Topic 1- part 2 - “Systems in time”

Discrete Time Systems (DTS)

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In this slides, WE WILL SEE:

1. What is a system?
2. Properties of a system

1. What is a system?

What is a system? in CT

- **Any transformation, any mapping of a signal into other signal:**

$$\boxed{y(t) = F\{x(t)\}}$$

$$y(t) = f(x(t), x(t-1), x(t+2)....)$$

- **even with feedback (autoregressive systems):**

$$y(t) = f(x(t), x(t-1), ..., y(t-1), y(t-2))$$

What is a system? in CT

A system is also called as *filter*

What is a system? in DT

- **Any transformation, any mapping of a signal into other signal:**

$$\boxed{y[n] = F\{x[n]\}}$$

$$y[n] = f(x[n], x[n-1], x[n+1], x[n-2]...)$$

- **even with feedback (autoregressive systems):**

$$y[n] = f(x[n], x[n-1]..., y[n-1], y[n-2]\ldots)$$

What is a system? in DT

A system is also called as *filter*

Examples of systems in continuous time

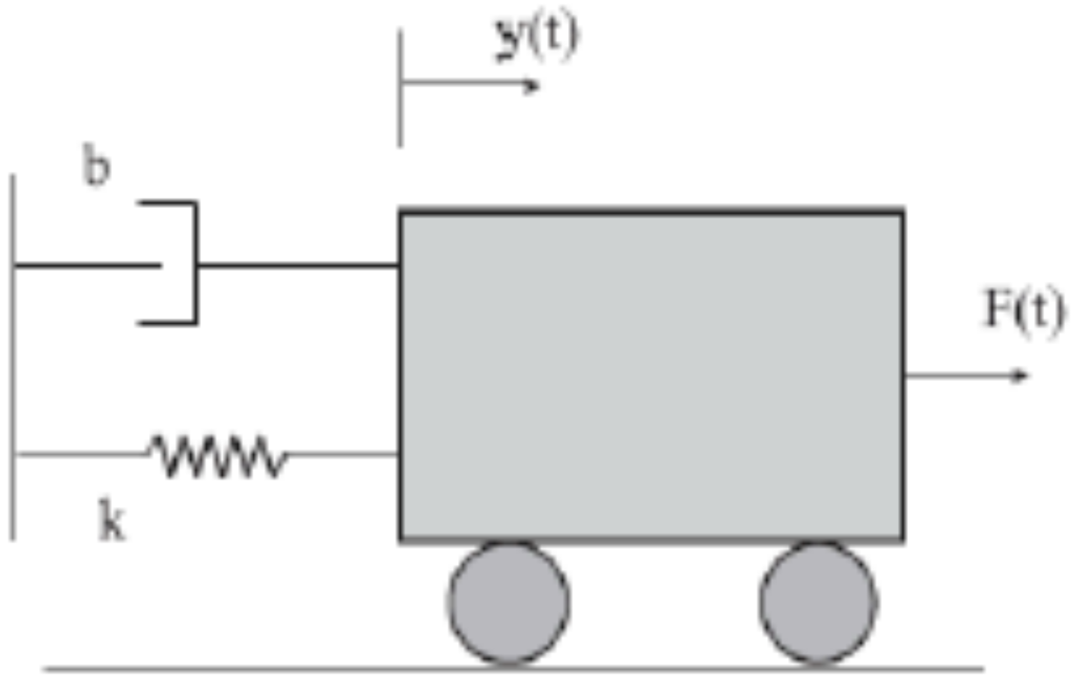
System definition

- A **system** can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

Example: dynamical systems

- A simple dynamical system: a little car on a surface, tide to the wall by a spring.
- Law of forces: DIFFERENTIAL EQUATIONS

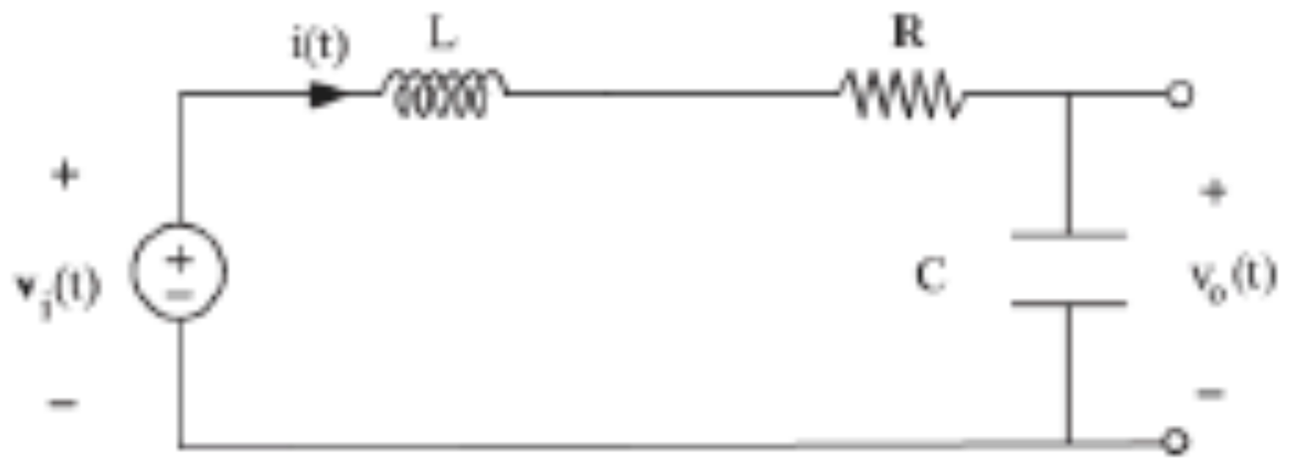
$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = F(t)$$



Examples of systems in continuous time

Example: a circuit system

- RLC circuit. The input is $v_i(t)$, an arbitrary signal.
- The output $v_o(t)$ will be a transformation of the input. Is there a equation relating them?



DIFFERENTIAL EQUATIONS

$$LC\frac{d^2v_0(t)}{dt^2} + RC\frac{dv_0(t)}{dt} + v_0(t) = v_i(t)$$

- It is a second order diferential equation. Note the similarity with the mechanical system..
- The signal and systems tools can be used in many applications.

Examples of systems in continuous time

Example: Integrator Systems

- We have an integrator system, which input is the signal $x(t) = tu(t)$. Therefore, for $t < 0$:

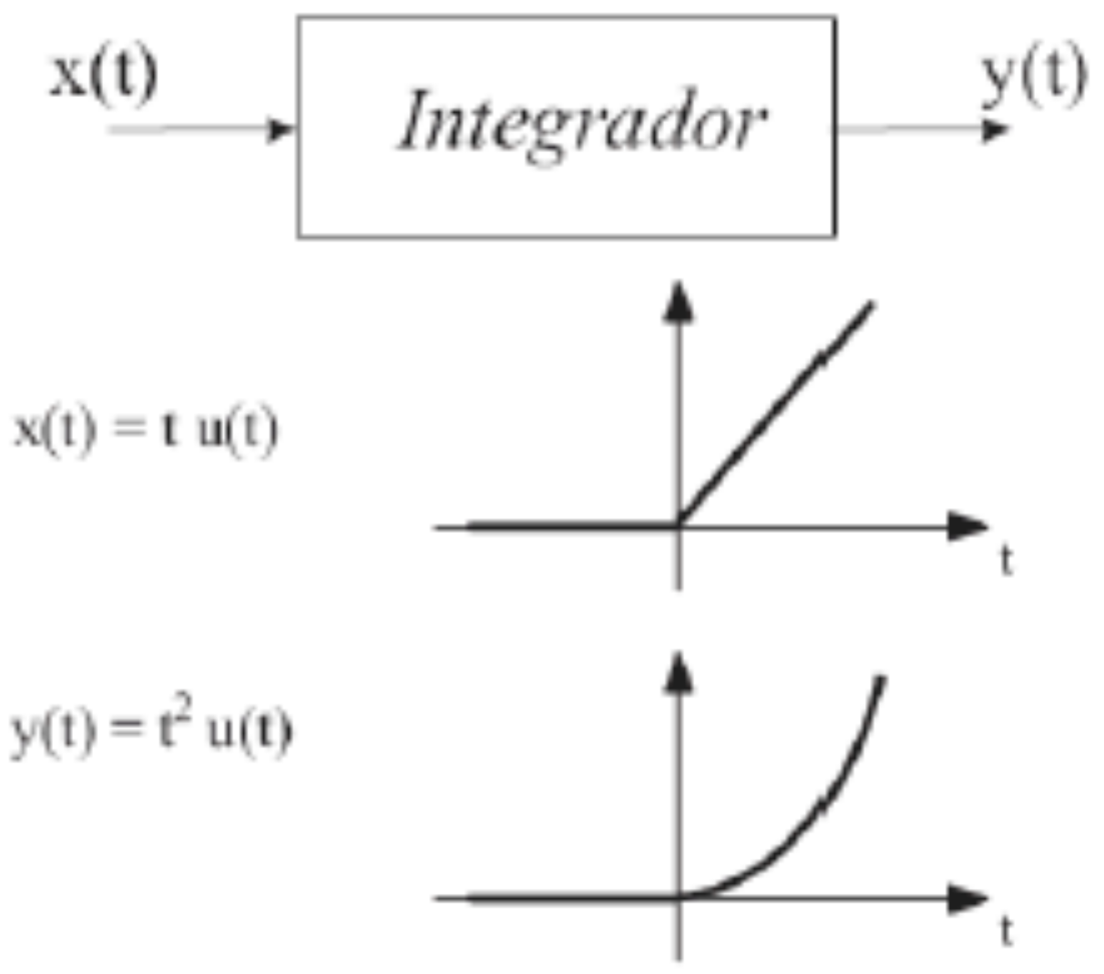
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} 0 d\tau = 0$$

whereas for $t \geq 0$:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{0}^{t} \tau d\tau = \left[\frac{\tau^2}{2}\right]_0^t = \frac{t^2}{2}$$

- The output can be expressed using the unit step signal:

$$y(t) = \frac{1}{2}t^2u(t)$$

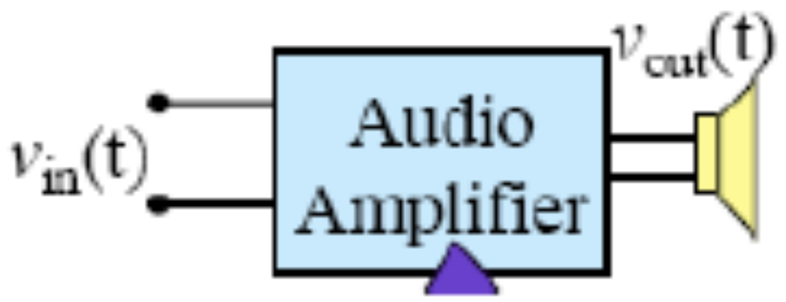


Examples of systems in continuous time

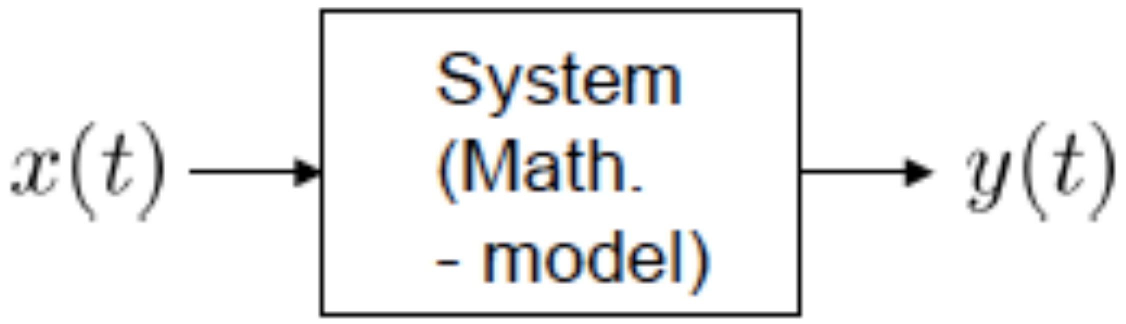
Continuous systems: models

- System: any operation/transformation over the signal

Physical Model

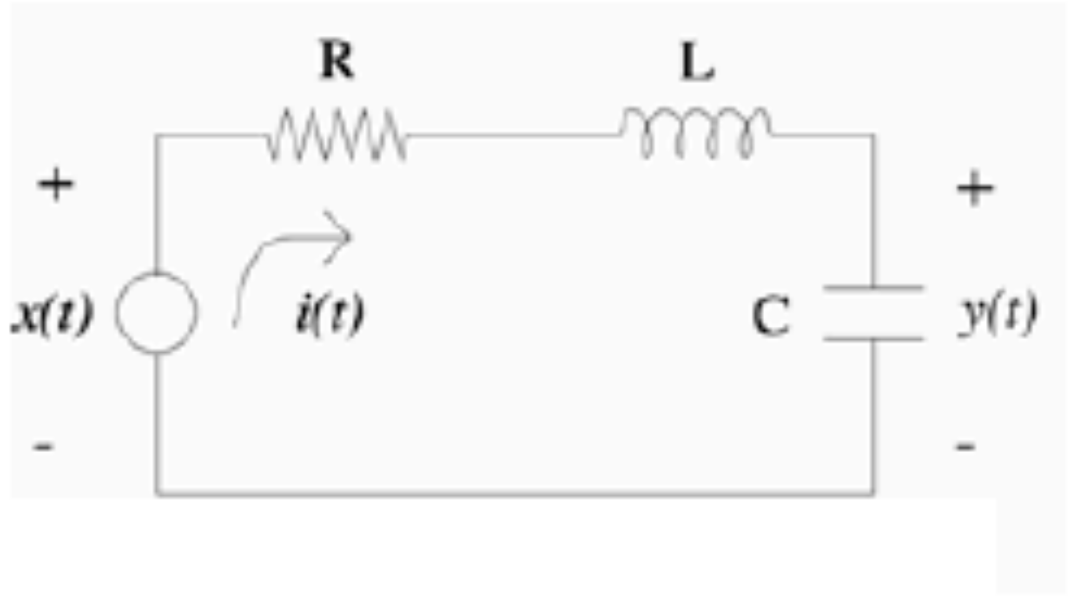


System Model



- Example 1 (CT): circuit RLC, described by differential equations

DIFFERENTIAL EQUATIONS



$$R\ i(t) + L\frac{d_i(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

*Note: different physical models can be represented by the same mathematical model (e.g., differential equations).

Examples of systems in continuous time:
DIFFERENTIAL EQUATIONS

Linear ordinary differential equations (L-ODE)

Linear differential equations with constant coefficients and null initial conditions:

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

(N) Initial conditions:

$$y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0^-} = ... = \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=0^-} = 0$$

Examples of systems in discrete time

$$y[n] = x[n]$$

$$y[n] = x[n-1]^2 + x[n-2]$$

$$y[n] = \log(x[n-5])$$

$$y[n] = x[n]^2$$

$$y[n] = x[n-1]$$

$$y[n] = x[n+2]$$

$$y[n] = y[n-1] + x[n] + 0.5x[n-3]$$

Examples of systems in discrete time:
DIFFERENCE EQUATIONS

Linear difference equations with constant coefficients

$$\sum_{i=0}^{L} b_i y[n-i] = \sum_{r=0}^{R} c_r x[n-r]$$

$n \geq 0$

$n = 0, 1, 2, 3...$

With L-INITIAL CONDITIONS (they are required)

$$y[-1], y[-2], ..., y[-L]$$

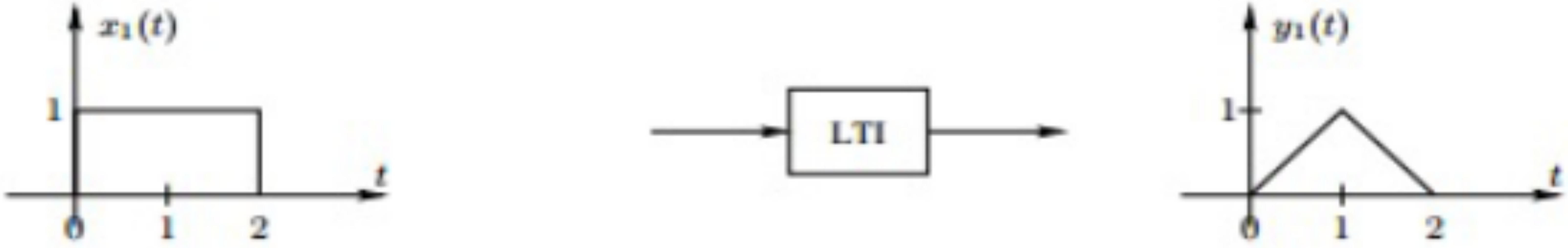
We need to know these L values !

2. Properties of a system

Properties of a systems

Properties of a system
(Causality, Linearity, temporal invariance, etc.)

- Why to know?
  - Important practical consequences for the analysis



Some properties of a system

- Some properties of a generic system:
- Memory
- Causality
- Stability
- Time invariance
- Linearity
- Invertibility (we do not see it, here)

→ **In this course, we will focus on Linear Time-Invariant (LTI) Systems**

Memory

Memory

Memory

- A system is said to be **memoryless** if its output, for each value of t, is dependent only on the input at that same time, that is, $y(t) = f(x(t))$.
  - **A system is "with memory"** ***in any other case*****.**
  - **Note that the memory can be** ***the past or the future*****….**

Examples

- Memoryless systems:
  - $y(t) = (2x(t) - x^2(t))^2$.
  - A resistor, in which $y(t) = Rx(t)$.
- Systems with memory:
  - A delay system, $y(t) = x(t-2)$.
  - A capacitor $v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i_C(\tau) d\tau$.

Without Memory

- If the output only depends on the input value at the same time instant.
- Mathematical definition:

$$x_1(t_0) = x_2(t_0) \rightarrow y_1(t_0) = y_2(t_0)$$

Examples:

$$y(t) = x(t)$$ → **memoryless**

$$y(t) = x(t-1)$$ → **with memory**

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$ → **with memory**

$$y(t) = 2(x(t) - x^2(t))^2$$ → **memoryless**

- y(t) must just depend on x(t)

Memory

(*)Determine if each of the following systems are memoryless or with memory:

1. $y(t) = t \cdot x(t)$. → memoryless
2. $y(t) = x(t+4)$. → with memory
3. $y(t) = \sum_{k=-3}^{0} x(t-k)$. → with memory
4. $y(t) = x(-t)$. → with memory
5. $y(t) = \cos(3t)x(t)$. → memoryless
6. $y(t) = x(t) + 0.5y(t-2)$. → with memory

- Note that the memory can be *the past or the future*….

Memory in a image or 2D signal

- **Bidimensional signal**

memoryless | **with memory** | **with memory** | **with memory**

Causality

Causal system

- The output y(t) in a time instant depends only to the values of the input x(t) until this time instant (no from “future” values of x(t)).

- **ALL** physical systems based on real time are causals, sice the time goes only forward….

- For Spatial signals/images, it is not the case. We can go up and down, left and right…

- It is not the case the analysis for recorded signals (we can go in the “future”…).

Causality

- and also outputs values in the past…

Causality

- A system is causal is the output at any time depends only on values of the input at the present time (same time) and in the past. Such a system is often referred to as being *physically feasibles* or *nonanticipative*.

- just to give an idea:

$$y(t) = f(x(t), x(t-1), x(t-2), ..., y(t-1), y(t-2), ...)$$

- we are talking of any mapping, any generic transformation, $f(...)$ or $f\{...\}$

- remember that “inputs” means also delayed version of the outputs

Non-causal and anti-causal

- **Non-Causal: the output depends on the past and future *jointly* (and can be also dependent on the present).**
- **Anti-Causal: the output depends on the future (and can be also dependent on the present).**
- *in different books (or slides, manuscripts etc.), there are different definitions…*

Causal system

- A system $x(t) \to y(t)$ is <u>causal</u>:

When: $x_1(t) \to y_1(t) \quad x_2(t) \to y_2(t)$

if: $x_1(t) = x_2(t) \quad \forall t \leq t_0$

then $y_1(t) = y_2(t) \quad \forall t \leq t_0$

– If two input signals are the same until t0, the output signals are the same unitl t0.

- <u>Anti-causal</u>: the output y(t) only depends on the future values of the input x(t).
- <u>Non-causal</u>: the output y(t) only depends on the past values and on the future vaules of the input x(t).

*Note: for linear and invariant systems (LTI) there is another (easier) method to see that.

Causal, Anti-Causal, Non-causal: examples

- **Causal system:**

$$y(t) = f(x(t), x(t-1), x(t-2))$$

- **Anti-causal system (depends on the future):**

$$y(t) = f(x(t+2)) \qquad y(t) = f(x(t), x(t+1)) \qquad y(t) = f(x(t), y(t+5))$$

- **Non-causal system (depends on the future and the past):**

$$y(t) = f(x(t-1), x(t+3)) \qquad y(t) = f(x(t), x(t-5), x(t+3))$$

Causality: examples

- The system $y(t) = x(t) - x(t-1)$ is causal.
- The system $y(t) = 2x(t+3)$ is anticausal.
- The system $y(t) = x(t-1) - x(t+3)$ is noncausal.

Causality: examples

1. $y(t) = x(-t)$. → Non-causal (…anti-causal…)
2. $y(t) = x(t) \cdot \cos(t+1)$. → Causal
3. $y(t) = Ax(t)$. → Causal
4. $y(t) = \int_{-\infty}^{t+2} x(\tau) d\tau$. → Non-causal

Are they causal?

- Examples:

$$y(t) = x(t+1)$$ → **Anti-causal**

$$y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)d\tau$$ → **Causal**

$$y(t) = x(t) - x(t+0.5)$$ → **Anti-causal**

$$y(t) = t^2(x(t) - x(t+0.5))$$ → **Anti-causal**

Causality: examples

$$y(t) = x(at)$$

Non-causal

$$\forall a \quad \text{with } a \neq 1$$

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https://www.youtube.com/watch?v=A5SlTkKfUz0&list=PLBlnK6fEyqRhG6s3jYIU48CqsT5cyiDTO&index=88

Causality: examples

$$y(t) = x(t - b)$$

$b \geq 0$

Causal

$b < 0$

Anti-causal

Causal systems and memoryless systems

- Every memoryless system is causal

Stability

Stability

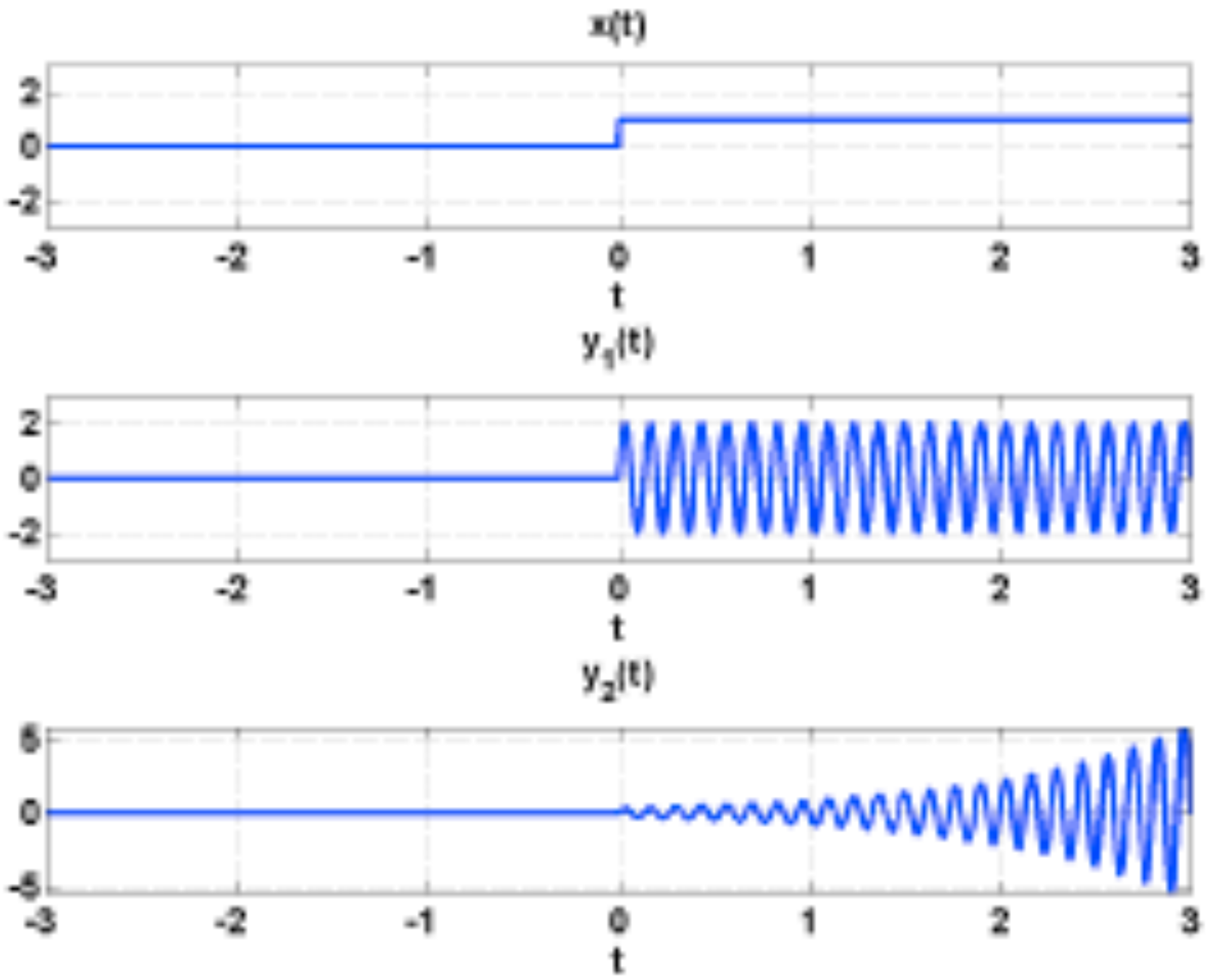
Stability

- A system is sadi to be **stable** when bounded inputs leads to bounded outputs, for any time, t. Mathematically, this property is expressed as (BIBO):

$$|x(t)| < K_x < \infty \Rightarrow |y(t)| < K_y < \infty \qquad \forall t \text{ !!!!}$$

- A system is unstable whenever we are able to find a *specific* bounded input that leads to an unbounded output. Finding one such example enable sus to conclude that the given system is unstable.

Stability



The corresponding system that produces these outputs is:

→ **Stable**

→ **Unstable**

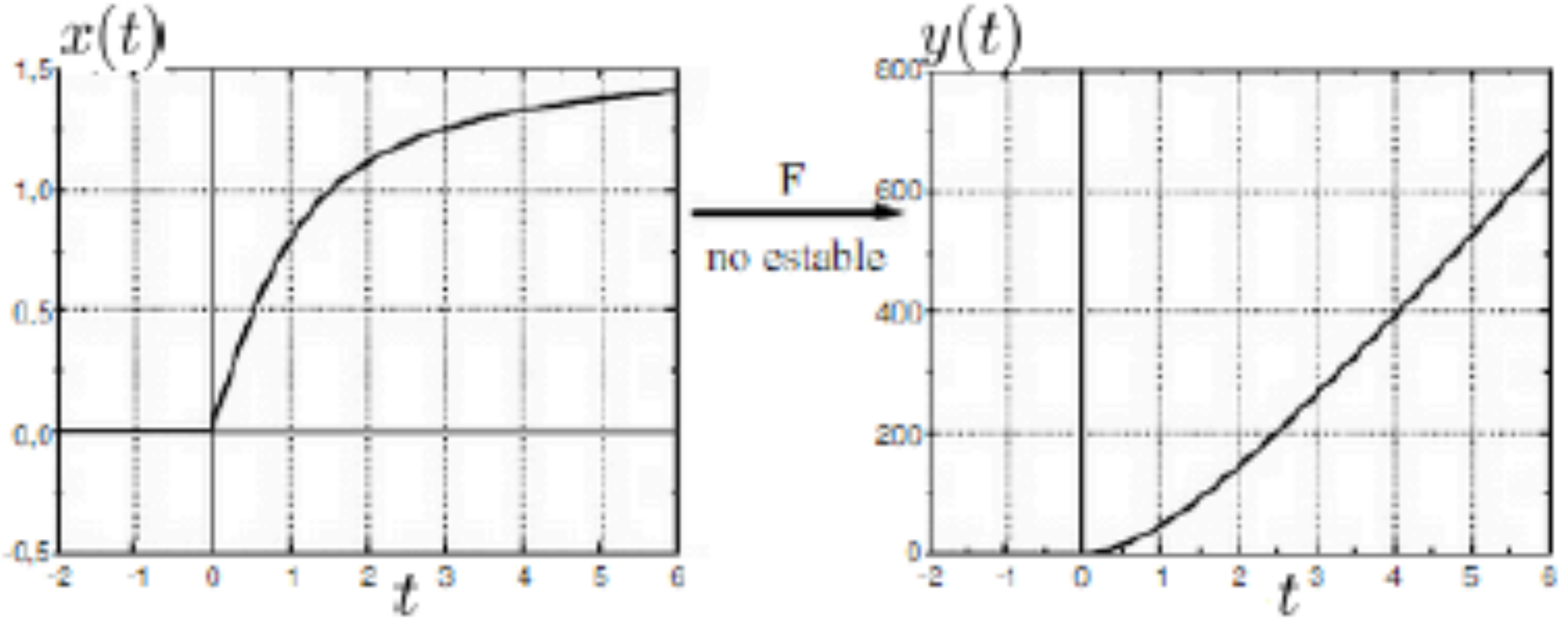
Stability

➢ Stable system: bounded inputs generate bounded outputs.

$$\forall t, \quad |x(t)| < B_x \quad (B_x \in \Re^+) \Longrightarrow \forall t, \quad |y(t)| < B_y \quad (B_y \in \Re^+)$$

➢ Example:

Consider the system: $x(t) \rightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$



The system is not stable, since there exists several possible bounded inputs which produce ouputs that diverge (think in any input signal where the result of the integral is an increasing area)

Exercise: study this system $\quad y(t) = \frac{1}{2T}\int_{-T}^{T} x(t-\tau)d\tau$

Stability: examples

1. $y(t) = [x(t)]^2$. → **Stable**
2. Derivative system: $y(t) = \frac{dx(t)}{dt}$. → **Stable**
3. Integrator system: $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$. → **Unstable**
4. $y(t) = t \cdot x(t)$. → **Unstable**
5. $y(t) = x(-t)$ → **Stable**
6. $y(t) = x(t-2) + 3x(t+2)$. → **Stable**
7. $y(t) = Impar\,(x(t))$. → **Stable**
8. $y(t) = e^{x(t)}$. → **Stable**

There are inputs x(t)

which generate unbounded y(t)

Stability: examples

$$y(t) = \frac{1}{x(t)+1} \longrightarrow \textbf{Unstable}$$

Temporal invariance

Temporal invariance (TI)

- Definition:

Consider: $x(t) \to y(t)$

then: $x(t - t_0) \to y(t - t_0)$

Examples: are they temporal invariant?

$y(t) = sin(x(t))$ → **time invariant**

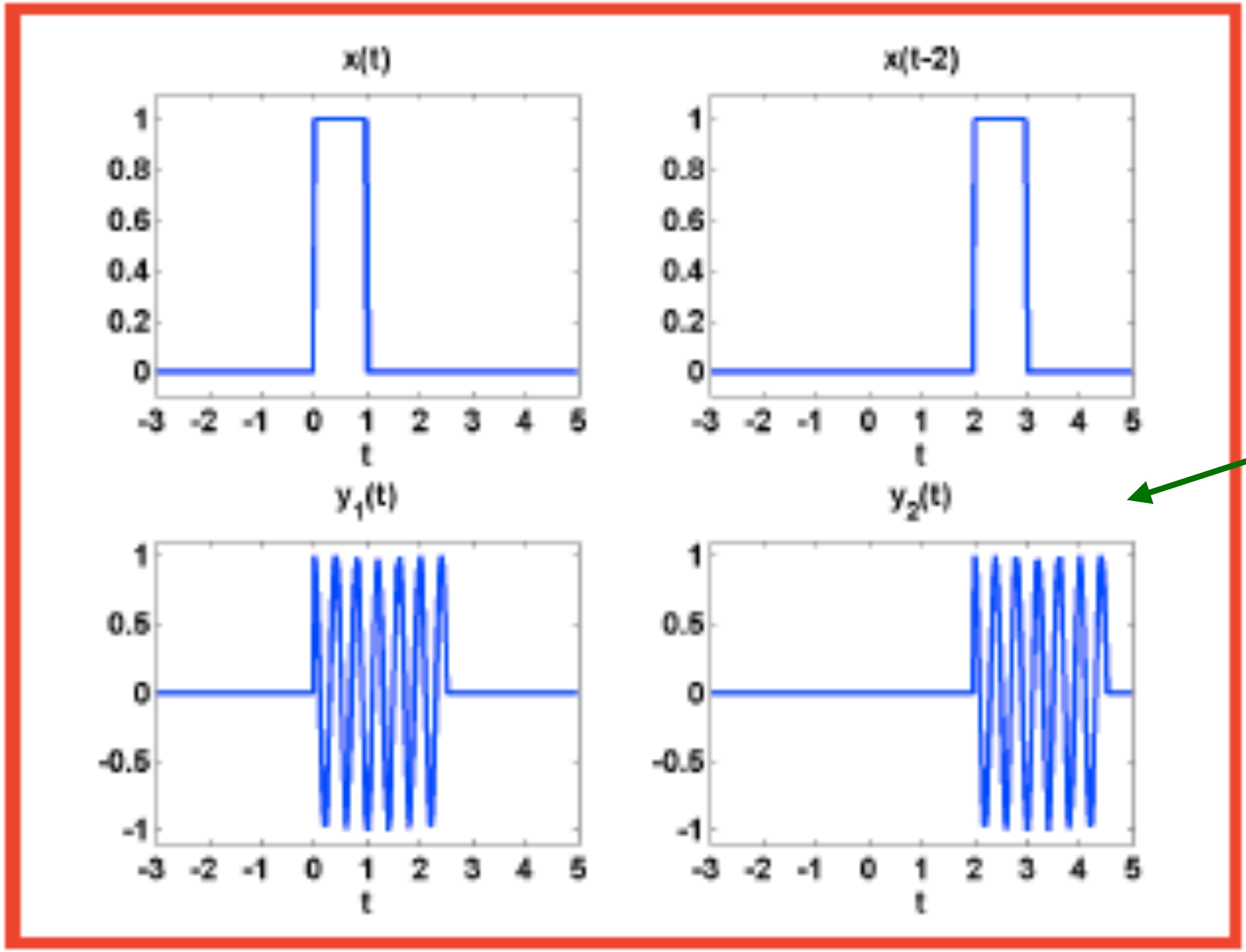
$y(t) = t \cdot x(t)$ → **time variant**

Time invariance

Time Invariance (I)

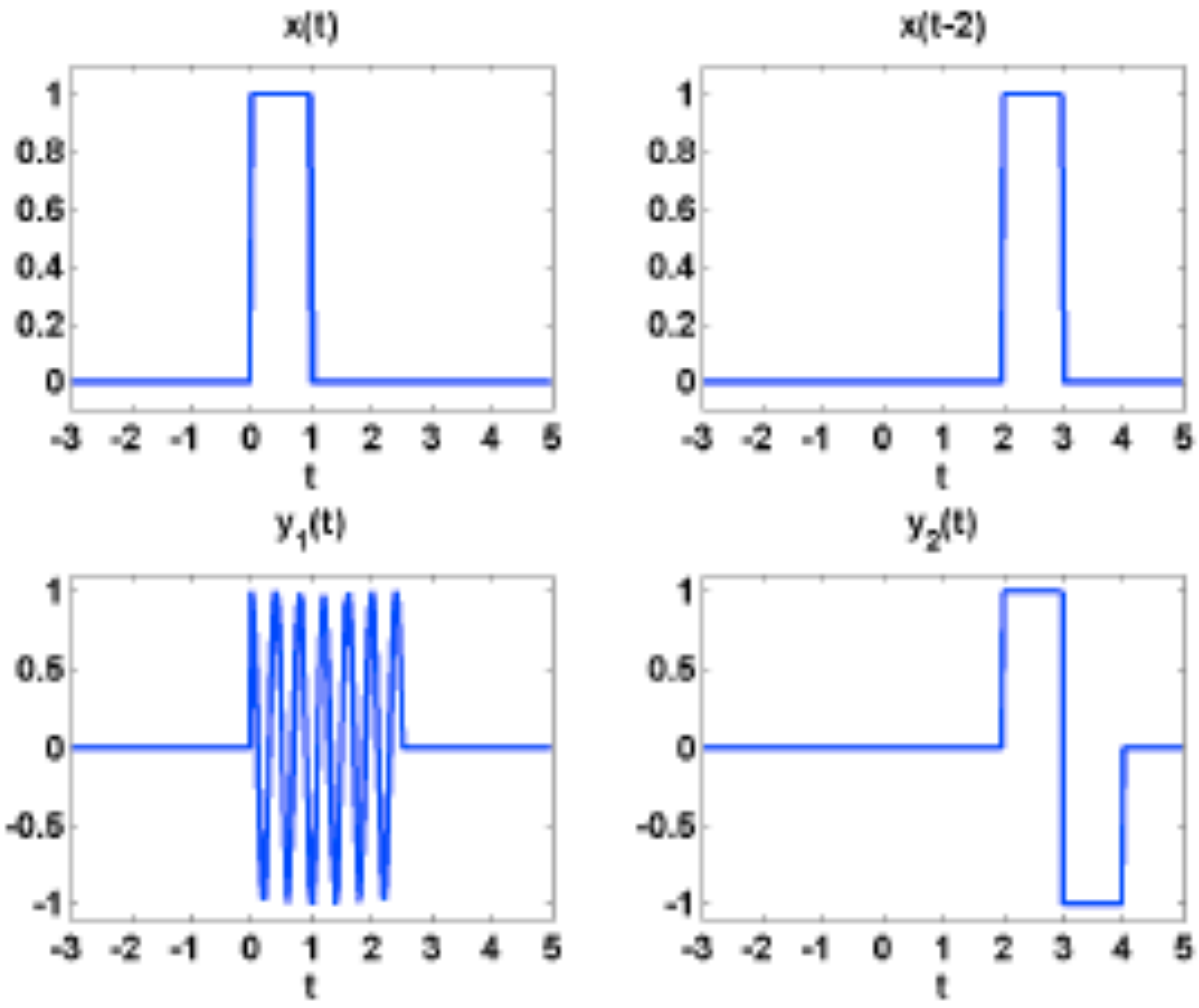
- A system is time invariant if the behavior and characteristicas of the system are fixed over time.
- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.
- The system is said to be *time variant* otherwise.

Time invariance



Output of a time invariant

Time invariance



Output of a

Time variant system

Time invariance: method

1. Let be $x_1(t)$ an arbitrary input, and let be $y_1(t)$ the output for this particular input.
2. The output is shifted by a given t_0, $y_1(t - t_0)$.
3. Then, consider a second input, $x_2(t)$, which is obtained by shifting $x_1(t)$ in time, $x_2(t) = x_1(t - t_0)$. The corresponding output is $y_2(t)$.
4. We have to compare both outputs $y_2(t) \stackrel{?}{=} y_1(t - t_0)$, if the equality holds, then the system is time invariant.

Time invariance: examples

1. $y(t) = \cos[x(t)]$. → time invariant
2. $y(t) = t + x(t)$. → time variant
3. $y(t) = tx(t)$. → time variant
4. $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$. → time variant
5. $y(t) = \frac{dx(t)}{dt}$. → time invariant

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$ → time invariant

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https://www.youtube.com/watch?v=P4_iWrawCZs
https://www.youtube.com/watch?v=PZfZGbbBuxk

Periodic input in a TI system

- If the input is periodic, the output will be periodic,

$$x(t+T) = x(t)$$
$$x(t) \rightarrow y(t)$$

Due to the system TI then:

$$x(t+T) \rightarrow y(t+T)$$

Since

$$x(t+T) = x(t)$$

Then the output are the same,

$$i.e., y(t) = y(t+T)$$

the output is also periodic

Linearity

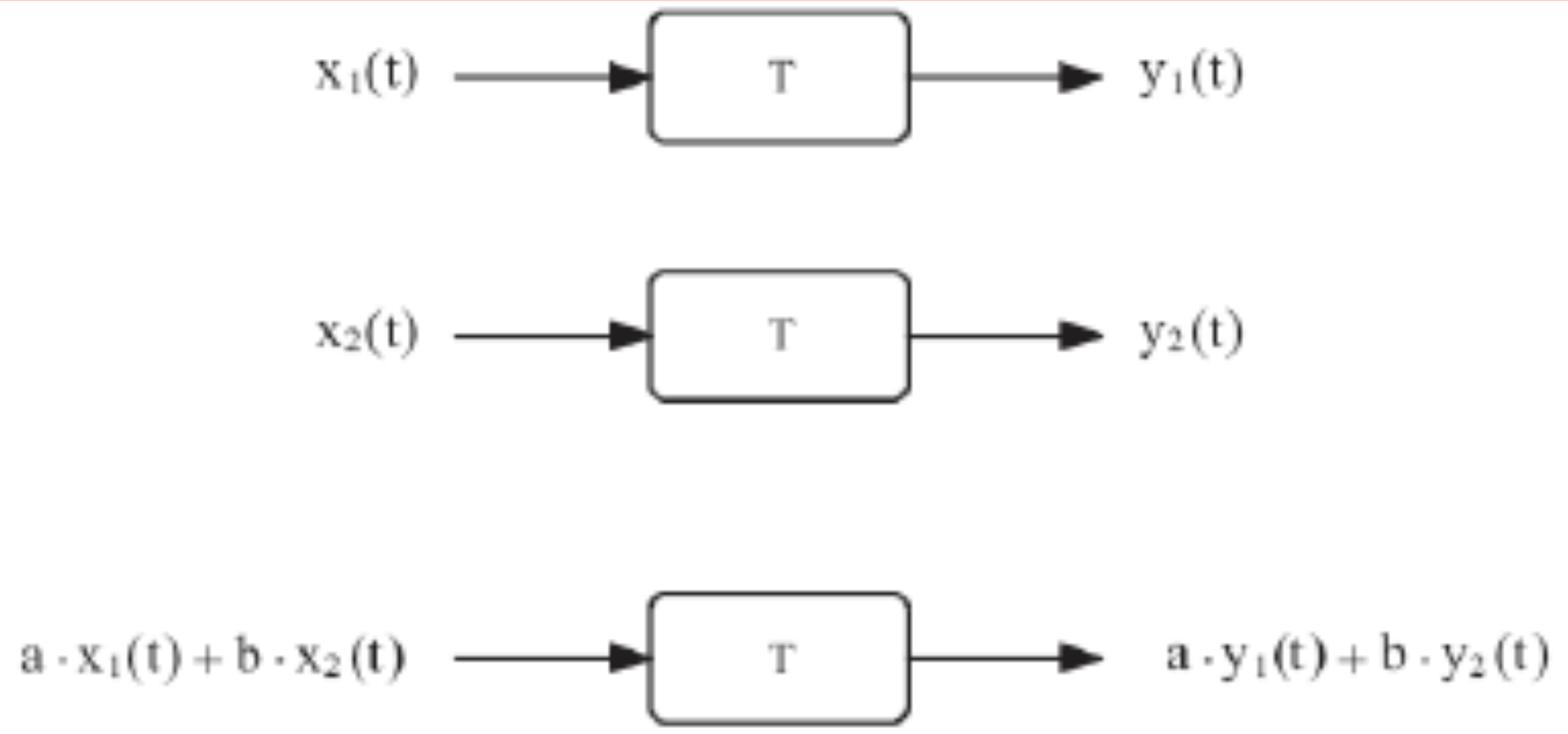
Linearity

Linearity

- A systems is said to be **linear** when it possesse the property of superposition. The property of superposition has two properties: *additivity* and *scaling* or *homogeneity*:
  1. Additivity: the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
  2. Scaling: the response to $ax_1(t)$ is $ay_1(t)$ (whit $a \in \mathbb{C}$).
- The two properties can be combined. A system is linear when the response to $ax_1(t) + bx_2(t)$ is $ay_1(t) + by_2(t)$.

Linearity

- Note, as a consequence, we can show that fo linear system an input which is zero for all time results in an output which is zero for all time.



Linearity

- Definition:

$$x_1(t) \rightarrow y_1(t) \qquad x_2(t) \rightarrow y_2(t)$$
$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

  - For linear systems: zero input → zero output

Examples:

$y(t) = Ax(t)$ → linear

$y(t) = tx(t)$ → linear

$y(t) = x^2(t)$ → non-linear

$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ → linear

$y(t) = sin(x(t))$ → non-linear

Linearity: examples

1. $y(t) = t \cdot x(t).$ → linear
2. $y(t) = x^2(t).$ → non-linear
3. $y(t) = 2x(t) + 3$ → non-linear

(4) $y(t) = 2t + x(t)$ → non-linear

(5) $y(t) = x(\sin(t))$ → linear

(6) $y(t) = x(t-1) + x(t+1)$ → linear

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https://www.youtube.com/watch?v=kAy5LHGPIRY
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Important properties

Comments on system properties

Study these properties at home

- Every memoryless system is causal
- The outuput of a linear system for a zero input is a zero output.
- If a system is time invariant, periodic inputs lead to periodic outputs.

Basic Properties of a DT system

- Memory:
  - The ouput at any time t depends only to the input at time t.

- Causalidad:
  - Outputs depend only from the present and from the past (no future)
  - For LTI systems, y[n] should NOT depend on x[k] with k>n
- Stability:
  - An input is bouded (finite amplitud) produces a bounded output (finite amplitud)

Basic Properties of a DT system

- **Temporal invariance**:

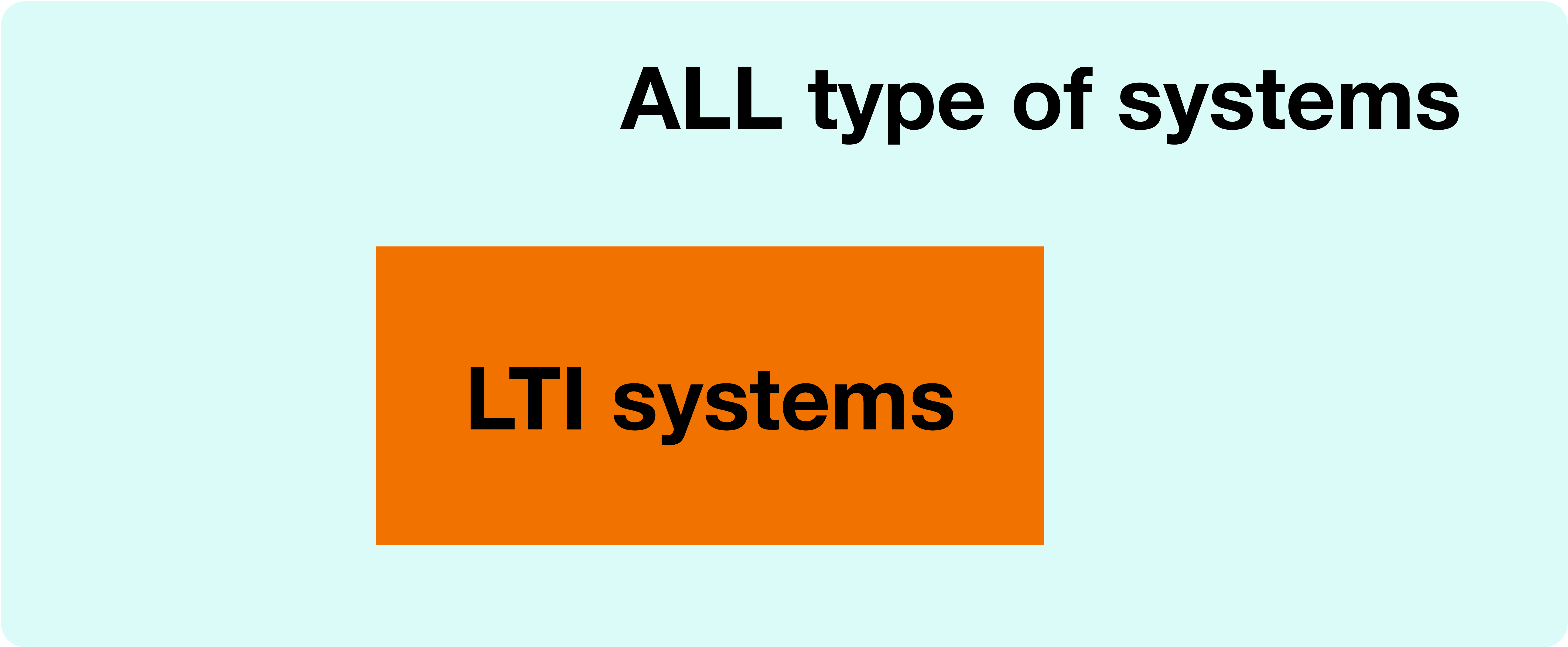
$$\text{If: } x[n] \rightarrow y[n]$$
$$\text{Then: } x[n-n_0] \rightarrow y[n-n_0]$$

- **Linearity**:.

$$\text{If: } x_1[n] \rightarrow y_1[n] \qquad x_2[n] \rightarrow y_2[n]$$
$$\text{Then: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

Summary: what will see…

- We will focus on: LINEAR TIME INVARIANT (LTI) SYSTEMS
- LTI systems in time
- LTI systems in transformed domain (frequency domain etc.)



Questions?