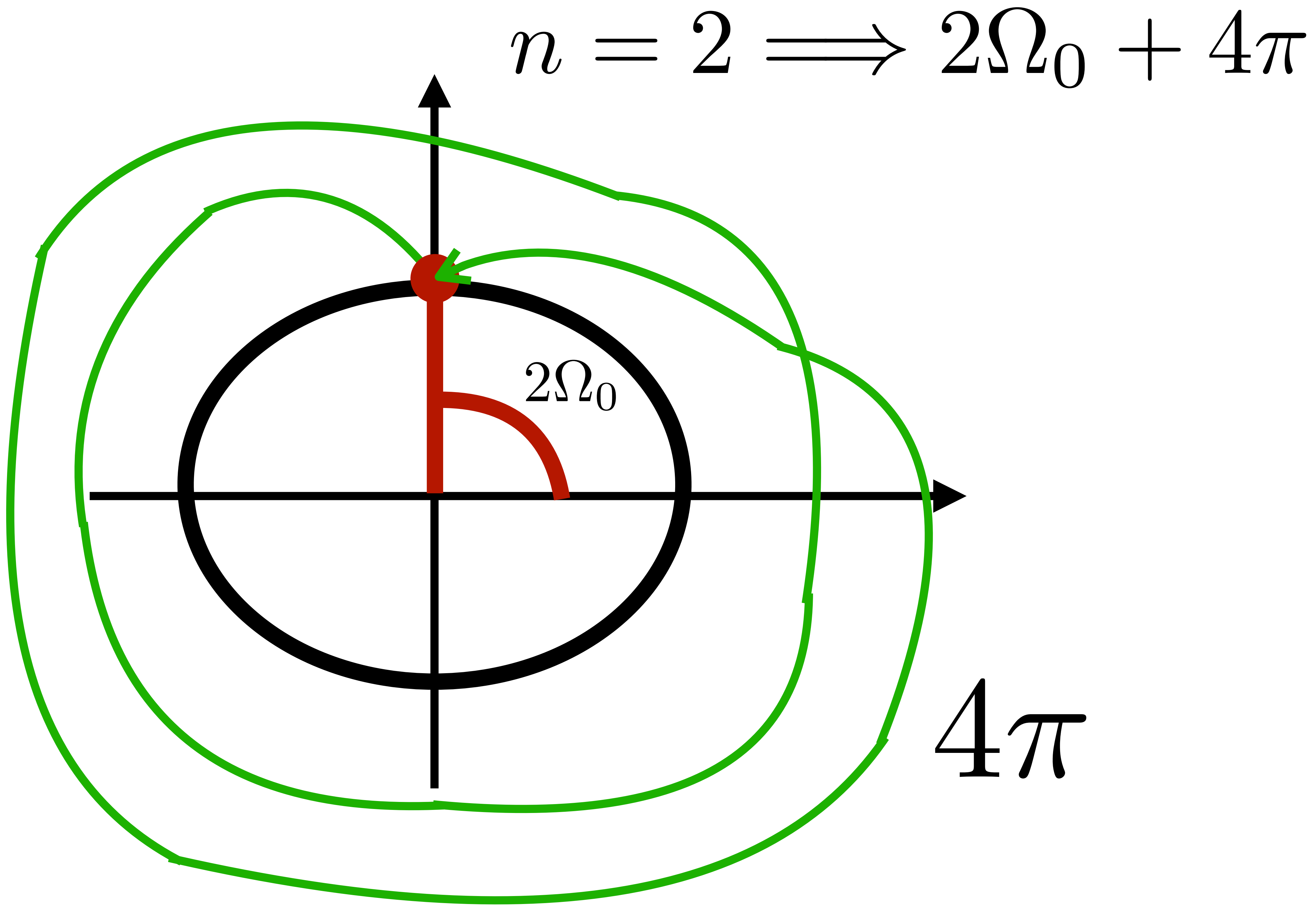
Where N and m are integers without common factors. In this case the period is *N*.

On property of sine and cosine (discrete time)

$$\Omega_0 = \frac{\pi}{4} \Rightarrow 45^\circ$$

$$(\Omega_0 + 2\pi)n$$





Since n is an integer, it defines only a unique point !!!!

On property of sine and cosine (discrete time)

In continuos time this is NOT true !!
due t can take any real value !!!

$$(\Omega_0 + 2\pi)t$$

Try to do the plot yourself (making some example choosing some possible value of "t")
(POSSIBLE QUESTION OF EXAM)

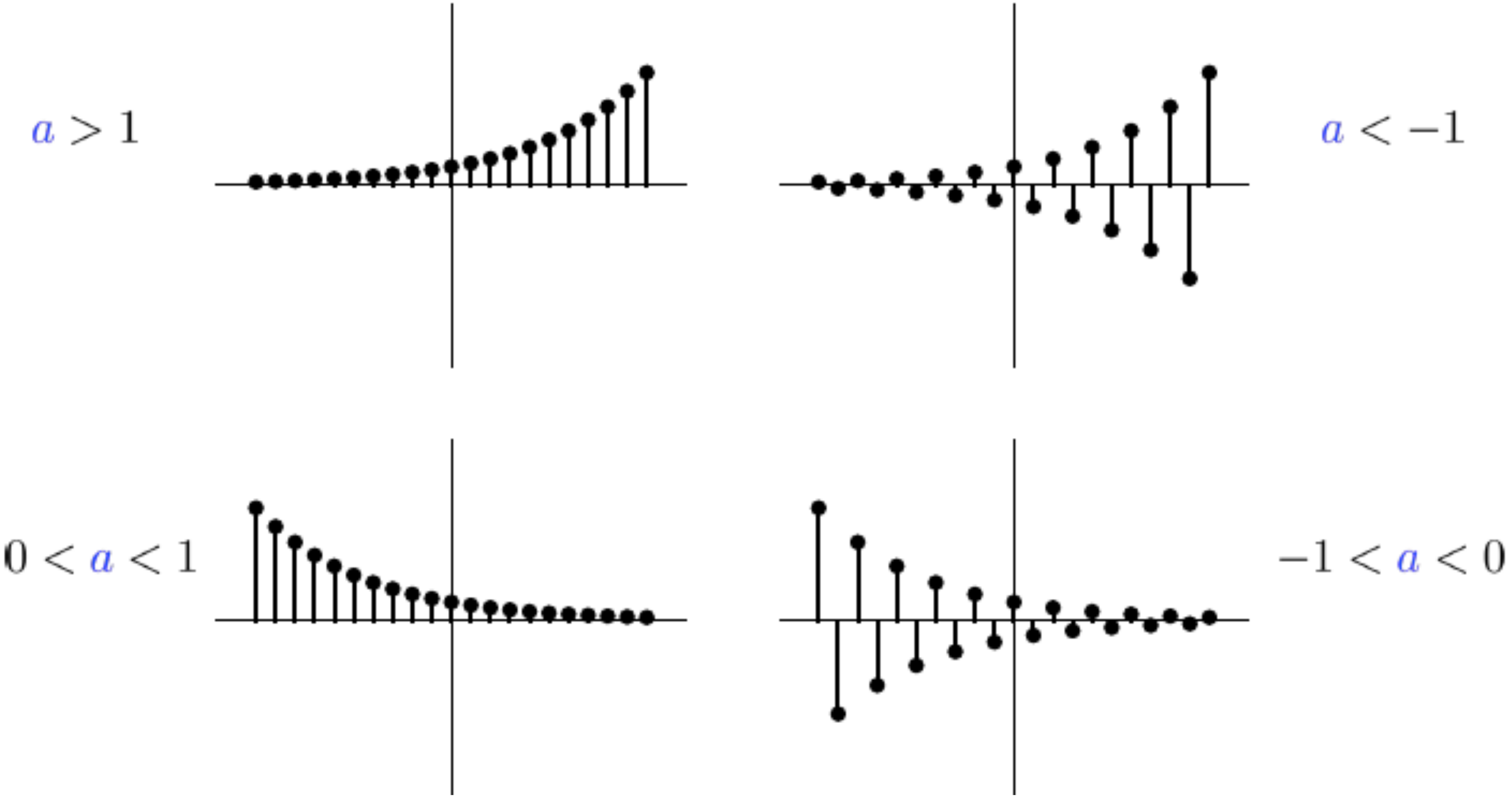
In the plot, we generate different points….

Important signals

Basic signals: Real exponential/power function

VERY IMPORTANT SLIDE !!!!

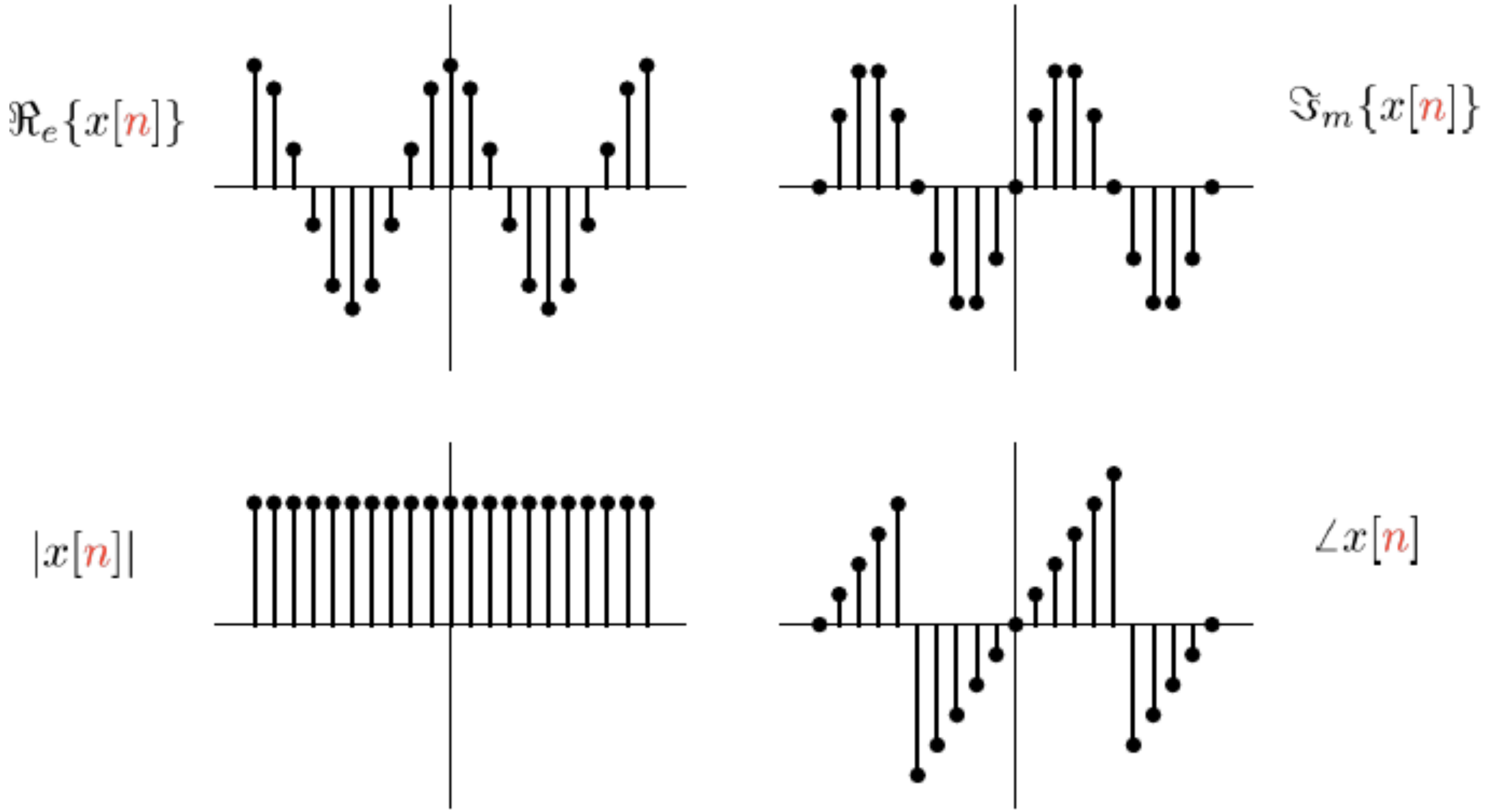
$$x[n] = Ca^n$$



Important signals

Basic signals: Complex exponential

$$x[n] = e^{j\Omega_0 n}$$



WARNING:
IN DISCRETE TIME IT CAN BE ALSO NON-periodic !

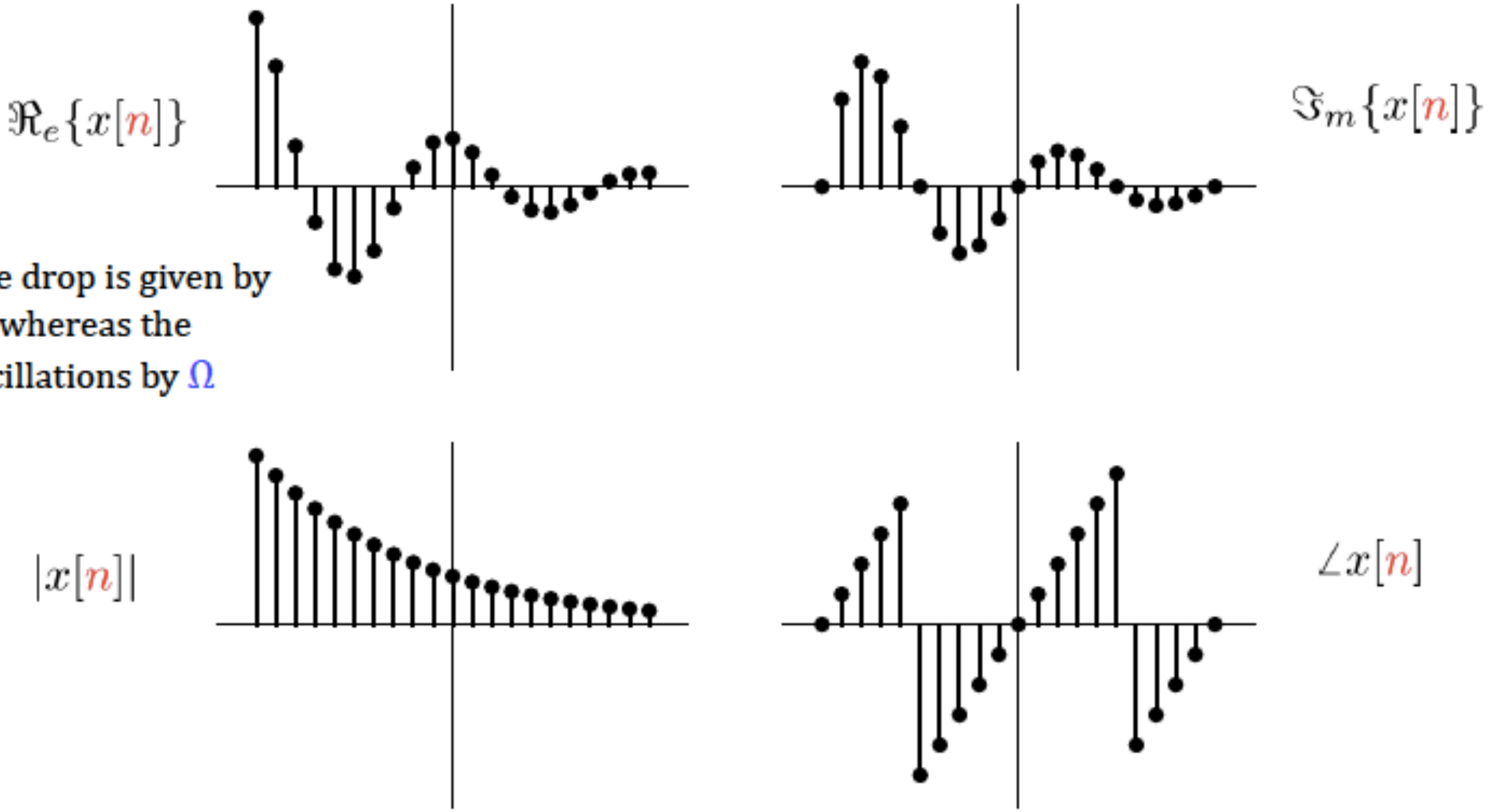
We will come back on this point in another class

Important signals

Basic signals: Complex exponential with “damping”/envelope

VERY IMPORTANT SLIDE !!!!

$$x[n] = e^{(a+j\Omega)n} = e^{an}e^{j\Omega n}$$



The drop is given by a, whereas the oscillations by Ω

1.3.3 Some properties in DT

Some properties of signals in DT

- Odd and even signals:
  - *Even signal x[n]*: $x[n] = x[-n]$
  - *Odd signal x[n]*: $x[n] = -x[-n]$
- Periodicity
  - Signal *x*[*n*] is periodic with periodic N if: $x[n] = x[n - N]$

N must be an integer !!!!

- Decomposition by Deltas
  - Any discrete signal can be expressed as a train of deltas:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Parameters: energy, power etc.

- Area, mean value, energy, power:

AREA

$$A_x = \sum_{n-\infty}^{\infty} x[n]$$
$$= ... + x[-1] + x[0] + x[1] + ...$$

ENERGY

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
$$= ... + |x[-1]|^2 + |x[0]|^2 + |x[1]|^2 ...$$

MEAN VALUE

$$\bar{x} = \langle x[n] \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$$

POWER

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

*Remark: with the notation |.| we have denoted the module of a vector (or a complex number), then the definition is valid also for complex signal.

Parameters: energy, power etc.

- We can also define the energy in a finite interval $-N \leq n \leq N$ as

$$E_N = \sum_{n=-N}^{N} |x[n]|^2$$

- Then the energy of the signal, E, is:

$$E = \lim_{N \to \infty} E_N$$

- Also the power can be expressed as function of E_N ,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} E_N$$

Parameters: energy, power etc.

- A discrete signal can be:
  - Energy signal: (finite energy and zero power) $E_x > 0, P_x = 0$
  - Power signal: $E_x = \infty, P_x < \infty$
  - Signal with infinite power: $E_x = \infty, P_x = \infty$

- We focus on energy ang power signals.

Questions?