Topic 2.4: Generalized Fourier Transform

Linear systems and circuit applications

Señales y Sistemas

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

Transformations for signal in continuous time

For Periodic signals | **For non-periodic signals**

Fourier Series (FS) | **Stand. Fourier Transform (FT)** | **Laplace Transform (FT)**

also for some Signals with Infinite Energy

Generalized Fourier Transform (GFT)

Mathematically, it is not completely valid… or we need other definition of Fourier Transformation….

Existence of the Fourier Transform

- The integral below must exist:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- **Sufficient conditions (signal with finite energy):**

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty$$

it ensures the existence of the direct transformation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

it ensures the existence of the direct and inverse transformations in L_2 sense

Fourier Transform: other form

- Other way of writing the FT:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)dt$$

- Then the FT can be seen as these two integrals
- These two integrals must exist, to have - x(t) - a FT

Periodic signals have not FT

- **Periodic signals have infinite energy !!!**
- **so they have not FT**
- You can also think in this way: the integral of the type below does not exist when x(t) is periodic…

$$\int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt \qquad \text{does not exist or infinity}$$

in a standard sense

- think to the limits -InF o +InF

Huge confusion

- Some books say that exists FT of periodic signal, it is not true…. mathematically, it is a nightmare…
- **For a periodic signal, the only true/proper mathematical tool is the Fourier Series (FS).**
- We will describe something that contains the same information of the FS - for periodic signals - (actually, it needs the information of the FS) that can be considered a generalization of the FT, but mathematically it is not a true-FT in a standard sense…

Huge confusion

- The **Generalized Fourier Transform** that will present it a properly defined only under the **Distribution Theory**.
- This requires more theory and other “calculus rules” (in an extended sense)
- Technically speaking, all the transformation involving the Dirac Deltas are properly valid under the **Distribution Theory.**

Huge confusion

- For the Laplace Transform, we consider standard calculus rules of integrals…and for Zeta Transform, standard sum rules

Other example

- Let us consider: $x(t) = e^{j\omega_0 t}$
- (this signal is periodic, then with infinite energy)
- A **good student** that have studied standard calculus (and does not know *Distribution Theory*) **MUST** write

$$X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \begin{cases} \nexists & \text{if } \omega \neq \omega_0, \\ \infty & \text{if } \omega = \omega_0. \end{cases}$$

Other example

- Indeed:

$$\int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{+\infty} e^{j(\omega_0 - \omega)t} dt$$

$$= \left[\frac{1}{j(\omega_0 - \omega)} e^{j(\omega_0-\omega)t} \right]_{-\infty}^{+\infty} = \nexists \quad \text{when } \omega_0 \neq \omega$$

Other example

- A **bad student** could write

$$X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \begin{cases} 0 & \text{if } \omega \neq \omega_0, \\ \infty & \text{if } \omega = \omega_0. \end{cases}$$

- In an standard sense, **we cannot write**

$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(\omega - \omega_0)$$

Other example

- Someone believes that the solution (or just makes confusion) is the *Cauchy Principal Value*, *that is another "thing":*

$$P.V. ==> \lim_{R\to\infty}\int_{-R}^{+R} e^{j\omega_0 t}e^{-j\omega t}dt = 0$$

- ***BUT:***

$$\lim_{R\to\infty}\int_{-R}^{+R} e^{j\omega_0 t}e^{-j\omega t}dt \neq \int_{-\infty}^{+\infty} e^{j\omega_0 t}e^{-j\omega t}dt$$

Other example

- …and, more generally, the FT is not defined as a Cauchy P.V.
- Other example of P.V.:

$$P.V. ==> \lim_{R\to\infty} \int_{-R}^{+R} t\ dt = 0$$

$$\int_{-\infty}^{+\infty} t\ dt = \nexists$$

….in the Laplace's world: "come back to normality"

- All these considerations are confirmed in *all books,* when they talk regarding Laplace Transform.
- Indeed the (Bilateral) Laplace Transform of the following signals does not exist:

$$\left.\begin{array}{l}\cos(\omega_0 t) \\ \sin(\omega_0 t)\end{array}\quad e^{j\omega_0 t}\right\}\quad \text{They have not Laplace Transform}$$

Some possible “generalized definition”

- Some partial “patch” (“parche”), is to consider the following definition: we can Generalized Fourier Transform of x(t) the function X(w) such that:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_G(\omega)e^{j\omega t}d\omega$$

- i.e., we use the inverse Fourier Transform. The direct formula cannot works but maybe the inverse formula one can work…

Some possible "generalized definition"

- Then, if we accept the Dirac Delta as a mathematical operator with some special rule, as an example we can see that

$$X_G(\omega) = 2\pi\delta(\omega - \omega_0)$$

- is the Generalized Fourier Transform of $x(t) = e^{j\omega_0 t}$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

Some possible "generalized definition"

- Then, we can write

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{GF}} 2\pi\delta(\omega - \omega_0)$$

$$1 \xleftrightarrow{\mathcal{GF}} 2\pi\delta(\omega) \qquad (\omega_0 = 0)$$

Some possible "generalized definition"

- Then, if a periodic signal x(t) has a Fourier Series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\mathcal{GF}\{x(t)\} = \mathcal{GF}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\}$$

$$\mathcal{GF}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{GF}\left\{e^{jk\omega_0 t}\right\}$$

Some possible "generalized definition"

$$\mathcal{GF}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{GF}\left\{e^{jk\omega_0 t}\right\}$$

$$X_G(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Generalized Fourier Trasform (GFT) - for a periodic signal

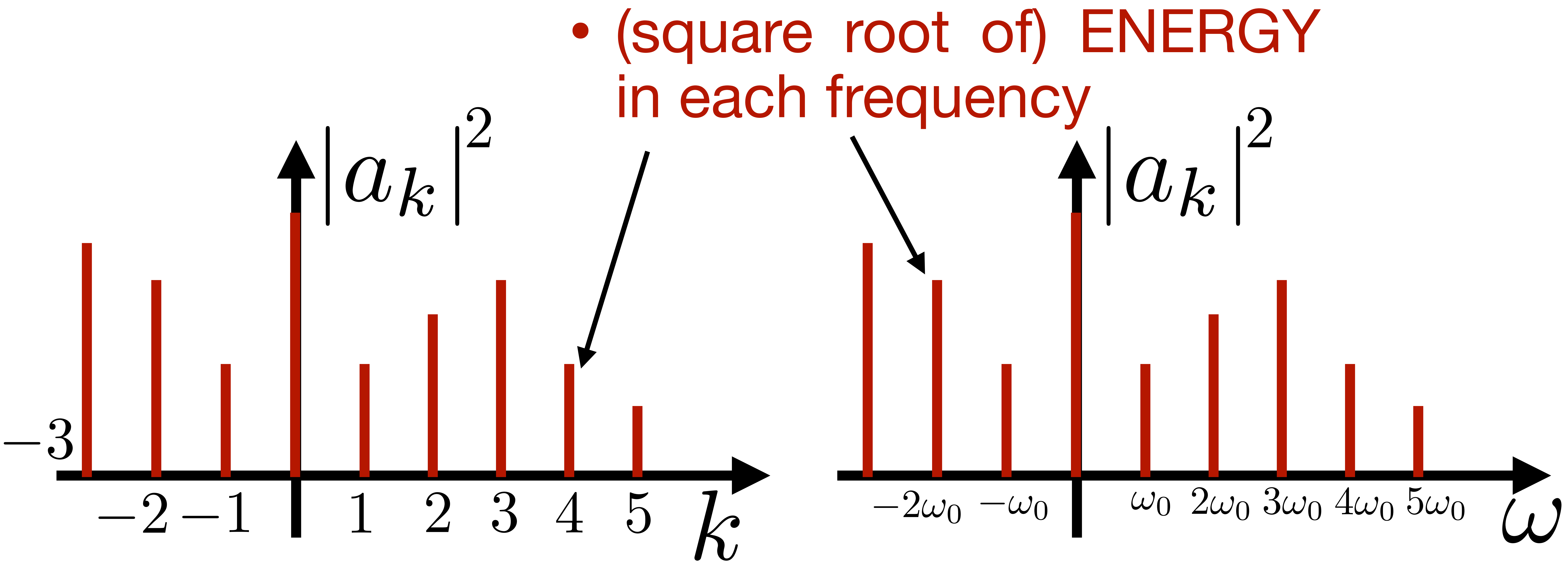
For periodic signals

$$x(t) = x(t+T_0) \overbrace{- - - - - >}^{FS} a_k$$

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

Recall something of the FS…

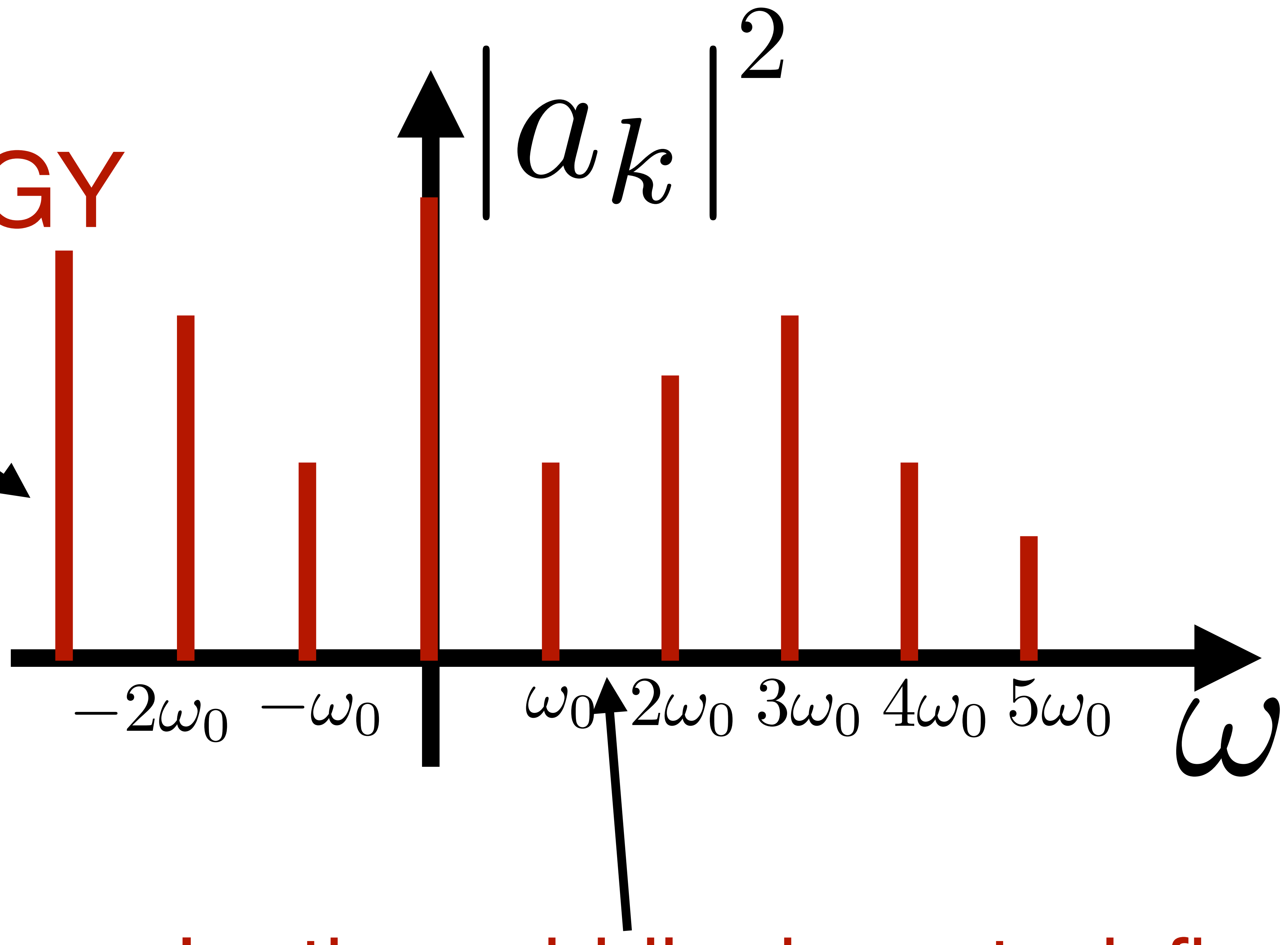
- The coefficients a_k's are complex numbers



Recall something of the FS…

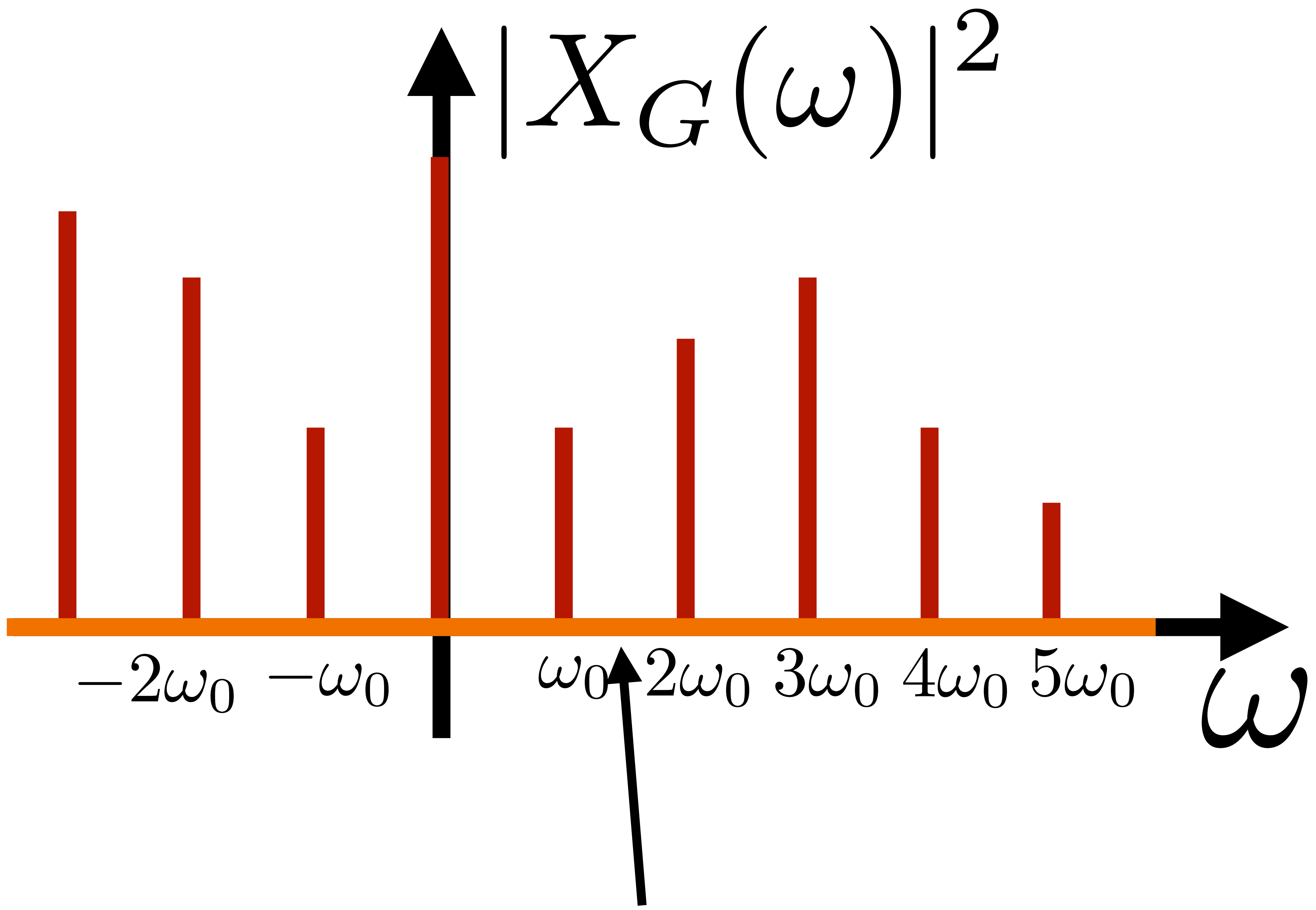
- Important observation:

- (square root of) ENERGY in each frequency



- In the middle is not defined… these frequencies are not contained

Recall something of the FS…



- If we consider a function that is equal to a_k at each kw_0 and "0" otherwise (there), then "almost nothing changes"…

Generalized Fourier Trasform (GFT)

For periodic signals

$$X_G(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

- It contains only the information of the FS
- we need to do the FS, to have the GFT

Example: GFT of the cosine

$$x(t) = \cos(\omega_0 t) \rightarrow \begin{array}{l} a_1 = a_{-1} = \frac{1}{2} \\ a_k = 0, \quad k \neq 1, -1 \end{array}$$

$$X_G(\omega) = 2\pi \left[\frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0) \right]$$

$$X_G(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Example: GFT of the sine

$$x(t) = \sin(\omega_0 t)$$

$$X_G(\omega) = 2\pi \left[\frac{1}{2j}\delta(\omega - \omega_0) - \frac{1}{2j}\delta(\omega + \omega_0) \right]$$

$$X_G(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

$$X_G(\omega) = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

The GFT for other “non-periodic” signals

- Given its definition, GTF can be used for other signals **that have not the FT.**
- **For instance, a constant has FS (it is like periodic with infinite period…)**
- **or a sum of two periodic signal that is not always periodic…**

Example: The GFT of a constant

$$x(t) = 1 \Longrightarrow a_0 = 1 \qquad \text{the rest are zeros...}$$

$$X_G(\omega) = 2\pi\delta(\omega)$$

The GFT of a sum of two periodic signals

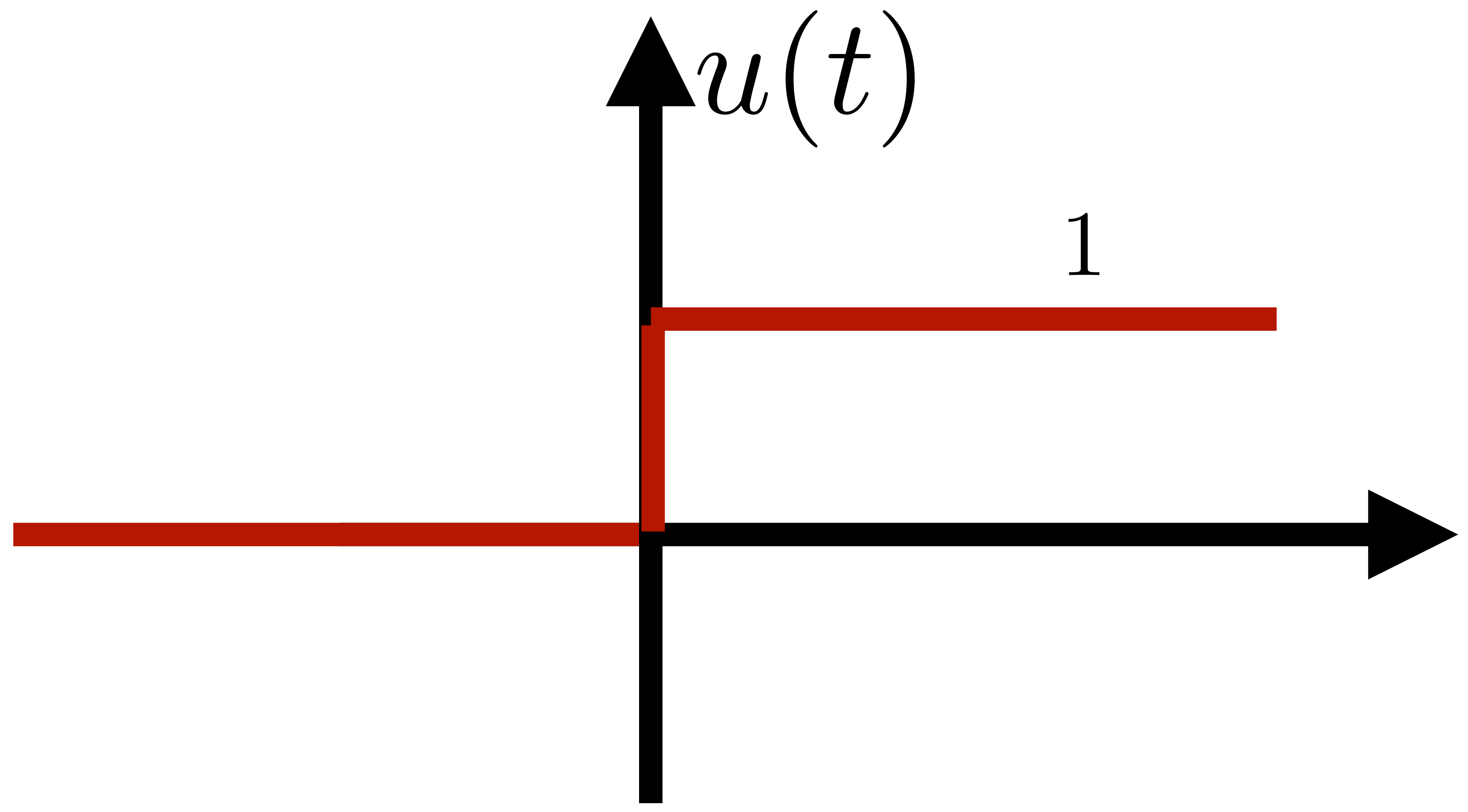
$$z(t) = x_1(t) + x_2(t)$$

- it could be non-periodic
- periodic
- periodic

$$Z_G(\omega) = X_{1,G}(\omega) + X_{2,G}(\omega)$$

Even more confusion….and more difficult

- Consider the step function (escalón) - infinite energy:

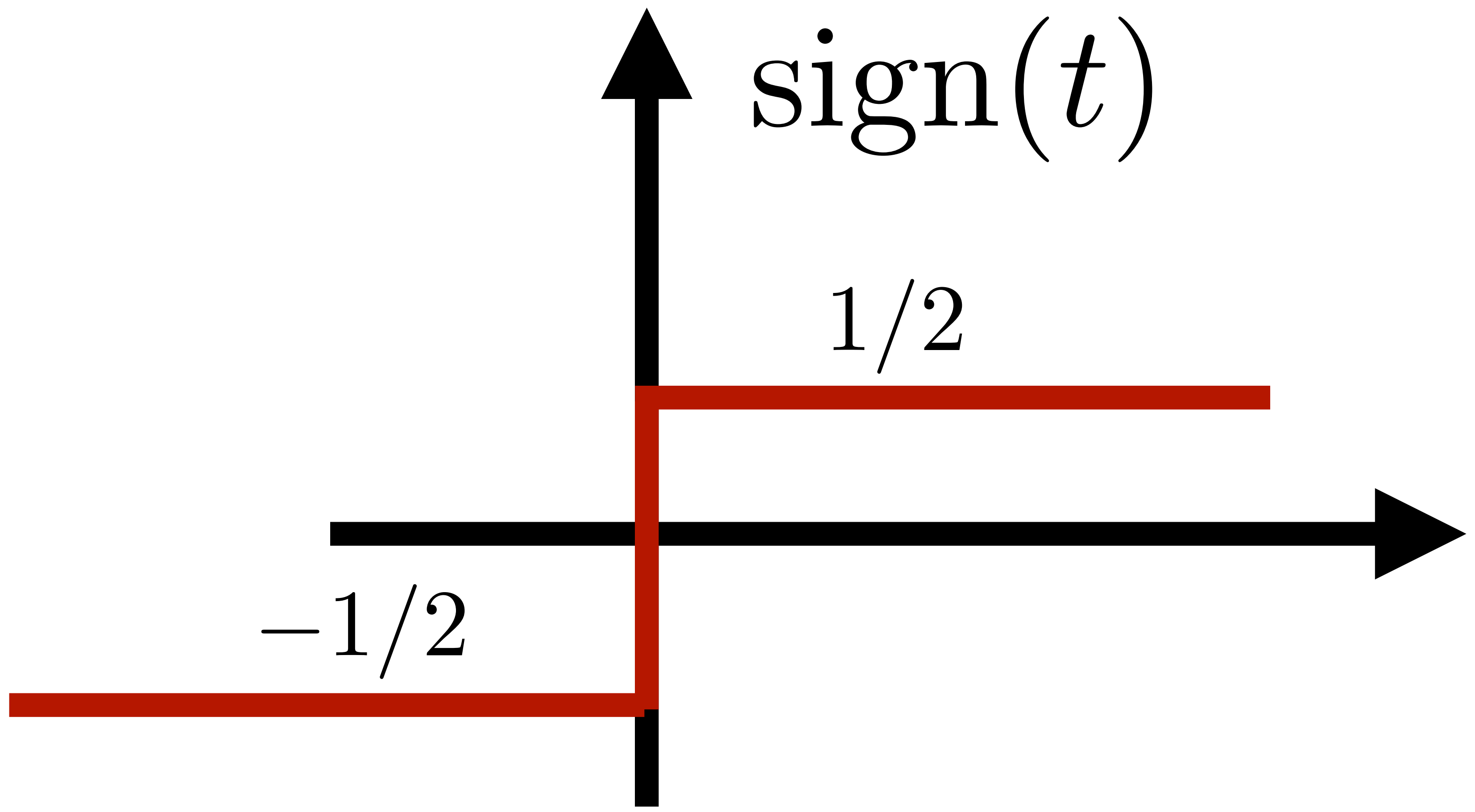


$$u(t) \overset{\mathcal{GF}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$$

- Many people prove it using FT properties, but the properties are valid on if FT exists !!

Even more confusion….

- Sign function - infinite energy:



$$\text{sign}(t) \overset{\mathcal{GF}}{\longleftrightarrow} \frac{1}{j\omega}$$

- **The Sign function has not (Bilateral) Laplace Transform**

Avoid confusion and recall

- Even if people says the opposite, the GTF is not coming from the standard definition of the FT. It requires the DISTRIBUTION THEORY for a proper mathematical definition.
- Usually people call it simply “FT” generating a lot of confusion.
- The LAPLACE TRANSFORM extends the Standard FT.

Questions?