Topic 2.1 - part 1
Why a “transformed” domain?
for signal in continuous time (CT)

Discrete Time Systems (DTS) and Señales y Sistemas

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Why a transformed domain?

- **Why do we “pass” to another domain different from the time domain?**
- **the answer in the next slides (at least the first main reason)… for signal in continuous time (CT)**

Why a transformed domain?

- **HERE FIRST MAIN EXPLANATION/MOTIVATION: system point of view, eigenfunctions and eigenvalues…**

- **The second main reason (in other slides): signal point of view, spectral analysis, signal decomposition…**

Recall eigenvalues/eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

- eigenvector

- eigenvalue

Why a transformed domain?

- Many signals in nature and communications can be expressed as **linear combinations of real/complex exponentials.**
- **For periodic signals**, this will be completely clear after looking the **Fourier series**.

$$e^{st} --- \exp(st)$$

Brief recall and explanation….

- Note the position of "*s*" and "*t*" can be switched:

$$e^{st}$$

$$t \in \mathbb{R} - -s \in ?$$

Brief recall and explanation….

- **"s" is a complex number:**

$$s \in \mathbb{C}$$

$$s = \sigma + j\omega$$

Brief recall and explanation….

$$s = \sigma + j\omega$$

- Frequency

- This “physical” interpretation is the reason for the second main reason that we will see in other slides…

Brief recall and explanation….

$$e^{st} = e^{(\sigma + j\omega)t}$$

$$= e^{\sigma t + j\omega t}$$

$$= e^{\sigma t} e^{j\omega t}$$

- envelope (real exponential)

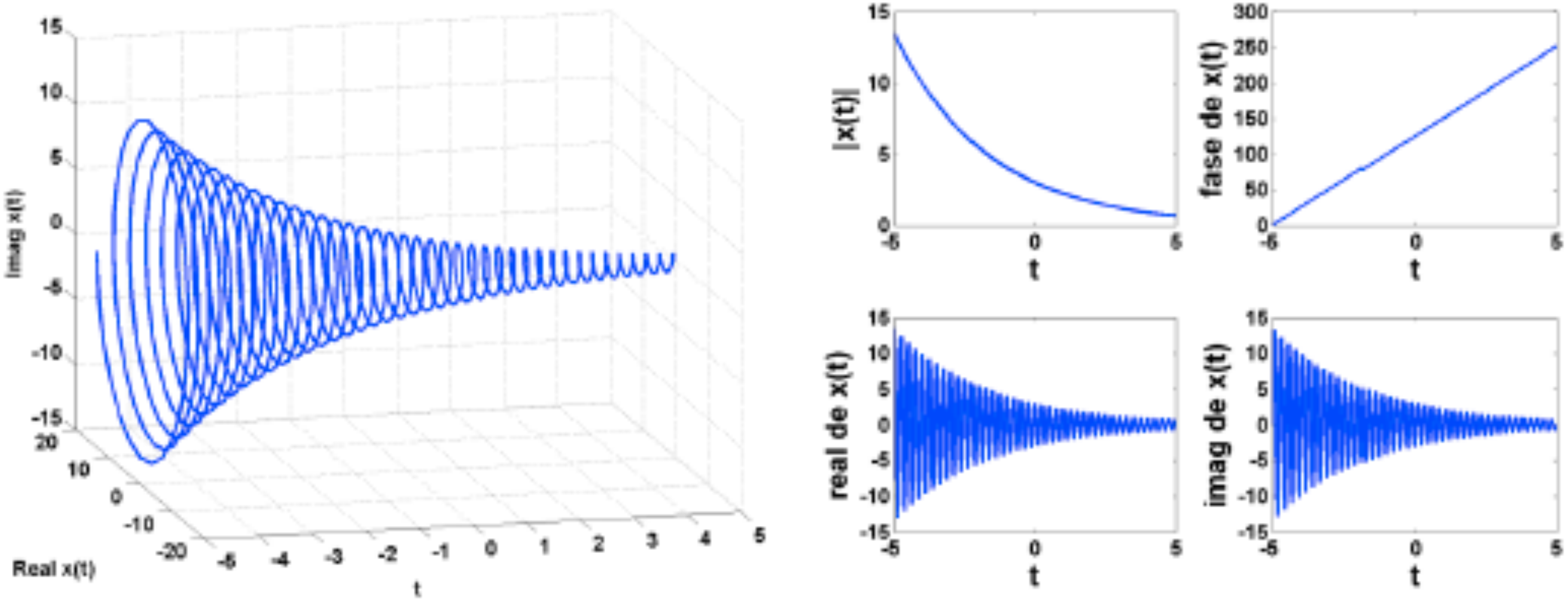
- Periodic part

$$T = \frac{2\pi}{\omega}$$

Brief recall and explanation….

- From Laura Martínez’s slides:

- Depending upon the values of these parameters the complex exponential can exhibit different characteristics.

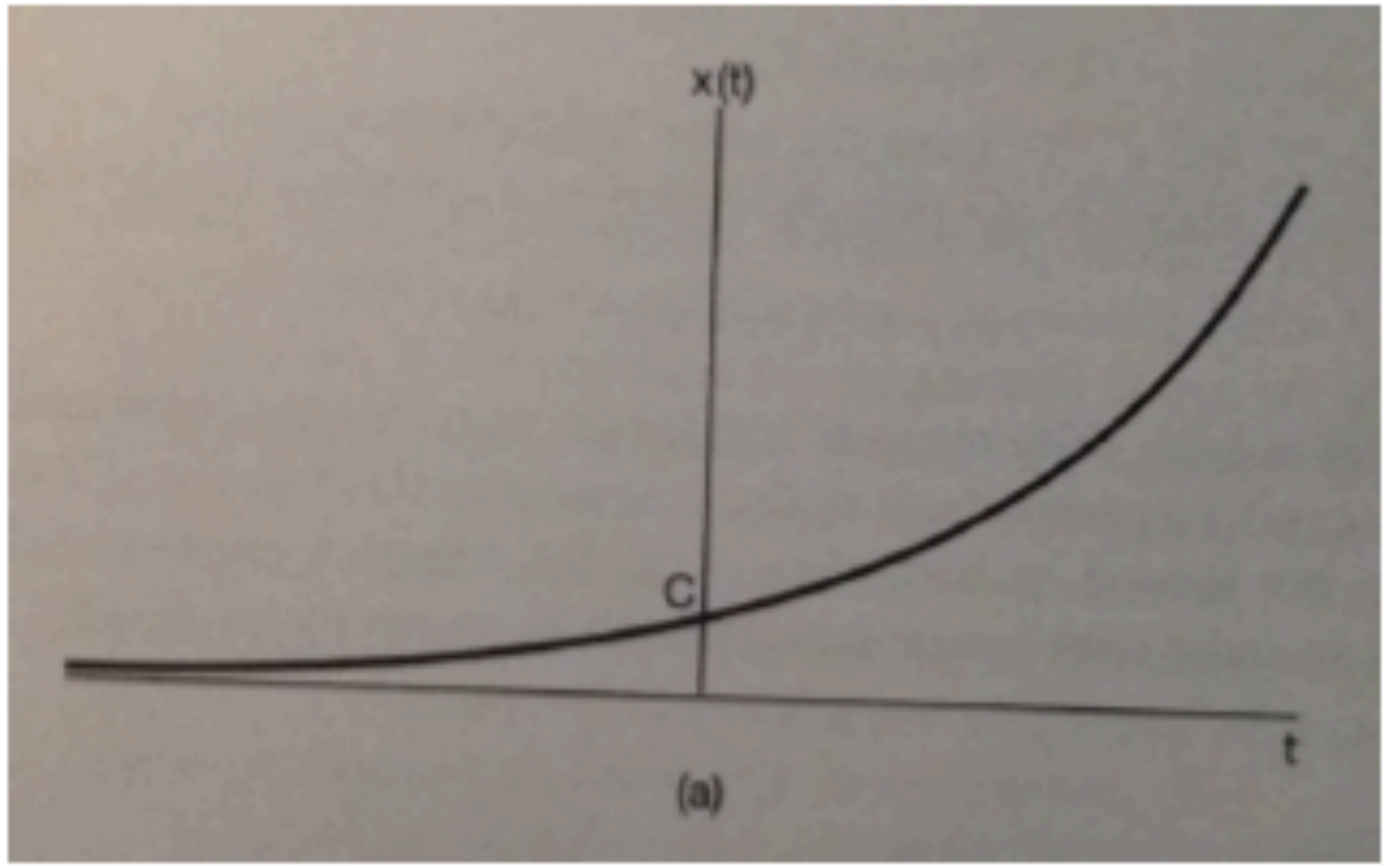


Brief recall and explanation….

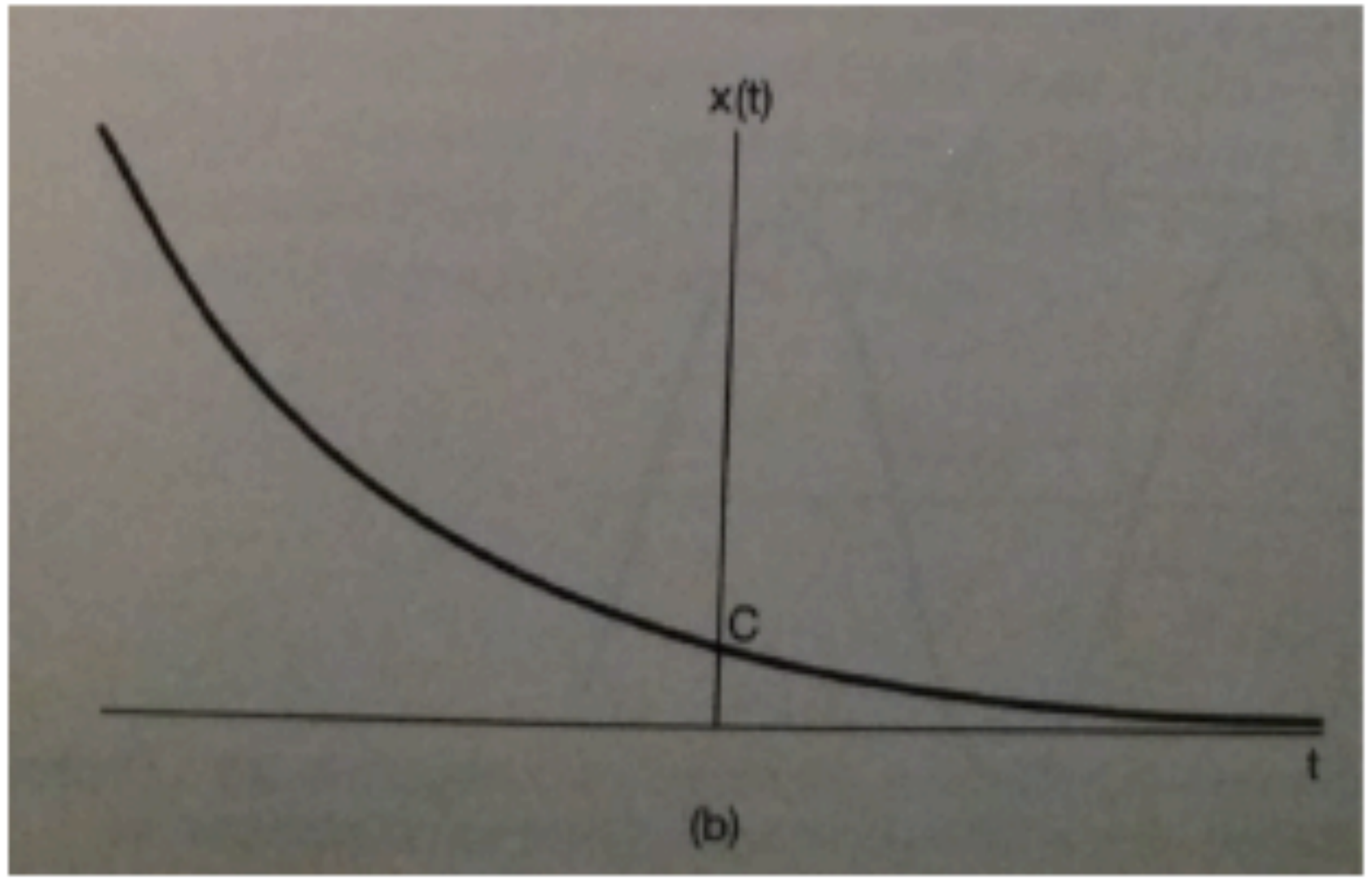
- From Laura Martínez’s slides:

Real exponential:

- When $C, a \in \mathbb{R} \rightarrow x(t) = Ce^{\sigma t}$



Growing exponential($\sigma > 0$)



Decaying exponential ($\sigma < 0$)

Brief recall and explanation….

- Recall the Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

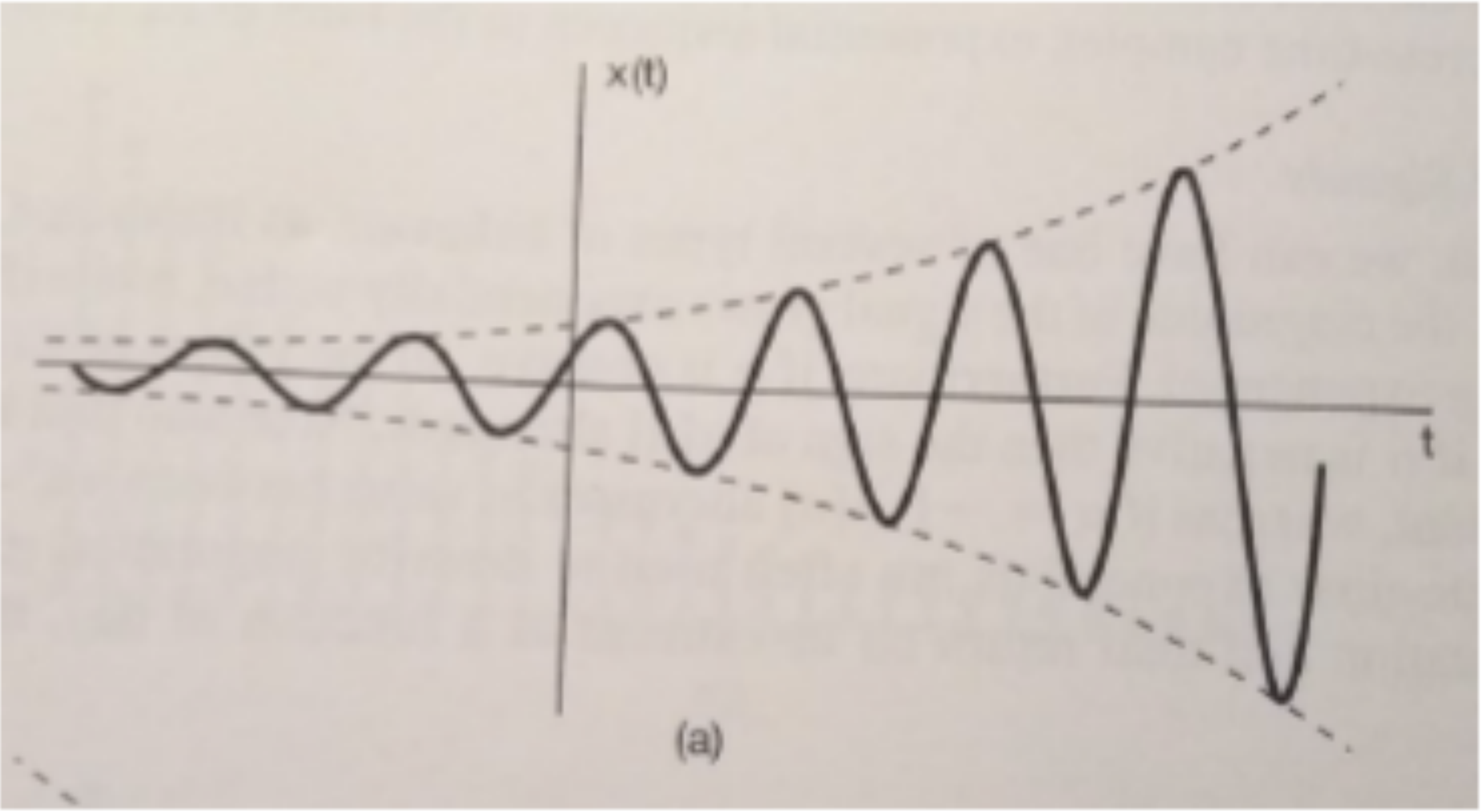
$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- Already linear combination of exponentials…

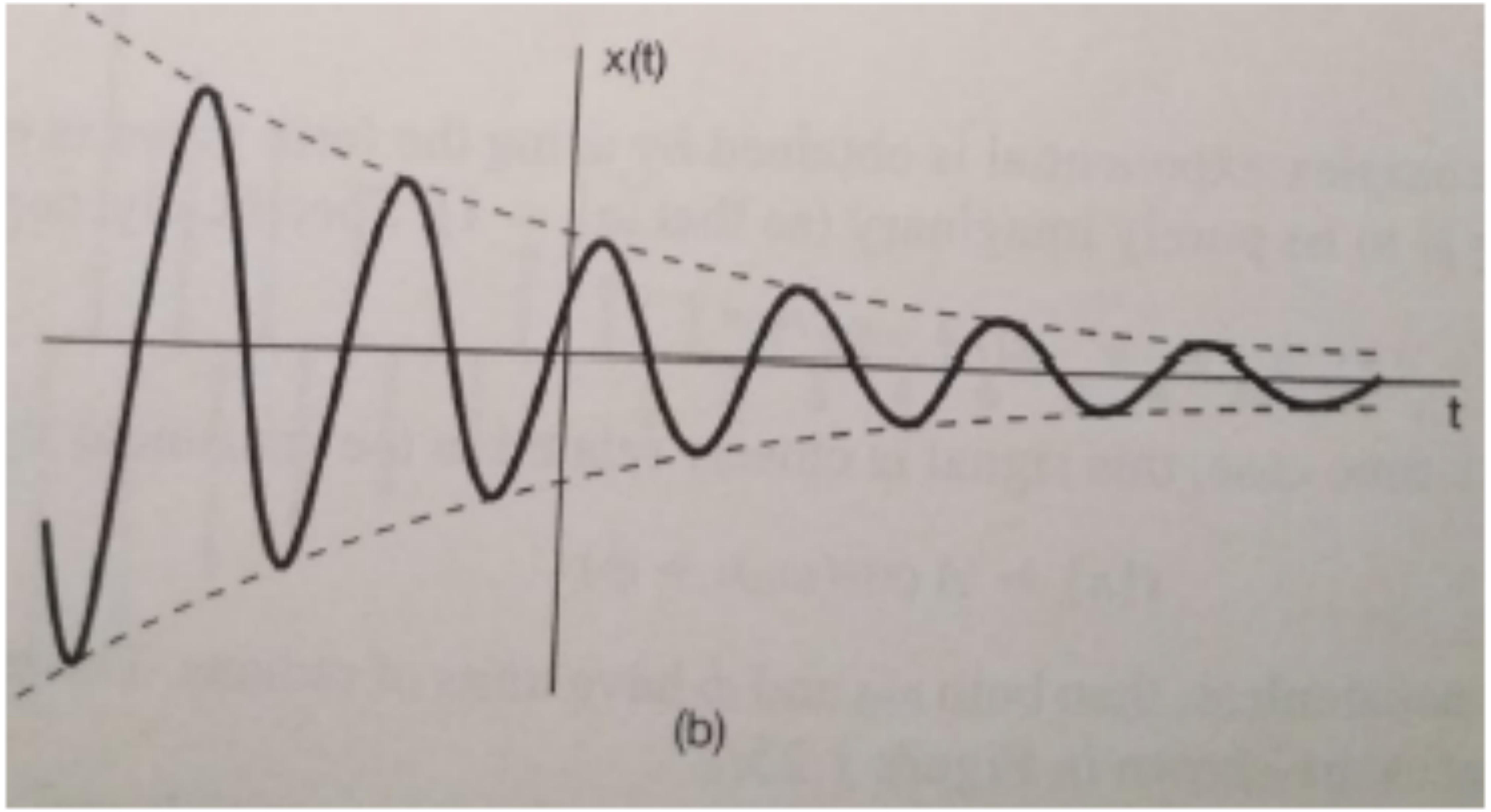
Brief recall and explanation….

- From Laura Martínez's slides:

$$e^{st} = e^{\sigma t} e^{j\omega t}$$



Growing sinusoidal signal (σ > 0)



Decaying exponential (σ < 0)

Why a transformed domain?

- Let us consider an exponential as input of an LTI system:

$$x(t) = e^{st}$$

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Why a transformed domain?

- Let us consider an exponential as input of an LTI system:

$$y(t) = e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$y(t) = H(s) e^{st}$$

$$y(t) = H(s) x(t)$$

Laplace Transform

- Laplace Transform of the impulse response:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$y(t) = H(s)x(t)$$

$$x(t) = e^{st}$$

Exponentials as eigenfunctions of LTI systems

Very important slide….

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

s: complex number

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = [\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau] e^{st} = H(s) e^{st}$$

$H(s)$: eigenvalue

e^{st}: eigenfunction

H(s) is also complex number, in general

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

Laplace Transform

Laplace Transform

- Laplace transformation of the impulse response:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

- **Note that this integral does not always exist….it depends on:**
- **the function h(t) inside**
- **and the value of "s"** (recall that is a complex number)

Laplace Transform

- Laplace transformation of the a generic function/signal:

$$X(s) = \int_{-\infty}^{+\infty} x(\tau)e^{-s\tau}d\tau$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

Laplace Transform

- Laplace transformation is defined in the complex domain and takes complex values:

$$s \in \mathbb{C} \qquad X(s) \in \mathbb{C}$$

then usually the people plot and study:

$$|X(s)| \qquad \mathrm{Real}[X(s)]$$

$$\mathrm{phase}[X(s)] \qquad \mathrm{Imag}[X(s)]$$

Why a transformed domain?

- Let us consider an exponential as input of an LTI system:

$$x(t) = e^{st}$$

- then the output is:

$$y(t) = H(s)x(t)$$

$$|y(t)| = |H(s)||x(t)|$$

Why a transformed domain?

$$|y(t)| = |H(s)||x(t)|$$

- Then, when:

$$|H(s)| \to 0 \qquad |y(t)| = 0$$

$$|H(s)| \to \infty \qquad |y(t)| \to \infty$$

Why a transformed domain?

- The Laplace transform can destroy the output signal,
- or makes it to diverge…. (explosion - instability)

$$|H(s)| \to 0 \qquad |y(t)| = 0$$

$$|H(s)| \to \infty \qquad |y(t)| \to \infty$$

Why a transformed domain?

- The Laplace transform of the impulse response h(t) says almost everything regarding the LTI system:

We want to study then

$$H(s)$$

as function of s.

Example: linear combination of real exponentials and h(t)=u(t)

Example: output calculation using the system function

- Consider a LTI system characterized by $h(t) = u(t)$. Calculate the output when the input is:

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} + Ce^{s_3 t}$$

- We start calculating the system function:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-s\tau}d\tau = \int_{0}^{\infty} e^{-s\tau}d\tau =_{Real(s)>0}$$

$$= \frac{1}{-s}[e^{-s\tau}]_0^{\infty} = \frac{1}{-s}[0-1] = \frac{1}{s}$$

- Using the linearity property:

$$y(t) = H(s_1)Ae^{s_1 t} + H(s_2)Be^{s_2 t} + H(s_3)Ce^{s_3 t} = \frac{A}{s_1}e^{s_1 t} + \frac{B}{s_2}e^{s_2 t}\frac{C}{s_3}e^{s_3 t}$$

where we assume that $Real(s_1), Real(s_2), Real(s_3) > 0$.

Examples - questions

- Consider:

$$\int_{-\infty}^{+\infty} h(t)e^{-2t}dt$$

- what is it: H(….) ?

Examples - questions

$$H(2) = \int_{-\infty}^{+\infty} h(t)e^{-2t}dt$$

Examples - questions

- Consider:

$$\int_{-\infty}^{+\infty} h(t)e^{-jt}dt = ?$$

- what is it: H(….) ?

Examples - questions

$$H(j) = \int_{-\infty}^{+\infty} h(t) e^{-jt} dt$$

Examples - questions

- Consider:

$$\int_{-\infty}^{+\infty} h(t)e^{-(2+j)t}dt = ?$$

- what is it: H(….) ?

Examples - questions

$$H(2+j) = \int_{-\infty}^{+\infty} h(t)e^{-(2+j)t}dt$$

$$\sigma = 2 \qquad \omega = 1$$

STANDARD Fourier transform

- Recall:

$$s = \sigma + j\omega$$

- Frequency

- From Laplace Transform to Fourier Transform:

$$\sigma = 0$$

STANDARD Fourier transform

$$s = 0 + j\omega$$

- **The FOURIER TRASFORM IS A SPECIAL CASE OF THE LAPLACE TRANSFORM WITH:**

$$\sigma = 0$$

STANDARD Fourier transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

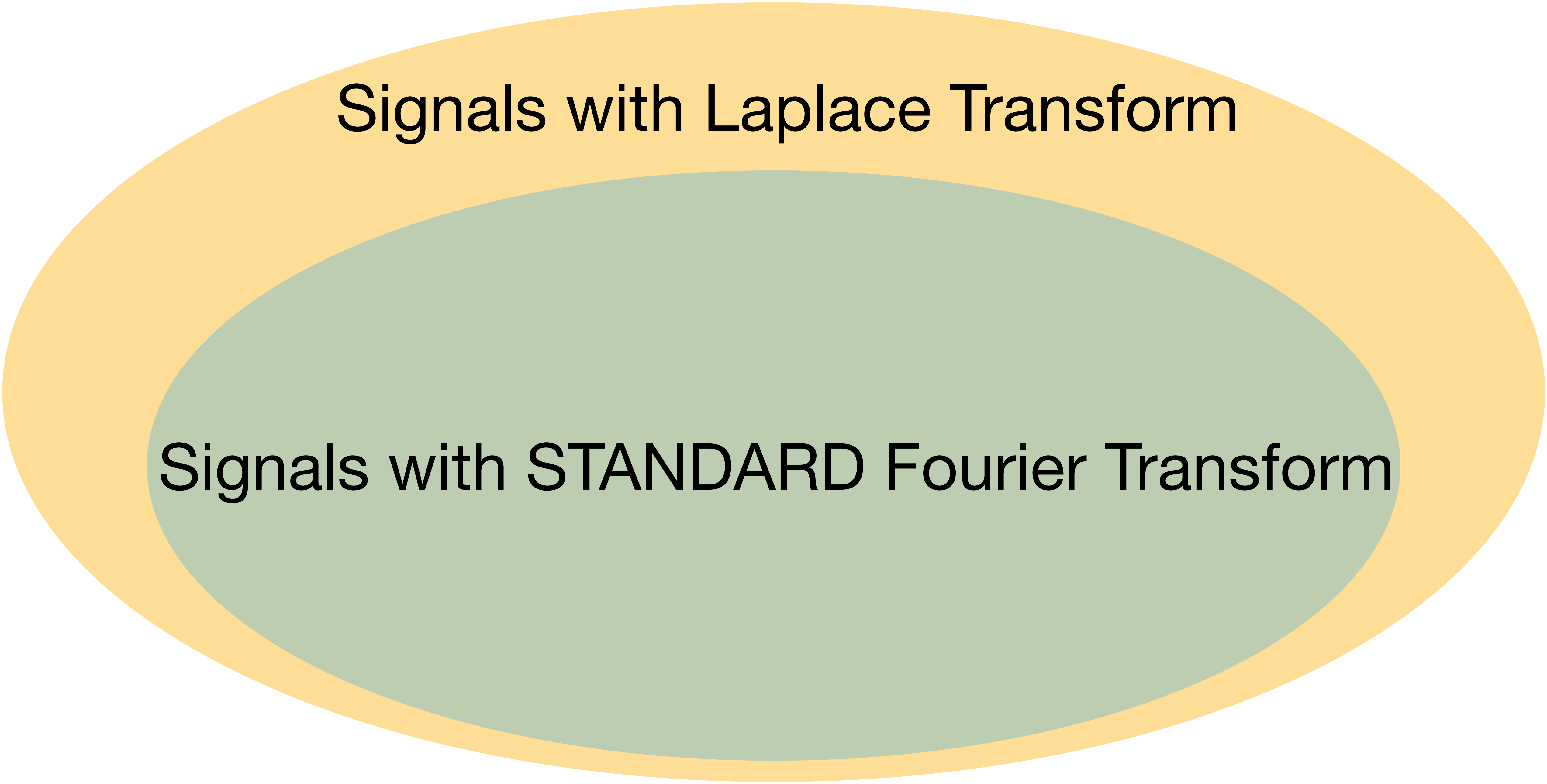
Laplace and STANDARD Fourier transforms

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

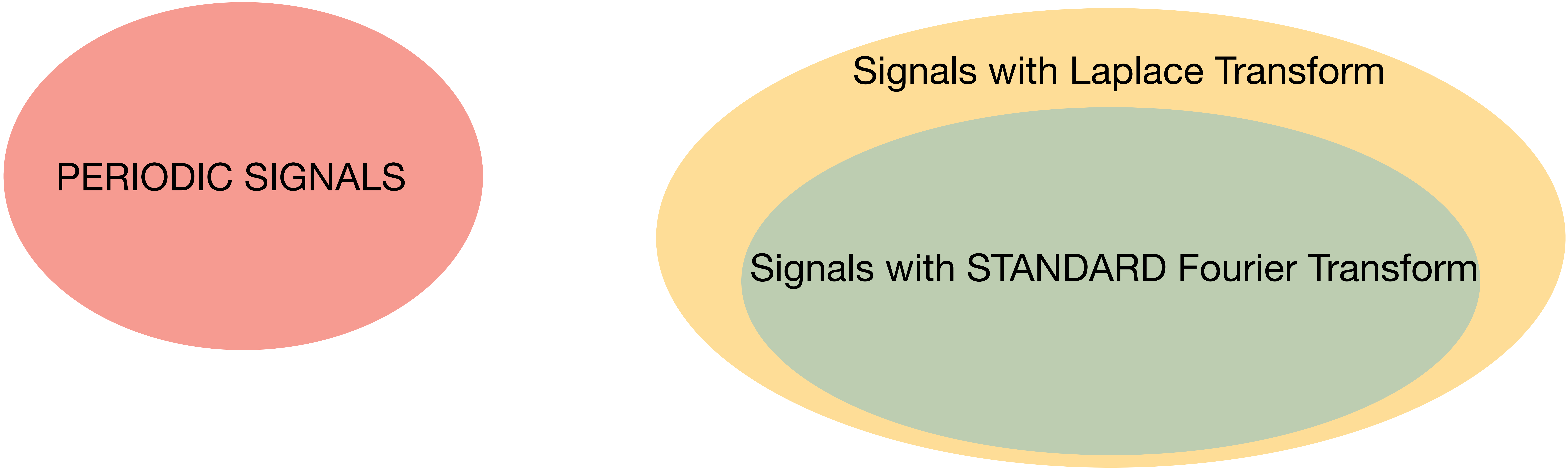
Laplace and STANDARD Fourier transforms

- **Some signals/functions do not have Laplace Transform.**
- **And even less signals/functions have STANDARD Fourier Transform…**



Laplace and STANDARD Fourier transforms

- **PERIODIC SIGNALS HAVE NOT LAPLACE/FOURIER**



Periodic signals in transformed domains

- **PERIODIC SIGNALS HAVE NOT LAPLACE/STANDARD FOURIER.**

- We will also see how to deal with periodic signals.
- For them we have the **FOURIER SERIES** and
- the **Generalized Fourier Transform.**

Questions?