Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 2

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About Vector Multiplication

Dot and Cross Product

- When we introduced the arithmetic operations,

  Why the product of two vectors
  was not defined?

- Vector multiplication could be defined in a manner analogous to the vector addition

  By componentwise multiplication

- However, such a definition is not very useful in our context
- Instead, we shall define and use two different concepts of a product of two vectors:
  - The Euclidean inner product, or dot product, defined for two vectors in $\mathbb{R}^n$ (where n is arbitrary)
  - The cross or vector product, defined only for vectors in $\mathbb{R}^3$

The Dot Product of Two Vectors: Algebraic Construction

Definition 3.1

- Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors
- The dot (or inner or scalar) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Dot product takes two vectors
and produces a single real number
(not a vector)

Example 1

In $\mathbb{R}^3$ we have

$$\begin{aligned} (1, -2, 5) \cdot (2, 1, 3) &= (1)(2) + (-2)(1) + (5)(3) = 15 \\ (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{k}) &= (3)(1) + (2)(0) + (-1)(-2) = 5 \end{aligned}$$

The Dot Product of Two Vectors: Algebraic Construction

Properties of Dot Products

If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are any vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), and $k \in \mathbb{R}$ is any scalar

1. $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ commutativity
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ distributivity
4. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

The Dot Product of Two Vectors: Geometric Interpretation

Definition 3.2

- If $\mathbf{a} = (a_1, a_2, a_3)$ then the length of $\mathbf{a}$ (also called the norm or magnitude) is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

- Using the distance formula, the length of the arrow from the origin to (a_1, a_2, a_3) is

$$\sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2}$$

- On the other hand

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$$

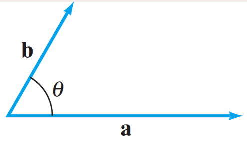
- Thus

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \text{ or } \|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

The Dot Product of Two Vectors: Geometric Interpretation

Theorem 3.3

- Let **a** and **b** are two nonzero vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) drawn with their tails at the same point
- Let θ, where $0 \leq \theta \leq \pi$, be the angle between **a** and **b**



- Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

Note

- If either **a** or **b** is the zero vector, then θ is indeterminate (i.e., can be any angle)

Angles Between Vectors

Corollary of Theorem 3.3

- Theorem 3.3 may be used to find the angle between two nonzero vectors **a** and **b**

$$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

- The use of the inverse cosine is unambiguous, since we take $0 \leq \theta \leq \pi$

Angles Between Vectors

Example 2

- If $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$, then formula gives

$$\theta = \cos^{-1} \frac{(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{j} - \mathbf{k})}{\|\mathbf{i} + \mathbf{j}\| \|\mathbf{j} - \mathbf{k}\|} = \cos^{-1} \frac{1}{(\sqrt{2} \cdot \sqrt{2})} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$$

Angles Between Vectors

Orthogonality

- If **a** and **b** are nonzero, then Theorem 3.3 implies

$$\cos\theta = 0 \text{ if and only if } \mathbf{a}\cdot\mathbf{b} = 0$$

- We have $\cos\theta = 0$ just in case $\theta = \frac{\pi}{2}$

Remember that $0 \le \theta \le \pi$

- We call **a** and **b** perpendicular (or orthogonal) when $\mathbf{a}\cdot\mathbf{b} = 0$
- If either **a** or **b** is the zero vector, the angle θ is undefined
- Since $\mathbf{a}\cdot\mathbf{b} = 0$ if **a** or **b** is $\mathbf{0}$, we adopt the standard convention

The zero vector
is perpendicular to every vector

Angles Between Vectors

Example 3

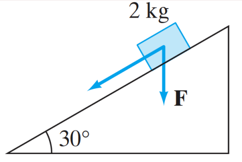
- The vector $\mathbf{a} = \mathbf{i} + \mathbf{j}$ is orthogonal to the vector $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = (1)(1) + (1)(-1) + (0)(1) = 0$$

Vector Projections

Motivation example

- Suppose that a 2 kg object is sliding down a ramp
- The ramp has a 30° incline with the horizontal



- If we neglect friction, the only force acting on the object is gravity

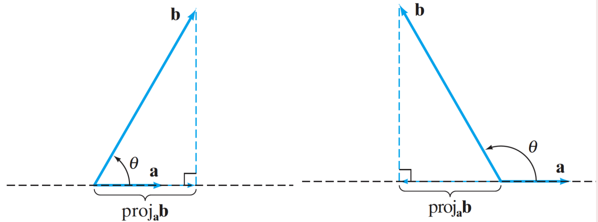
What is the component of the gravitational force in the direction of motion of the object?

- To answer questions of this nature, we need to find the projection of one vector on another

Vector Projections

Projection of one vector on another: intuitive idea

- Let **a** and **b** be two nonzero vectors
- Imagine dropping a perpendicular line from the head of **b** to the line through **a**



- The projection of **b** onto **a** , denoted proj$_{\mathbf{a}}\mathbf{b}$, is the vector represented by the arrow in figure

Vector Projections

Projection of one vector on another: precise formula

- Recall that

  A vector is determined by
  magnitude (length) and direction

- The direction of $\text{proj}_{\mathbf{a}}\mathbf{b}$ is either
  - The same as that of **a** or
  - Opposite to **a** if the angle θ between **a** and **b** is more than $\frac{\pi}{2}$
- Using trigonometry

$$|\cos\theta| = \frac{\|\text{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|}$$

- The absolute value sign around $\cos\theta$ is needed in case

$$\frac{\pi}{2} \leq \theta \leq \pi$$

Vector Projections

Projection of one vector on another: precise formula

- Since

$$|\cos\theta| = \frac{\|\text{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|}$$

- Hence, with a bit of algebra and Theorem 3.3, we have

$$\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \|\mathbf{b}\||\cos\theta| = \frac{\|\mathbf{a}\|\|\mathbf{b}\||\cos\theta|}{\|\mathbf{a}\|} = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}$$

Thus, we know the magnitude and direction of $\text{proj}_{\mathbf{a}}\mathbf{b}$

Vector Projections

Proposition 3.4

- Let k be any scalar and $\mathbf{a}$ any vector
- Then
  1. $\|k\mathbf{a}\| = |k|\|\mathbf{a}\|$
  2. A unit vector (i.e., a vector of length 1) in the direction of a nonzero vector $\mathbf{a}$ is given by

$$\frac{\mathbf{a}}{\|\mathbf{a}\|}$$

This provides a
compact formula for $\text{proj}_{\mathbf{a}}\mathbf{b}$

Vector Projections

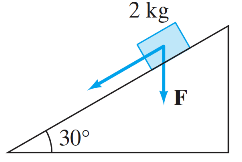
Compact formula for $\text{proj}_{\mathbf{a}}\mathbf{b}$

$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{b} &= \pm\left(\frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}\right)\times\frac{\mathbf{a}}{\|\mathbf{a}\|} = \pm\frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta|}{\|\mathbf{a}\|}\frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\end{aligned}$$

Vector Projections

Example 4

- Suppose that a 2 kg object is sliding down a ramp
- The ramp has a 30° incline with the horizontal

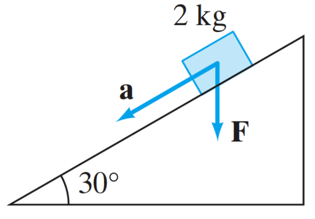


- If we neglect friction, the only force acting on the object is gravity

What is the component of the gravitational force in the direction of motion of the object?

Vector Projections

Example 4

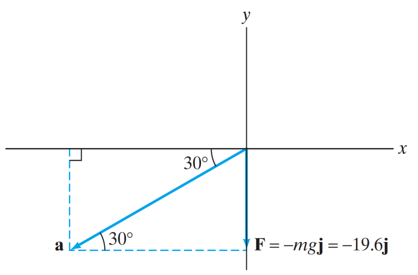


- We need to calculate $\text{proj}_{\mathbf{a}}\mathbf{F}$
- $\mathbf{F}$ is the gravitational force vector
- $\mathbf{a}$ points along the ramp as shown in figure

Vector Projections

Example 4

- The coordinate situation is shown in figure

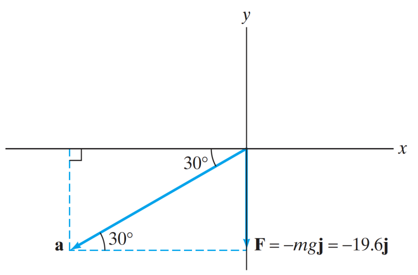


- From trigonometric considerations, we must have $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ such that

$$a_1 = -\|\mathbf{a}\|\cos 30^\circ \text{ and } a_2 = -\|\mathbf{a}\|\sin 30^\circ$$

Vector Projections

Example 4

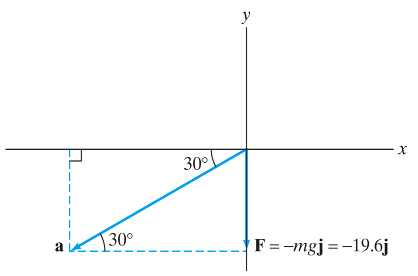


- We are really only interested in the direction of **a**
- There is no loss in assuming that **a** is a unit vector
- Thus

$$\mathbf{a} = -\cos 30^\circ \mathbf{i} - \sin 30^\circ \mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

Vector Projections

Example 4



- Taking $g = 9.8\text{m}/\text{sec}^2$, we have $\mathbf{F} = -2g\mathbf{j} = -19.6\mathbf{j}$
- Therefore

$$\text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{\mathbf{a}\cdot\mathbf{F}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)\cdot(-19.6\mathbf{j})}{1}\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$

Vector Projections

Example 4

$$\text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{\mathbf{a}\cdot\mathbf{F}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)\cdot(-19.6\mathbf{j})}{1}\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$$

$$= 9.8\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right) \approx -8.49\mathbf{i} - 4.9\mathbf{j}$$

- And the component of **F** in this direction is

$$\|\text{proj}_{\mathbf{a}}\mathbf{F}\| = \|-8.49\mathbf{i} - 4.9\mathbf{j}\| = 9.8 \text{ N}$$

Vector Projections

Normalization of a vector

- Unit vectors, that is, vectors of length 1, are important in that they capture the idea of direction

They all have the same length

- Proposition 3.4 shows that every nonzero vector **a** can have its length adjusted to give a unit vector

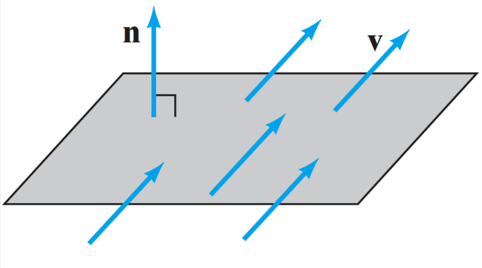
$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- **u** points in the same direction as **a**
- This operation is referred to as normalization of the vector **a**

Vector Projections

Example 5

- A fluid is flowing across a plane surface with uniform velocity vector **v**
- Let **n** be a unit vector perpendicular to the plane surface

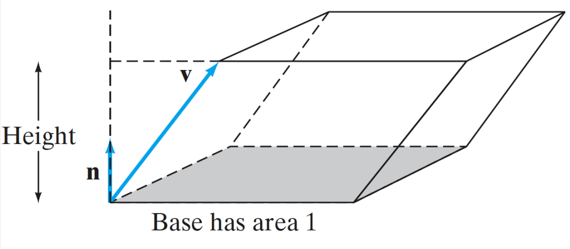


- Find (in terms of **v** and **n**) the volume of the fluid that passes through a unit area of the plane in unit time

Vector Projections

Example 5

- Suppose one unit of time has elapsed
- Then, over a unit area of the plane (a unit square), the fluid will have filled a “box” as in figure



- The box may be represented by a parallelepiped
- The volume we seek is the volume of this parallelepiped

Vector Projections

Example 5

- The volume of this parallelepiped is

$$\text{Volume} = (\text{area of base})\ (\text{height})$$

- The area of the base is 1 unit by construction
- The height is given by $\text{proj}_{\mathbf{n}}\mathbf{v}$
- Since $\mathbf{n} \cdot \mathbf{n} = \|\mathbf{n}\|^2 = 1$

$$\text{proj}_{\mathbf{n}}\mathbf{v} = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n}$$

- Hence

$$\|\text{proj}_{\mathbf{n}}\mathbf{v}\| = \|(\mathbf{n} \cdot \mathbf{v})\mathbf{n}\| = |\mathbf{n} \cdot \mathbf{v}|\|\mathbf{n}\| = |\mathbf{n} \cdot \mathbf{v}|$$

The Cross Product of Two Vectors

Motivation

- The cross product of two vectors in $\mathbb{R}^3$ is an "honest" product

it takes two vectors
and produces a third one

- However, the cross product possesses less "natural" properties

it cannot be defined for vectors in $\mathbb{R}^2$
without first embedding them in $\mathbb{R}^3$

- We will define the cross product first geometrically, and then deduce an algebraic formula

The Cross Product of Two Vectors in $\mathbb{R}^3$

Definition 4.1

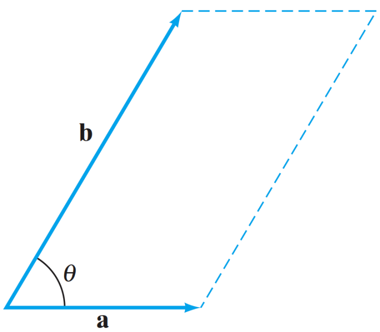
- Let **a** and **b** be two vectors in $\mathbb{R}^3$ (not $\mathbb{R}^2$)
- The cross product (or vector product) of **a** and **b**, denoted $\mathbf{a} \times \mathbf{b}$, is the vector whose length and direction are given as follows
- The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by **a** and **b** or is zero if either **a** is parallel to **b** or if **a** or **b** is **0**
- Alternatively, the following formula holds

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin \theta$$

where θ is the angle between **a** and **b**

The Cross Product of Two Vectors in $\mathbb{R}^3$

Definition 4.1



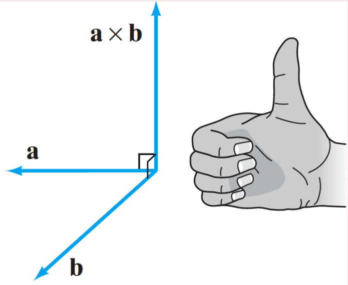
- The area of this parallelogram is

$$\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

The Cross Product of Two Vectors in $\mathbb{R}^3$

Definition 4.1

- The direction of $\mathbf{a} \times \mathbf{b}$ is such that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (when both $\mathbf{a}$ and $\mathbf{b}$ are nonzero)
- It is taken so that the ordered triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right-handed set of vectors



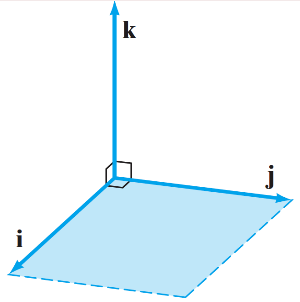
- If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, or if $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

The Cross Product of Two Vectors in $\mathbb{R}^3$

Example 1

- Compute the cross product of the standard basis vectors for $\mathbb{R}^3$

- First consider $\mathbf{i} \times \mathbf{j}$ as shown in figure



- The vectors $\mathbf{i}$ and $\mathbf{j}$ determine a square of unit area

The Cross Product of Two Vectors in $\mathbb{R}^3$

Example 1

- Compute the cross product of the standard basis vectors for $\mathbb{R}^3$

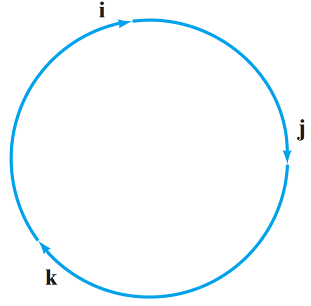
- The vectors $\mathbf{i}$ and $\mathbf{j}$ determine a square of unit area
- Thus
$$\|\mathbf{i} \times \mathbf{j}\| = 1$$
- Any vector perpendicular to both $\mathbf{i}$ and $\mathbf{j}$ must be perpendicular to the plane in which $\mathbf{i}$ and $\mathbf{j}$ lie
- Hence, $\mathbf{i} \times \mathbf{j}$ must point in the direction of $\pm k$
- The right-hand rule implies that $\mathbf{i} \times \mathbf{j}$ must point in the positive k direction
- Since $\|\mathbf{k}\| = 1$, we conclude that
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

The Cross Product of Two Vectors in $\mathbb{R}^3$

Example 1

- Compute the cross product of the standard basis vectors for $\mathbb{R}^3$

- The same argument establishes that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- To remember these basic equations, draw $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in a circle



Properties of the Cross Product; Coordinate Formula

Properties of the Cross Product

- Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$ and let $k \in \mathbb{R}$ be any scalar
- Then
  1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutativity)
  2. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributivity)
  3. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (distributivity)
  4. $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$

Properties of the Cross Product; Coordinate Formula

Properties the Cross Product Does Not Fulfil

- Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$ and let $k \in \mathbb{R}$ be any scalar
- In general, the cross product is not commutative

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

- In general, the cross product does not fulfil associativity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

Example

Let $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

Properties of the Cross Product; Coordinate Formula

Coordinate formula for the cross product

- Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
- Then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= \cdots \\ &= -a_2 b_1\mathbf{k} + a_3 b_1\mathbf{j} + a_1 b_2\mathbf{k} - a_3 b_2\mathbf{i} - a_1 b_3\mathbf{j} + a_2 b_3\mathbf{i} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \end{aligned}$$

- There is a more elegant way to understand this formula

We explore this reformulation next

Properties of the Cross Product; Coordinate Formula

Coordinate formula for the cross product

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Example 2

$$\begin{aligned}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} + 2\mathbf{k}) &= (3 \cdot 2 - (-2) \cdot 0)\mathbf{i} + (-2 \cdot 2 - 1 \cdot 2)\mathbf{j} \\ &+ (1 \cdot 0 - 3 \cdot 2)\mathbf{k} = 6\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}\end{aligned}$$

Matrices and Determinants: A First Introduction

Matrices

- A matrix is a rectangular array of numbers
- Examples of matrices are

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If a matrix has m rows and n columns, we call it $m \times n$
- Thus, the three matrices just mentioned are, respectively, 2×3, 3×2 and 4×4
- To some extent, matrices behave algebraically like vectors
- Mainly interesting is the the notion of a determinant
- It is a real number associated to an $m \times n$ square matrix

Matrices and Determinants: A First Introduction

Definition 4.2: Determinants

- Let A be a 2×2 or 3×3 matrix
- Then the determinant of A, denoted $\det A$ or $|A|$, is the real number computed from the individual entries of A as follows:

1. 2×2 case

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Matrices and Determinants: A First Introduction

Definition 4.2: Determinants

- Let A be a 2×2 or 3×3 matrix
- Then the determinant of A, denoted det A or $|A|$, is the real number computed from the individual entries of A as follows:

2. 3×3 case

If

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

then

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

Matrices and Determinants: A First Introduction

Definition 4.2: Determinants

- Let A be a 2×2 or 3×3 matrix
- Then the determinant of A, denoted det A or $|A|$, is the real number computed from the individual entries of A as follows:

3. 3×3 case in terms of 2×2 determinants

If

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

then

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

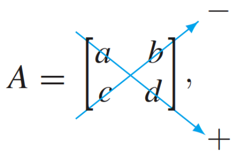
Matrices and Determinants: A First Introduction

Diagonal Approach for 2×2 and 3×3 Determinants

- We write (or imagine) diagonal lines running through the matrix entries

It is not valid
for higher-order determinants

1. 2×2 case



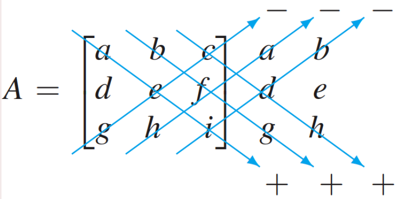
$$|A| = ad - bc$$

Matrices and Determinants: A First Introduction

Diagonal Approach for 2 × 2 and 3 × 3 Determinants

2. 3 × 3 case

We need to repeat the first two columns for the method to work



$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

Matrices and Determinants: A First Introduction

Connection Between Determinants and Cross Products

- If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

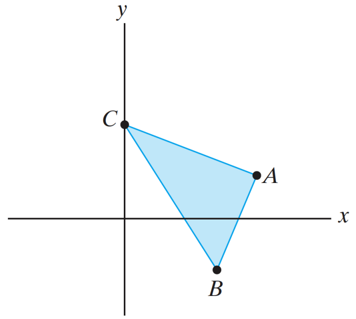
Example 3

$$\begin{aligned} (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} - 4\mathbf{j} - 5\mathbf{k} \end{aligned}$$

Areas and Volumes

Example 4

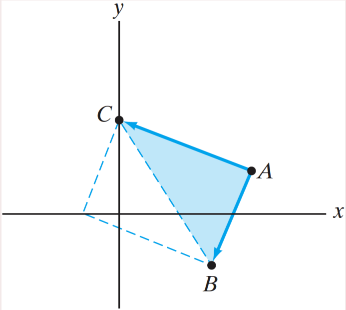
Use vectors to calculate the area of the triangle whose vertices are $A(3, 1)$, $B(2, -1)$, and $C(0, 2)$ as shown in figure



Areas and Volumes

Example 4

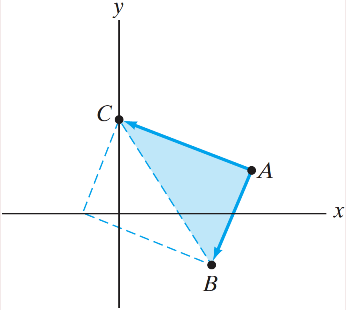
- The trick is to recognize that any triangle can be thought of as half of a parallelogram



- Now, the area of a parallelogram is obtained from a cross product

Areas and Volumes

Example 4



- $\overrightarrow{AB} \times \overrightarrow{AC}$ is a vector whose length measures the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\text{Area of } \nabla ABC = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

Areas and Volumes

Example 4

- To use the cross product, we must consider $\overrightarrow{AB}, \overrightarrow{AC} \in \mathbb{R}^3$
- We simply take the k-components to be zero

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{i} - 2\mathbf{j} = -\mathbf{i} - 2\mathbf{j} - 0\mathbf{k} \\ \overrightarrow{AC} &= -3\mathbf{i} + \mathbf{j} = -3\mathbf{i} + \mathbf{j} + 0\mathbf{k} \end{aligned}$$

- Therefore

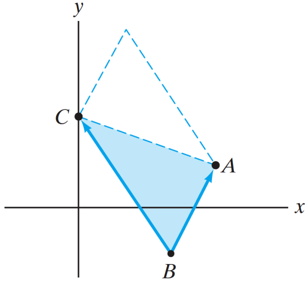
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ -3 & 1 & 0 \end{vmatrix} = -7\mathbf{k}$$

$$\text{Area of } \nabla ABC = \frac{1}{2}\|-7\mathbf{k}\| = \frac{7}{2}$$

Areas and Volumes

Example 4

- There is nothing sacred about using A as the common vertex
- We could just as easily have used B or C, as shown in figure

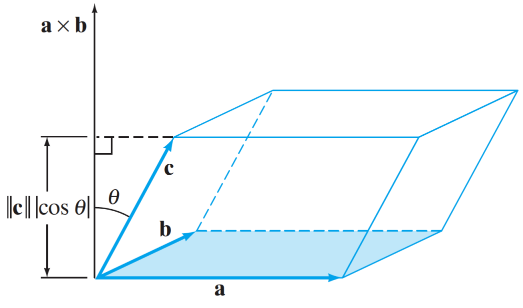


$$\begin{aligned}\text{Area of } \nabla ABC &= \frac{1}{2}\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \frac{1}{2}\|(\mathbf{i} + 2\mathbf{j}) \times (-2\mathbf{i} + 3\mathbf{j})\| \\ &= \frac{1}{2}\|7\mathbf{k}\| = \frac{7}{2}\end{aligned}$$

Areas and Volumes

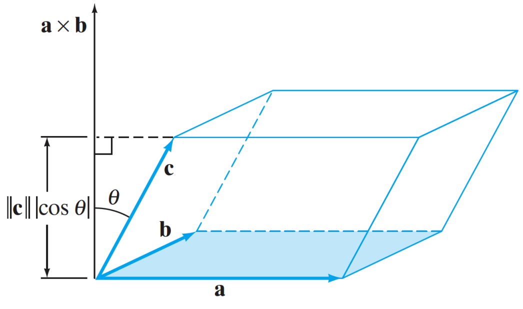
Example 5

Find a formula for the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**



Areas and Volumes

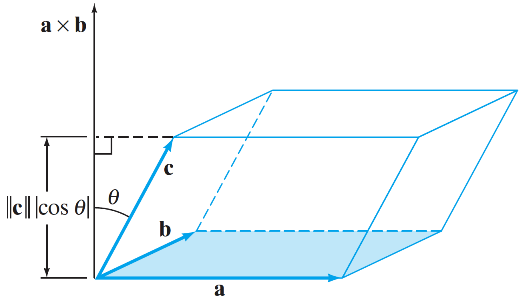
Example 5



- The volume of a parallelepiped is equal to the product of the area of the base and the height
- The base is the parallelogram determined by **a** and **b**
- Its area is $\|\mathbf{a} \times \mathbf{b}\|$

Areas and Volumes

Example 5

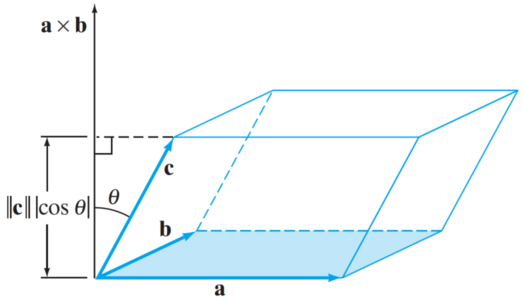


- The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to this parallelogram
- The height of the parallelepiped is $\|\mathbf{c}\| |\cos\theta|$
- θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$

The absolute value is needed in case $\theta > \frac{\pi}{2}$

Areas and Volumes

Example 5



Volume of parallelepiped = (area of base)(height)

$= \|\mathbf{a} \times \mathbf{b}\| \|\mathbf{c}\| |\cos\theta| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$

Areas and Volumes

Example 5

$$\text{Volume of parallelepiped} = (\text{area of base})(\text{height})$$
$$= \|\mathbf{a} \times \mathbf{b}\| \|\mathbf{c}\| |\cos\theta| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$

For example, the parallelepiped determined by the vectors

$$\mathbf{a} = \mathbf{i} + 5\mathbf{j}, \;\; \mathbf{b} = -4\mathbf{i} + 2\mathbf{j} \;\text{ and }\; \mathbf{c} = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$\begin{aligned} \text{Volume of parallelepiped} &= |((\mathbf{i} + 5\mathbf{j}) \times (-4\mathbf{i} + 2\mathbf{j})) \cdot (\mathbf{i} + \mathbf{j} + 6\mathbf{k})| \\ &= |22\mathbf{k} \cdot (\mathbf{i} + \mathbf{j} + 6\mathbf{k})| = |22(6)| = 132 \end{aligned}$$

Areas and Volumes

Triple Scalar Product

- The real number $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ appearing in Example 5 is known as the triple scalar product of the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$
- Since $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ represents the volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}| = |(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}|$$

- In fact, the absolute value signs are not needed

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

Areas and Volumes

Triple Scalar Product

- There is a convenient formula for calculating triple scalar products
- If

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \text{ and } \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

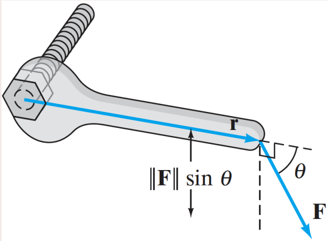
- Then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt

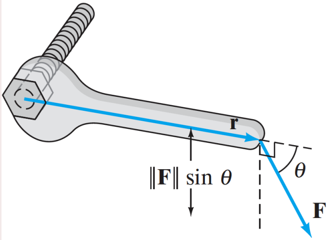


- You apply some force to the end of the wrench handle farthest from the bolt
- The bolt move in a direction perpendicular to the plane determined by the handle and the direction of your force

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



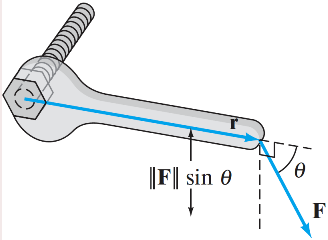
- To measure exactly how much the bolt moves, we need the notion of torque (or twisting force)
- Letting F denote the force you apply to the wrench

Amount of torque = (length of wrench)(component of F $\perp$ wrench)

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



- Let **r** be the vector from the center of the bolt head to the end of the wrench handle
- Then

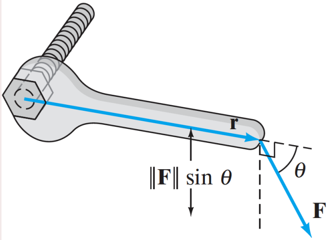
$$\text{Amount of torque} = \|\mathbf{r}\| \|\mathbf{F}\| sin\theta$$

where θ is the angle between **r** and **F**

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



- That is, the amount of torque is

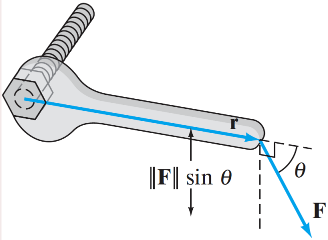
$$\|\mathbf{r} \times \mathbf{F}\|$$

- And the direction of $\mathbf{r} \times \mathbf{F}$ is the same as the direction in which the bolt moves

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



- Hence, it is quite natural to define the torque vector $\mathbf{T}$ to be

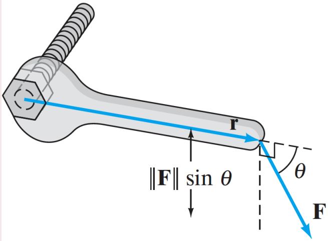
$$\mathbf{T} = \mathbf{r} \times \mathbf{F}$$

- This torque vector $\mathbf{T}$ is a concise way to capture the physics of this situation

Torque

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



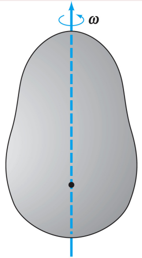
- Note that if **F** is parallel to **r**, then $\mathbf{T} = \mathbf{0}$

If you try to push or pull the wrench,
the bolt does not turn

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

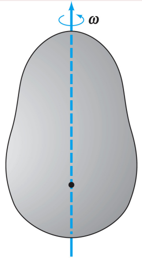


What is the relation between
the (linear) velocity of a point of the object
and the rotational velocity?

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

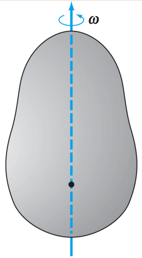


- First, we need to define a vector ω, the angular velocity vector of the rotation
- This vector points along the axis of rotation, and its direction is determined by the right-hand rule

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

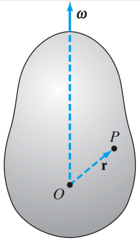


- The magnitude of ω is the angular speed (measured in radians per unit time) at which the object spins
- Assume that the angular speed is constant in this discussion

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

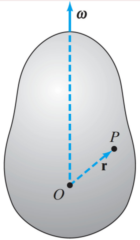


- Fix a point O (the origin) on the axis of rotation
- Let $\mathbf{r}(t) = \overrightarrow{OP}$ be the position vector of a point P of the body, measured as a function of time

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure



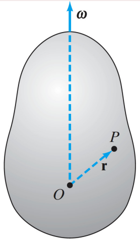
- The velocity **v** of P is defined by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$$

Rotation of a Rigid Body

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure



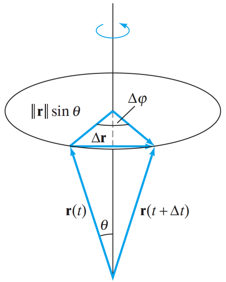
- $\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$

The vector change in position between times t and $t + \Delta t$

- Our goal is to relate $\mathbf{v}$ and ω

Rotation of a Rigid Body

Spinning an object about an axis



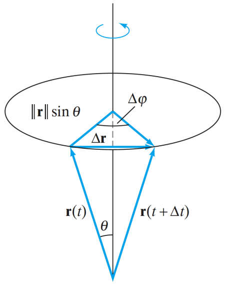
- As the body rotates, the point P (at the tip of the vector $\mathbf{r}$) moves in a circle whose plane is perpendicular to ω
- The radius of this circle is

$$\|\mathbf{r}(t)\| \sin\theta$$

where θ is the angle between ω and $\mathbf{r}$

Rotation of a Rigid Body

Spinning an object about an axis

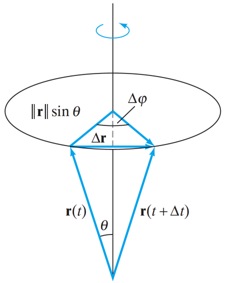


- Both $\|\mathbf{r}(t)\|$ and θ must be constant for this rotation

The direction of $\mathbf{r}(t)$
may change with t, however

Rotation of a Rigid Body

Spinning an object about an axis

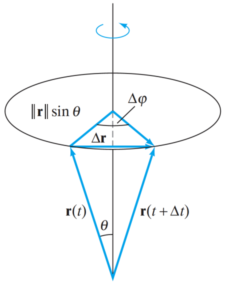


- If $t \approx 0$, then $\|\Delta\mathbf{r}\|$ is approximately the length of the circular arc swept by P between t and $t + \Delta t$
- That is,

$$\begin{aligned} \|\Delta\mathbf{r}\| &\approx \text{(radius of circle)(angle swept through by } P\text{)} \\ &= (\|\mathbf{r}\| \sin\theta)(\Delta\phi) \end{aligned}$$

Rotation of a Rigid Body

Spinning an object about an axis

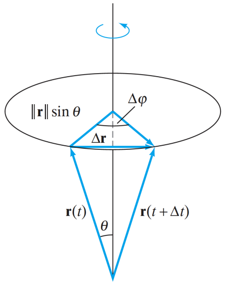


- Thus

$$\left\| \frac{\Delta \mathbf{r}}{\Delta t} \right\| \approx \|\mathbf{r}\| \sin\theta \frac{\Delta \phi}{\Delta t}$$

Rotation of a Rigid Body

Spinning an object about an axis

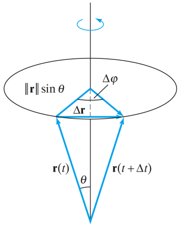


- Now, let $\Delta t \to 0$
- Then $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{v}$ and $\frac{\Delta \phi}{\Delta t} \to \|\omega\|$ by definition of the angular velocity vector ω
- Thus, we have

$$\|\mathbf{v}\| = \|\omega\|\|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|$$

Rotation of a Rigid Body

Spinning an object about an axis



$$\|\mathbf{v}\| = \|\omega\|\|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|$$

- It's not difficult to see intuitively that **v** must be perpendicular to both ω and **r**
- Right-hand rule should enable you to establish the vector equation

$$\mathbf{v} = \omega \times \mathbf{r}$$

Rotation of a Rigid Body

Spinning an object about an axis

- Apply to a bicycle wheel formula

$$\|\mathbf{v}\| = \|\omega\|\|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|$$

- It tells us that the speed of a point on the edge of the wheel is equal to the product of
  - The radius of the wheel, and
  - The angular speed

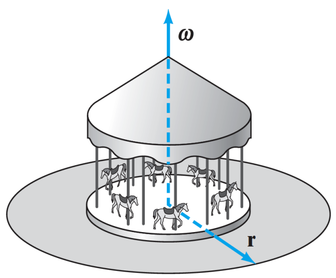
θ is $\frac{\pi}{2}$ in this case

- If the rate of rotation is kept constant, a point on the rim of a large wheel goes faster than a point on the rim of a small one

Rotation of a Rigid Body

Spinning an object about an axis

- In the case of a carousel wheel, this result tells you to sit on an outside horse if you want a more exciting ride



Summary of Products Involving Vectors

Scalar Multiplication: $k\mathbf{a}$

- Result is a vector in the direction of $\mathbf{a}$
- Magnitude is $\|k\mathbf{a}\| = |k|\|\mathbf{a}\|$
- Zero if $k = 0$ or $\mathbf{a} = \mathbf{0}$
- Commutative: $k\mathbf{a} = \mathbf{a}k$
- Associative: $k(l\mathbf{a}) = (kl)\mathbf{a}$
- Distributive: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ and $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$

Summary of Products Involving Vectors

Dot Product: $\mathbf{a} \cdot \mathbf{b}$

- Result is a scalar
- Magnitude is $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$; θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is maximized if $\mathbf{a} \parallel \mathbf{b}$
- Zero if $\mathbf{a} \perp \mathbf{b}$, $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
- Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- Associativity is irrelevant, since $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ doesn't make sense
- Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- If $\mathbf{a} = \mathbf{b}$ then $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

Summary of Products Involving Vectors

Cross Product: $\mathbf{a} \times \mathbf{b}$

- Result is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$; θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is maximized if $\mathbf{a} \perp \mathbf{b}$
- Zero if $\mathbf{a} \parallel \mathbf{b}$, $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
- Anticommutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- Not associative: In general $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- Distributive: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- If $\mathbf{a} \perp \mathbf{b}$ then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|$