Topic 1.5 - part 1 On differential equations and difference equations

Discrete Time Systems (DTS) and Señales y Sistemas

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In this slides, WE WILL SEE:

- We will see an observation regarding the *complete solution* of linear differential equations and linear difference equations…

- Linear differential equations

How express the output of an LTI system in CT

- Mathematically, the output the LTI systems can be expressed in two ways:

1. Linear differential equations with constant coefficients and null initial conditions.
2. Convolution integral.

- These two ways are equivalent.

Linear ordinary differential equations (L-ODE)

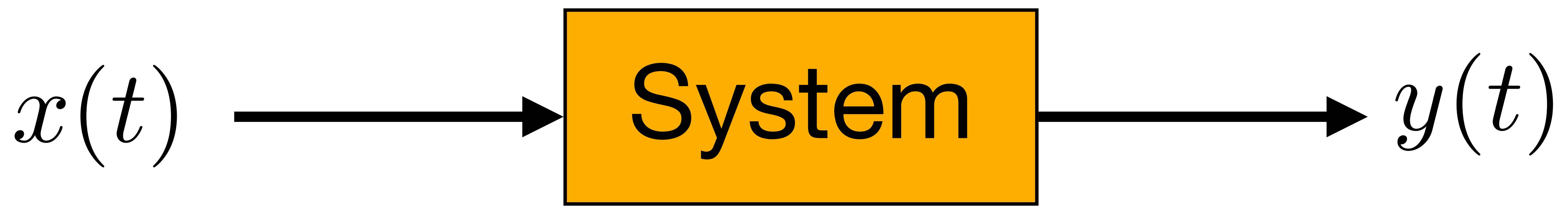
1. **Linear differential equations with constant coefficients and null initial conditions:**

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

(N) Initial conditions:

$$y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0^-} = \ldots = \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=0^-} = 0$$

- **x(t): input —–> y(t): output**



We are interested in this ***"forced" L-ODE with null initial conditions***

Linear ordinary differential equations (L-ODE)

1. **Linear differential equations with constant coefficients and null initial conditions, examples:**

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} = x(t) \longrightarrow \begin{array}{l} N = 2; \quad a_2 = 1; \quad a_1 = -2 \quad \text{the rest of } a_n \text{ are zero} \\ M = 0; \quad b_0 = 1; \quad \text{the rest of } b_m \text{ are zero} \end{array}$$

$$\frac{dy(t)}{dt} - 4.5y(t) = x(t) + j\frac{dx(t)}{dt} + 6\frac{d^2x(t)}{dt^2} \longrightarrow \begin{array}{l} N = 1; \quad a_1 = 1; \quad a_0 = -4.5 \quad \text{the rest of } a_n \text{ are zero} \\ M = 2; \quad b_0 = 1; b_1 = j; b_2 = 6 \quad \text{the rest of } b_m \text{ are zero} \end{array}$$

Brief overview of the solutions of L-ODE

- **First of all, define the *homogeneous L-ODE*:**

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = 0$$

- **with general *non-null* initial conditions:**

$$y(0) \neq 0 \quad \left.\frac{dy(t)}{dt}\right|_{t=0^-} \neq 0 \qquad \left.\frac{d^{M-1} y(t)}{dt^{N-1}}\right|_{t=0^-} \neq 0$$

Brief overview of the solutions of L-ODE

- The solution of an *homogeneous L-ODE* ***with NULL initial conditions is*****:**

$$y(t) = 0, \quad \forall t$$

- **But** with non-null initial conditions, the solution y(t) of an homogeneous L-ODE is non-zero (generally) and it is called ***transient solution,*** denoted here as:

$$y_o(t)$$

There are courses just devoted to study this solution…

Four cases…

1. homogeneous L-ODE with NULL initial conditions
2. homogeneous L-ODE with NON-NULL initial conditions
3. forced L-ODE (with input) with NULL initial conditions
4. forced L-ODE (with input) with NON-NULL initial conditions

The more general case is the last one

But we are interested in the third one

Forced L-ODE with NON-NULL initial conditions

- **General solution:**

$$y(t) = y_o(t) + y_f(t)$$

Solution of the homogenous equation with NON-NULL initial conditions ($y_o(t)$)

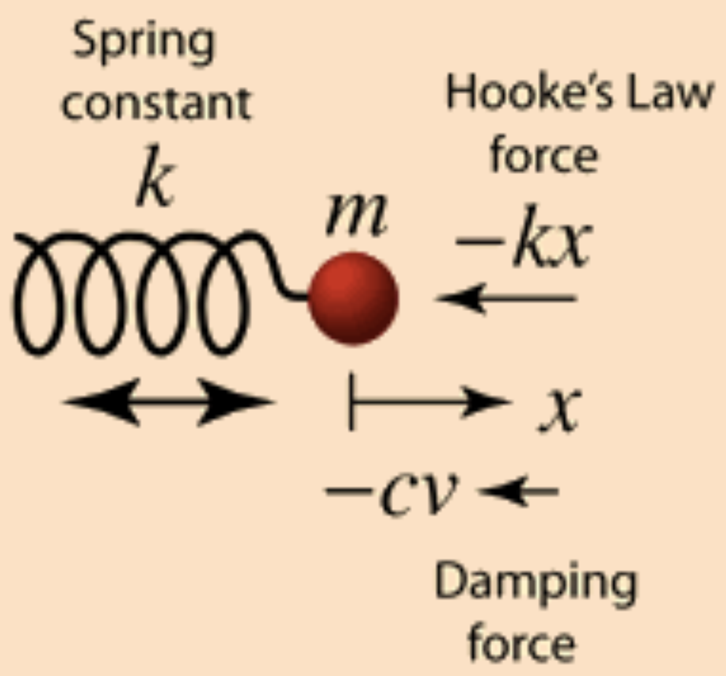
Solution of the forced equation with NULL initial conditions ($y_f(t)$)

Forced L-ODE with NON-NULL initial conditions

Driven Oscillator

If a damped oscillator is driven by an external force, the solution to the motion equation has two parts, a transient part and a steady-state part, which must be used together to fit the physical boundary conditions of the problem.

The motion equation is of the form



Newton's 2nd Law terms — Sinusoidal driving force

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos(\omega t + \varphi_d)$$

and has a general solution

$$x(t) = x_{transient} + x_{steady\ \ state}$$

In the underdamped case this solution takes the form

Transient solution — Steady-state solution

$$x(t) = A_h e^{-\gamma t}\sin(\omega' t + \varphi_h) + A\cos(\omega t - \varphi)$$

Determined by initial position and velocity — Determined by driving force

The initial behavior of a damped, driven oscillator can be quite complex. The parameters in the above solution depend upon the initial conditions and the nature of the driving force, but deriving the detailed form is an involved algebra problem. The form of the parameters is shown below.

different notation

$$y(t) = y_o(t) + y_f(t)$$

Forced L-ODE with NULL initial conditions

- **In this case:**

$$y(t) = 0 + y_f(t)$$

Solution of the homogenous equation with NULL initial conditions

Solution of the forced equation with NULL initial conditions

- ***<u>In this course, we focus on this scenario.</u>***

Forced L-ODE with NULL initial conditions

- **In this case:**

$$y(t) = y_f(t)$$

→ **Solution of the forced equation with NULL initial conditions**

- ***In this course, we focus on this solution.***
- all the books/notes use the notation y(t) but is actually y_f(t)

1. homogeneous L-ODE with NULL initial conditions

(no input, initial cond=zero)

$$y_o(t) = 0$$

$$y_f(t) = 0$$

$$y(t) = 0$$

2. homogeneous L-ODE with NON-NULL initial conditions

(no input, initial cond. non-zeros)

$$y_f(t) = 0$$

$$y(t) = y_o(t)$$

3. forced L-ODE with NULL initial conditions

(with input, initial cond=zero)

$$y_o(t) = 0$$

$$y(t) = y_f(t)$$

CASE OF THIS COURSE !!!

4. forced L-ODE with NON-NULL initial conditions

(with input, initial cond. non-zeros)

$$y(t) = y_o(t) + y_f(t)$$

Convolution integral

- **Solution of the forced equation with NULL initial conditions can be as:**

$$\boxed{y(t) = y_f(t)} = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- *but what is the function/signal h(t) ?*

Convolution integral

- **Convolution:**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- *h(t) is called* **IMPULSE RESPONSE (respuesta al impulso)**
- **impulse = delta function**

Convolution integral

- **Other notation:**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- *h(t) is called* **IMPULSE RESPONSE (respuesta al impulso)**
- **impulse = delta function**

Impulse Response

- Reason of the name: **h(t) is the response of the system when the input is a delta (an impulse).**
- h(t) is the output, i.e., y(t)=h(t), of the system when x(t)=delta(t):

$$x(t) = \delta(t) \Longrightarrow y(t) = h(t)$$

Impulse Response

- h(t) is the output, i.e., y(t)=h(t), of the system when x(t)=delta(t):

$$x(t) = \delta(t) \Longrightarrow y(t) = h(t)$$

- indeed, for the delta's properties:

$$y(t) = \int_{-\infty}^{+\infty} \delta(\tau) h(t-\tau) d\tau = h(t)$$

Impulse Response

- **Then, h(t) summaries an LTI system (and its properties).**

- Then, h(t) “summaries”/is equivalent a forced L-ODE with null initial conditions.

Summary: what we saw in these slides

- An LTI system in continuous time can be expressed (mathematically):
- Forced L-ODE *with constant coefficients* and *with NULL initial conditions*
- Convolution integral where h(t) is the impulse response

- **Difference equations (also part of Topic 6)**

Linear difference equations with constant coefficients

$$\sum_{i=0}^{L} b_i y[n-i] = \sum_{r=0}^{R} c_r x[n-r]$$

$n \geq 0$

$n = 0, 1, 2, 3...$

With L-INITIAL CONDITIONS (they are required)

$$y[-1], y[-2], ..., y[-L]$$

We need to know these L values !

Example

$$y[n] - 0.2y[n-1] - 0.2y[n-2] = x[n] + x[n-1]$$

$$L = 2 \qquad R = 1$$

$$b_0 = 1, b_1 = -0.2, b_2 = -0.2 \qquad c_0 = 1, x_1 = 1$$

Additionally consider the L- INITIAL CONDITIONS:

$$y[-1] = 8, y[-2] = -5$$

and the following input signal:

$$x[-1] = 6, x[0] = -2, x[1] = -3,$$

$$\text{the rest of values are zero for } n \neq -1, 0, 1$$

Example

Obtain the first 3 values of the output signal y[n], i.e., y[0], y[1] and y[2]:

$$y[n] - 0.2y[n-1] - 0.2y[n-2] = x[n] + x[n-1]$$

$$y[-1] = 8, y[-2] = -5$$

$$x[-1] = 6, x[0] = -2, x[1] = -3,$$

the rest of values are zero for $n \neq -1, 0, 1$

Example

$$y[n] = 0.2y[n-1] + 0.2y[n-2] + x[n] + x[n-1]$$

$$n = 0$$

$$y[0] = 0.2y[0-1] + 0.2y[0-2] + x[0] + x[0-1]$$

$$y[0] = 0.2 \cdot 8 + 0.2 \cdot (-5) - 2 + 6$$

$$y[0] = 4.6$$

$$n = 1$$

$$y[1] = 0.2y[1-1] + 0.2y[1-2] + x[1] + x[1-1]$$

$$y[1] = 0.2y[0] + 0.2y[-1] + x[1] + x[0]$$

$$y[1] = 0.2 \cdot 4.6 + 0.2 \cdot 8 - 3 - 2$$

$$y[1] = -2.48$$

Example

Solution:

$$y[0] = 4.6 \quad y[1] = -2.48 \quad y[2] = -2.576$$

SOLUTION: y[n]

We have solved the difference equation for 3 times steps by applying the recursion 3 times….

For linear difference equations with constant coefficients general analytical solutions can be obtained (as for the linear differential equations with constant coefficient).

GENERAL SOLUTION: y[n]

$$y[n] = y_o[n] + y_f[n]$$

Solution of the homogeneous system:
With x[n]=0 and generic initial conditions (free-response; "transitory", if stable...)

"Forced" solution:
Generic x[n] but null initial conditions.

Regarding y_o[n]....

$$y[n] = y_o[n] + y_f[n]$$

Solution of the homogeneous system: with x[n]=0 and generic initial conditions (free-response; “transitory”, if stable...)

- ✓ We have to study the ``characteristic’’ polynomial: related to the denominator of H(z) and the poles are roots of the ``characteristic’’ polynomial...
- ✓ There are courses only for this goal: Dynamical systems etc... (as with the differential equations)

Regarding y_o[n]....

$$y[n] = y_o[n] + y_f[n]$$

✓ **IF THE INITIAL CONDITIONS ARE NULL, THEN:**

$$y_o[n] = 0$$

Regarding y_o[n]....

✓ IF THE INITIAL CONDITIONS ARE NULL, THEN:

$$y_o[n] = 0$$

And so:

$$y[n] = y_f[n]$$

Regarding y_f[n]....

$$y_f[n] = x[n] * h[n]$$

In all this course, we consider “null initial conditions”, then we have:

$$\boxed{y[n] = y_f[n]} = x[n] * h[n]$$

How can we find the corresponding h[n]?

$$y[-1] = ... = y[-L] = 0$$

$$x[n] = \delta[n]$$ In discrete time, it is possible (it is much easier)

Recall that h[n] is the "impulse response":
It can be done "by hand" following the recursion...

There is also a general procedure to obtain the anylitical form of h[n]
(we do not considered it now...)

Questions?